Invariant annulus for homeomorphisms of the torus homotopic to Dehn twists

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Abstract

In this paper we consider torus homeomorphisms f homotopic to Dehn twists. We prove that if the vertical rotation set of f is reduced to zero, then there exists a compact connected essential "horizontal" set K, invariant under f. In other words, if we consider the lift $\widehat{f}$ of f to the cylinder, which has zero vertical rotation number, then all points have uniformily bounded motion under iterates of $\widehat{f}$. Also, we give a simple explicit condition which, when satisfied, implies that the vertical rotation set contains an interval and thus also implies positive topological entropy.

Key words: vertical rotation set, Dehn twists, omega limits, brick decompositions

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1 Introduction and main results

In this paper we study torus homeomorphisms f homotopic to Dehn twists, continuing the work started in [3]. One of the main motivations for our work is a recent example of F. Tal and A. Koropecki, where they present an area preserving torus homeomorphism h homotopic to the identity, such that its rotation set is only $(0,0)$, satisfying the following property:

- h has a lift to the plane, denoted $\widetilde{h}$, such that $\widetilde{h}$ has fixed points and some points in the plane have unbounded $\widetilde{h}$-orbits in every direction

In other words, this example implies that the existence of sub-linear displacement does not imply linear displacement, at least in the homotopic to the identity class. The main motivation for our work is to show that maps homotopic to Dehn twists have a different behavior. Before presenting our results, we need some definitions.

Definitions:

1. Let $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus and let $p : \mathbb{R}^2 \longrightarrow \mathrm{T}^2$ and $\pi : \mathbb{R}^2 \longrightarrow S^1 \times \mathbb{R}$ be the associated covering maps. Coordinates are denoted as $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$, $(\widehat{x}, \widehat{y}) \in S^1 \times \mathbb{R}$ and $(x, y) \in \mathrm{T}^2$.

2. Let $DT(\mathrm{T}^2)$ be the set of homeomorphisms of the torus homotopic to a Dehn twist $(x, y) \longrightarrow (x + k_{Dehn} y \bmod 1, y \bmod 1)$, for some $k_{Dehn} \in \mathbb{Z}^*$, and let $DT(S^1 \times \mathbb{R})$ and $DT(\mathbb{R}^2)$ be the sets of lifts of elements from $DT(\mathrm{T}^2)$ to the cylinder and plane. Homeomorphisms from $DT(\mathrm{T}^2)$ are denoted f and their lifts to the vertical cylinder and plane are respectively denoted $\widehat{f}$ and $\widetilde{f}$.

3. Let $p_{1,2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the standard projections; $p_1(\tilde{x}, \tilde{y}) = \tilde{x}$ and $p_2(\tilde{x}, \tilde{y}) = \tilde{y}$. Projections on the cylinder are also denoted by p_1 and p_2.

4. Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, the so called vertical

rotation set can be defined as follows, see [5] and [9]:

$$\rho_V(\widehat{f}) = \bigcap_{i \geq 0} \overline{\bigcup_{n \geq i} \left\{ \frac{p_2 \circ \widehat{f}^n(\widehat{z}) - p_2(\widehat{z})}{n} : \forall \widehat{z} \in S^1 \times \mathbb{R} \right\}} \tag{1}$$

This set is a closed interval (maybe a single point, but never empty) and all interior points are realized by compact f-invariant sets, see [5], [1] and [2]. Our main theorem is concerned with the case when $\rho_V(\widehat{f}) = \{0\}$.

Theorem 1 *: Given $f \in DT(\mathrm{T}^2)$ and $\widehat{f} \in DT(S^1 \times \mathbb{R})$, suppose that $\rho_V(\widehat{f}) = \{0\}$. Then there exists a compact connected $\widehat{f}$-invariant subset of $S^1 \times \mathbb{R}$ that separates the ends of the cylinder.*

Note that no area preservation hypothesis appear in our theorem. The following corollary is almost immediate:

Corollary 1 *: Given $f \in DT(\mathrm{T}^2)$ and $\widehat{f} \in DT(S^1 \times \mathbb{R})$, suppose that $\rho_V(\widehat{f}) = [a, p/q]$, for some rational p/q and some real a smaller than p/q. Then there exists $M > 0$ such that for all $\widehat{z} \in S^1 \times \mathbb{R}$, $p_2 \circ \widehat{f}^n(\widehat{z}) - p_2(\widehat{z}) - np/q < M$, for all integers $n > 0$.*

The next corollary gives an explicit criteria which implies non-degenerate vertical rotation set and thus by a result analogous to the one in [8], implies positive topological entropy (see for instance [5] and [1]).

