Dynamics of homeomorphisms of the torus homotopic to Dehn twists

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Abstract

In this paper we consider torus homeomorphisms f homotopic to Dehn twists. We prove that if the vertical rotation set of f is reduced to zero, then there exists a compact connected essential "horizontal" set K, invariant under f. In other words, if we consider the lift $\widehat{f}$ of f to the cylinder, which has zero vertical rotation number, then all points have uniformly bounded motion under iterates of $\widehat{f}$. Also, we give a simple explicit condition which, when satisfied, implies that the vertical rotation set contains an interval and thus also implies positive topological entropy. As a corollary of the above results, we prove a version of Boyland's conjecture to this setting: If f is area preserving and has a lift $\widehat{f}$ to the cylinder with zero Lebesgue measure vertical rotation number, then either the orbits of all points are uniformly bounded under $\widehat{f}$, or there are points in the cylinder with positive vertical velocity and others with negative vertical velocity.

Key words: vertical rotation set, omega limits, brick decompositions

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1 Introduction and main results

In this paper we study homeomorphisms f of the torus homotopic to Dehn twists. These homotopy classes are in some way simpler to analyze than the identity case. One of the reasons for this is the fact that there is no sense in defining a two dimensional rotation set for torus maps homotopic to Dehn twists, instead a vertical rotation set is defined, see expression (1).

Many important conjectures for homotopic to the identity maps have their analogs in this setting. For instance, how is the rotation interval of a minimal Dehn twist homeomorphism? Does the set of minimal Dehn twist C^r-diffeomorphisms ($r \geq 2$) have no interior? If f is a Dehn twist homeomorphism which preserves area and has zero Lebesgue measure vertical rotation number, is it true that either f is more or less like an annulus homeomorphism or the vertical rotation interval has no empty interior?

One of the main motivations for our work is a recent example of F. Tal and A. Koropecki, where they present an area preserving torus homeomorphism h homotopic to the identity, such that its rotation set is only $(0,0)$, satisfying the following property:

- h has a lift to the plane, denoted $\widetilde{h}$, such that $\widetilde{h}$ has fixed points and some points in the plane have unbounded $\widetilde{h}$-orbits in every direction

In other words, this example implies that the existence of sub-linear displacement does not imply linear displacement, at least in the homotopic to the identity class. In this work we show that maps homotopic to Dehn twists have a different behavior. Before presenting our results, we need some definitions.

Definitions:

1. Let $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus and let $p : \mathbb{R}^2 \longrightarrow \mathrm{T}^2$ and $\pi : \mathbb{R}^2 \longrightarrow S^1 \times \mathbb{R}$ be the associated covering maps. Coordinates are denoted as $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$, $(\widehat{x}, \widehat{y}) \in S^1 \times \mathbb{R}$ and $(x, y) \in \mathrm{T}^2$.

2. Let $DT(\mathrm{T}^2)$ be the set of homeomorphisms of the torus homotopic to a Dehn twist $(x, y) \longrightarrow (x + ky \bmod 1, y \bmod 1)$, for some $k \in \mathbb{Z}^*$, and let

$DT(S^1 \times \mathbb{R})$ and $DT(\mathbb{R}^2)$ be the sets of lifts of elements from $DT(\mathrm{T}^2)$ to the cylinder and plane. Homeomorphisms from $DT(\mathrm{T}^2)$ are denoted f and their lifts to the vertical cylinder and plane are respectively denoted $\widehat{f}$ and $\widetilde{f}$.

3. Let $p_{1,2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the standard projections; $p_1(\tilde{x}, \tilde{y}) = \tilde{x}$ and $p_2(\tilde{x}, \tilde{y}) = \tilde{y}$. Projections on the cylinder are also denoted by p_1 and p_2.

4. Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, the so called vertical rotation set can be defined as follows, see [12]:

$$\rho_V(\widehat{f}) = \bigcap_{i \geq 1} \overline{\bigcup_{n \geq i} \left\{ \frac{p_2 \circ \widehat{f}^n(\widehat{z}) - p_2(\widehat{z})}{n} : \widehat{z} \in S^1 \times \mathbb{R} \right\}} \tag{1}$$

   This set is a closed interval (maybe a single point, but never empty) and it was proved in [1] and [3] (and much earlier in [6], although the first author discovered this only recently) that all numbers in its interior are realized by compact f-invariant subsets of T^2, which are periodic orbits in the rational case. From its definition, it is easy to see that

$$\rho_V(\widehat{f}^m + (0, n)) = m.\rho_V(\widehat{f}) + n \text{ for any integers } n, m.$$

5. Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, let μ be a f-invariant Borel probability measure. We define the vertical rotation number of μ as follows:

$$\rho_V(\mu) = \int_{\mathrm{T}^2} \phi(x, y) d\mu,$$

   where the vertical displacement function $\phi : \mathrm{T}^2