Corollary 2 *: Given $f \in DT(\mathrm{T}^2)$ and $\widehat{f} \in DT(S^1 \times \mathbb{R})$, there exists $M > 0$ such that if for some points $\widehat{z}_1, \widehat{z}_2 \in S^1 \times \mathbb{R}$, we have $p_2 \circ \widehat{f}^{n_1}(\widehat{z}_1) - p_2(\widehat{z}_1) < -M$ and $p_2 \circ \widehat{f}^{n_2}(\widehat{z}_2) - p_2(\widehat{z}_2) > M$, for certain positive integers n_1 and n_2, then 0 is an interior point of $\rho_V(\widehat{f})$.*

This corollary is a generalization of theorem 3 of [3], which says that given an area-preserving $f \in DT(\mathrm{T}^2)$ and an exact lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$ (exact means that the vertical rotation number of the Lebesgue measure is zero), if $\rho_V(\widehat{f})$ is not reduced to 0, then 0 is an interior point of $\rho_V(\widehat{f})$. Moreover, as the proof will show, M can be easily computed from f.

This paper is organized as follows. In the second section we present some background results we use, with references and in the third section we prove our results. From now on we assume, without loss of generality, that any $f \in DT(\mathrm{T}^2)$ we consider, is homotopic to a Dehn twist $(x, y) \longrightarrow (x + k_{Dehn}y \bmod 1, y \bmod 1)$ with $k_{Dehn} > 0$.

2 Basic Tools

2.1 Brick Decompositions of the plane

We define a brick decomposition of the plane as follows:

$$\mathbb{R}^2 = \bigcup_{i=0}^{\infty} D_i,$$

where each $D_i \in Brick\_Decomposition$ is the closure of a connected simply connected open set, such that ∂D_i is a polygonal simple curve and $interior(D_i) \cap interior(D_j) = \emptyset$, for $i \neq j$. Moreover, the decomposition is locally finite, that is, $\bigcup_{i=0}^{\infty} \partial D_i$ is a graph whose vertices have at most four edges adjacent to them and the number of elements of the decomposition contained in any compact subset of the plane is finite.

Given an orientation preserving homeomorphism of the plane $\widetilde{h}$, we say that the brick decomposition is free, if all its bricks are free, that is, $\widetilde{h}(D_i) \cap D_i = \emptyset$, for all $i \in \mathbb{N}$. Given two bricks, D and E,we say that there is a chain connecting them, if there are bricks

$$D = D_0,\ D_1, D_2, ..., D_{n-1}, D_n = E$$

such that $\widetilde{h}(D_i) \cap D_{i+1} \neq \emptyset$, for $i = 0, 1, ..., n-1$. If $D = E$, the chain is said to be closed.

In the following we will present a version of a theorem of J. Franks [6] due to Le Roux and Guillou, see [7], page 39:

Lemma 1 *: The existence of a closed chain of free closed bricks implies that* *there exists a simple closed curve* $\gamma \subset \mathbb{R}^2$, *such that*

$$index(\gamma, \widetilde{h}) = degree(\gamma, \frac{\widetilde{h}(z) - z}{\left\|\widetilde{h}(z) - z\right\|}) = 1.$$

This result is a clever application of Brouwer's lemma on translation arcs.

2.2 Some results from [3]

Here we present some results from [3]. In particular, all proofs can be found there.

2.2.1 On the sets $\mathbf{B}_S^-$ and $\mathbf{B}_N^+$

Consider a homeomorphism $f \in DT(\mathrm{T}^2)$, a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$ and a lift of $\widehat{f}$ to the plane, denoted $\widetilde{f} \in DT(\mathbb{R}^2)$. Let

$$H = S^1 \times \{0\},$$

$$H^- = S^1 \times]-\infty, 0] \text{ and } H^+ = S^1 \times [0, +\infty[.$$

If we consider the closed sets,

$$B^{-(+)} = \bigcap_{n \leq 0} \widehat{f}^n(H^{-(+)}),$$

we get that they are both closed and positively $\widehat{f}$-invariant. For each of these sets, consider the following subsets: $B_S^- \subset B^-$ and $B_N^+ \subset B^+$, each of which consisting of exactly all unbounded connected components of respectively, B^- and B^+. The sets B_S^- and B_N^+ are always closed and positively $\widehat{f}$-invariant (see [4]), but in some cases may be empty. The next lemmas tell us that under certain conditions, they really exist.

Lemma 2 *: Suppose* $0 \in \rho_V(\widehat{f})$ *and for any given* $M > 0$, *there exists a positive integer* i *and a point* $\hat{z} \in S^1 \times [0, 1]$ *such that* $p_2 \circ \hat{f}^i(\hat{z}) > M$. *Then* $B_N^+ \cap H \neq \emptyset$.

Remark: A similar argument implies that if $0 \in \rho_V(\widehat{f})$ and for any given $M > 0$, there exists a positive integer i and a point $\hat{z} \in S^1 \times [0, 1]$ such that $p_2 \circ \hat{f}^i(\hat{z}) < -M$, then $B_S^- \cap H \neq \emptyset$.

The next result is a generalization of the previous one. We will present a proof here, since it does not appear elsewhere.

Lemma 3 *: Suppose* $0 \in \rho_V(\widehat{f})$. *Then* B_N^+ *and* B_S^- *are not empty.*

Proof:

Suppose that for any given $M > 0$, there exist positive integers i^- and $i^+$and points $\hat{z}^-, \hat{z}^+ \in S^1 \times [0,1]$ such that $p_2 \circ \hat{f}^{i^-}(\hat{z}^-) < -M$ and $p_2 \circ \hat{f}^{i^+}(\hat{z}^+) > M$. In this case, the above lemma implies that B_S^- and B_N^+ are not empty.

If for some $M_0 > 0$, $\widehat{f}^n(S^1 \times [0,1]) \subset S^1 \times [-M_0, M_0]$ for all integers $n > 0$, then there exists a compact connected set $K \subset S^1 \times \mathbb{R}$, such that K separates the ends of the cylinder and $\widehat{f}(K + (0,l)) = K + (0,l)$, for all integers l. So again, B_S^- and B_N^+ are not empty.

If for some $M_0 > 0$, $\widehat{f}^n(S^1 \times [0,1]) \subset S^1 \times [-M_0, +\infty[$ for all integers $n > 0$, then there exists a compact connected set $K \subset S^1 \times \mathbb{R}$, such that K separates the ends of the cylinder, K is the boundary of an open connected set denoted O, which contains the upper end of the cylinder and $\widehat{f}(O) \subset O$. If $\underset{n \geq 0}{\cap} \widehat{f}^n(closure(O)) = \emptyset$, then $0 \notin \rho_V(\widehat{f})$. So $\underset{n \geq 0}{\cap} \widehat{f}^n(closure(O)) \neq \emptyset$ and as it is closed, connected, unbounded, $\widehat{f}$-invariant and contained in O, we get that B_N^+ is not empty. The same construction can be performed with O^c and analogously it gives that $\underset{n \leq 0}{\cap} \widehat{f}^n(O^c) \neq \emptyset$ and as it is also closed, connected, unbounded, $\widehat{f}$-invariant and contained in O^c, B_S^- is also not empty.

The remaining possibility can be treated in an analogous way. □

2.2.2 The ω-limit sets of $\mathbf{B}_S^-$ and $\mathbf{B}_N^+$

In this subsection we examine some properties of the set

$$\omega(B_S^-) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{i=n}^{\infty} \widehat{f}^i(B_S^-)} = \bigcap_{n=0}^{\infty} \widehat{f}^n(B_S^-). \tag{2}$$

Lemma 4 *:* $\omega(B_S^-)$ *is a closed,* $\widehat{f}$*-invariant set, whose connected components are all unbounded.*

Of course, since B_S^- is closed and positively $\widehat{f}$-invariant, we also have that $\omega(B_S^-) \subset B_S^-$. It is still possible that $\omega(B_S^-) = \emptyset$, and the next lemma tells us that in this case, things are easier.

Lemma 5 *: Suppose* $0 \in \rho_V(\widehat{f})$ *and* $\omega(B_S^-) = \emptyset$. *Then* $\rho_V(\widehat{f}) \supset [0, -\epsilon]$, *for some* $\epsilon > 0$.

Now, if we consider the set B_S^- for $\widehat{f}^{-1}$, denoted $B_S^-(inv)$, we get:

Lemma 6 *: The sets* $\omega(B_S^-)$ *and* $\omega(B_S^-(inv))$ *are equal.*

Clearly, similar results hold for B_N^+.

3 Proofs

3.1 Proof of theorem 1

Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, assume that $\rho_V(\widehat{f}) = \{0\}$. Lemmas 3 and 5 imply that $B_N^+ \neq \emptyset$, $B_S^- \neq \emptyset$ and the same holds for their ω-limits, $\omega(B_N^+) \neq \emptyset$ and $\omega(B_S^-) \neq \emptyset$.

The fact that $\omega(B_N^+) \neq \emptyset$ and $\omega(B_S^-) \neq \emptyset$ has the following consequence, which already appeared in [3]. From now on, let us fix some $\widetilde{f} \in DT(\mathbb{R}^2)$, which is a lift of $\widehat{f}$:

Lemma 7 *:There exists a real number* $M' > 0$ *such that for all* $\widetilde{x} \in \mathbb{R}$, $\widetilde{f}^n(\{\widetilde{x}\} \times [M', +\infty[) \cap \{\widetilde{x}\} \times [M', +\infty[= \emptyset$ *and* $\widetilde{f}^n(\{\widetilde{x}\} \times]-\infty, -M']) \cap \{\widetilde{x}\} \times]-\infty, -M'] = \emptyset$ *for all integers* $n > 0$.

Proof:

See lemmas 7 and 8 of [3]. □

Remark: The constant M' can be computed in terms of $\widetilde{f}$ in a simple way. To be precise, note that for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$, there are constants $A_f > 0$ and $B_f > 0$ such that

$$\left| p_2 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) - \widetilde{y} \right| < A_f \text{ and } \left| p_1 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) - \widetilde{x} - k_{Dehn}\widetilde{y} \right| < B_f. \qquad (3)$$

Using these constants and following the ideas in the proof of lemmas 7 and 8 of [3], we get that

$$M' = \frac{14 + 3B_f}{k_{Dehn}} + A_f + 2 \tag{4}$$

Now let us suppose by contradiction that there exists a point $\widehat{z}$ in the cylinder and an integer $n_0 > 0$ such that

$$\left| p_2(\widehat{f}^{n_0}(\widehat{z})) - p_2(\widehat{z}) \right| > 2M' + 8.$$

Without loss of generality, we can assume that $p_2(\widehat{z}) < -M' - 3$ and $p_2(\widehat{f}^{n_0}(\widehat{z})) > M' + 3$.

Let us also consider the fixed point free mapping of the plane

$$\widetilde{g}(\bullet) = \widetilde{f}^{n_0}(\bullet) - (0,1).$$

First, note that for all $\widetilde{x} \in \mathbb{R}$, $\widetilde{g}(\{\widetilde{x}\} \times [M'+2, +\infty[) \cap \{\widetilde{x}\} \times [M'+2, +\infty[= \emptyset$ and $\widetilde{g}(\{\widetilde{x}\} \times]-\infty, -M'-2]) \cap \{\widetilde{x}\} \times]-\infty, -M'-2] = \emptyset$. Moreover, using the fact that $\widetilde{g}$ is also the lift of a torus homeomorphism homotopic to a Dehn twist and a compacity argument, one can prove that there exists an integer $N > 0$, such that for all integers n, the sets

$$\begin{array}{c} F_n^- = [n/N, (n+1)/N] \times]-\infty, -M'-2] \\ \text{and} \\ F_n^+ = [n/N, (n+1)/N] \times [M'+2, \infty[\end{array} \tag{5}$$

are free under $\widetilde{g}$, that is, $\widetilde{g}(F_n^{+or-}) \cap F_n^{+or-} = \emptyset$, for all integers n. Moreover, the fact that $k_{Dehn} > 0$ (see the end of section 1) implies that there exists an integer $K_{crit} > 0$, such that for all integers n

$$\begin{array}{c} \widetilde{g}(F_n^+) \cap F_m^+ \neq \emptyset, \text{ for all } m \geq n + K_{crit} \\ \text{and} \\ \widetilde{g}(F_n^-) \cap F_m^- \neq \emptyset, \text{ for all } m \leq n - K_{crit}. \end{array}$$

These will be important bricks in a special brick decomposition of the plane in $\widetilde{g}$-free sets we will construct, which will be invariant under integer horizontal translations $(x,y) \to (x+1,y)$.

Clearly, such a construction is possible, because as $\widetilde{g}(\widetilde{x}+1, \widetilde{y}) = \widetilde{g}(\widetilde{x}, \widetilde{y}) + (1,0)$, we just have to decompose $S^1 \times [-M'-2, M'+2]$ into a union of bricks with sufficiently small diameter, so that their pre-images under π are $\widetilde{g}$-free.

To conclude our proof, we will show that this brick decomposition has a closed brick chain, a contradiction with the fact that $\widetilde{g}$ is fixed point free, see lemma 1. This idea was already used in the proof of theorem 4 of [2].

Consider a point $\widetilde{z} \in \pi^{-1}(\widehat{z})$ and a brick $F_{i_0}^-$ that contains $\widetilde{z}$. From our choices,

$$\widetilde{g}(F_{i_0}^-) \cap F_{i_1}^+ \neq \emptyset, \text{ for some integer } i_1.$$

As $\rho_V(\widehat{f}) = \{0\}$, there exists a point $\widehat{w} \in S^1 \times]M' + 2, +\infty[$ such that

$$p_2(\widehat{g}^n(\widehat{w})) \overset{n\to\infty}{\to} -\infty.$$

So, we can choose a point $\widetilde{w} \in F_{i_2}^+$, for some integer i_2, such that:

- $i_2 > i_1 + K_{crit}$, so $\widetilde{g}(F_{i_1}^+) \cap F_{i_2}^+ \neq \emptyset$;
- $\widetilde{g}^{n_2}(\widetilde{w}) \in F_{i_3}^-$, for some integers $n_2 > 0$ and $i_3 > i_0 + K_{crit}$;

As $\widetilde{g}(F_{i_3}^-) \cap F_{i_0}^- \neq \emptyset$, we get there exists a closed brick chain starting at $F_{i_0}^-$. As we said, this is a contradiction because the existence of a fixed point for $\widetilde{g}$ implies that $1/n_0 \in \rho_V(\widehat{f})$. Thus $\widehat{f}^n(S^1 \times [0,1]) \subset S^1 \times [-10 - 2M', 2M' + 10]$ for all integers $n > 0$ and there exists a compact connected set $K \subset S^1 \times \mathbb{R}$, such that K separates the ends of the cylinder, $\widehat{f}(K + (0,l)) = K + (0,l)$, for all integers l and $|p_2(K)| \leq 4M' + 20$. □

3.2 Proof of corollary 1

Without loss of generality, by considering $\widehat{f}^q - (0,p)$, we can suppose that $\rho_V(\widehat{f}) = [a, 0]$, for some $a < 0$. As in the proof of theorem 1, lemma 3 implies that $B_N^+ \neq \emptyset$, $B_S^- \neq \emptyset$ and $B_N^+(inv) \neq \emptyset$, $B_S^-(inv) \neq \emptyset$. If for instance $\omega(B_S^-) = \emptyset$, then lemma 6 implies that $\omega(B_S^-(inv)) = \emptyset$ and so lemma 5 implies that there exists $\epsilon > 0$ such that $\rho_V(\widehat{f}^{-1}) \supset [-\epsilon, 0]$, which gives $\rho_V(\widehat{f}) \supset [0, \epsilon]$, a contradiction. So, we can assume that $\omega(B_N^+) \neq \emptyset$ and $\omega(B_S^-) \neq \emptyset$. If we suppose that for every $M > 0$, there exists a point $\widehat{z} \in S^1 \times \mathbb{R}$ and an integer $n > 0$ such that

$$p_2(\widehat{f}^n(\widehat{z})) - p_2(\widehat{z}) > M,$$

then following exactly the same ideas used in theorem 1, we arrive at a contradiction which proves the corollary. □

3.3 Proof of corollary 2

As in theorem 1, let us fix a $\widetilde{f} \in DT(\mathbb{R}^2)$, which is a lift of $\widehat{f}$. First, we will show that if

$$M \geq M_0 \overset{def.}{=} (20 + 2B_f)/k_{Dehn} + 10 \text{ (see (3))},$$

then $\widehat{f}$ has a fixed point. In case $\widehat{f}$ is fixed point free, lemma 2 of [2] tells us that there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that $\widehat{f}(\gamma) \cap \gamma = \emptyset$ and $\gamma \subset S^1 \times [-m_D, m_D]$, where $m_D > 0$ is the smallest real number that satisfies

$$\begin{array}{c} \widetilde{f}(\{\widetilde{x}\} \times [m_D, +\infty[) \subset [\widetilde{x} + 10, +\infty[\times \mathbb{R} \\ and \\ \widetilde{f}(\{\widetilde{x}\} \times [-\infty, -m_D]) \subset]-\infty, \widetilde{x} - 10] \times \mathbb{R}, \end{array} \tag{6}$$

for all $\widetilde{x} \in \mathbb{R}$. A simple computation shows that if we take m_D equal $(10 + B_f)/k_{Dehn}$, then (6) is satisfied.

So, as $M \geq 2m_D + 10$, the corollary hypotheses imply that $\widehat{f}$ has a fixed point. Thus $0 \in \rho_V(\widehat{f})$ and lemma 3 implies that $B_N^+ \neq \emptyset$, $B_S^- \neq \emptyset$ and the same holds for the inverse of $\widehat{f}$, namely, $B_S^-(inv) \neq \emptyset$ and $B_N^+(inv) \neq \emptyset$. If $\omega(B_N^+) = \emptyset$, then lemma 5 implies that there exists $\delta > 0$ such that $\rho_V(\widehat{f}) \supset [0, \delta]$. Also, from lemma 6 we get that $\omega(B_N^+(inv)) = \emptyset$ and so again by lemma 5, there exists $\epsilon > 0$ such that $\rho_V(\widehat{f}^{-1}) \supset [0, \epsilon]$, which gives $\rho_V(\widehat{f}) \supset [-\epsilon, \delta]$ and the corollary is proved. So, again we can suppose that $\omega(B_S^-) \neq \emptyset$ and $\omega(B_N^+) \neq \emptyset$.

If $\rho_V(\widehat{f}) = [a, 0]$ for some $a \leq 0$, then if

$$M \geq M_1 \overset{def.}{=} 2M' + 8 = \frac{28 + 6B_f}{k_{Dehn}} + 2A_f + 12,$$

by the same argument used to prove theorem 1, we arrive at a contradiction. The same happens in the other possibility, that is, if $\rho_V(\widehat{f}) = [0, b]$, for some $b > 0$.

So, it is enough to choose

$M = \max\{M_0, M_1\} = M_1 = \dfrac{28 + 6B_f}{k_{Dehn}} + 2A_f + 12$ to finish the proof. □

References

[1] Addas-Zanata S. (2002): On the existence of a new type of periodic and quasi-periodic orbits for twist maps of the torus. *Nonlinearity* **15**, 1399-1416

[2] Addas-Zanata S. (2005): Some extensions of the Poincaré-Birkhoff theorem to the cylinder and a remark on mappings of the torus homotopic to Dehn twists. *Nonlinearity* **18**, 2243-2260.

[3] Addas-Zanata S., Garcia B. and Tal F. (2010): On the dynamics of homeomorphisms of the torus homotopic to Dehn twists. *preprint.*

[4] Addas-Zanata S. and Tal F. A. (2009): Homeomorphisms of the annulus with a transitive lift. online at *Math. Zeit.*

[5] Doeff H.E. (1997): Rotation Measures for homeomorphisms of the torus homotopic to a Dehn twist. *Ergod. Th. & Dynam. Sys.* **17,** 575-591

[6] Franks J. (1988): Generalizations of the Poincaré-Birkhoff theorem. *Ann. of Math.* **128,** 139-151

[7] Le Roux F.(2004): Homéomorphismes de surfaces: théorèmes de la fleur de Leau-Fatou et de la variété stable. *Astérisque* **292**, 210 pp.

[8] Llibre J. e Mackay R. (1991): Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergod. Th. & Dynam. Sys.* **11,** 115-128.

[9] Misiurewicz M and Ziemian K (1989): Rotation sets for maps of tori. *J. London Math. Soc.* **40** 490-506.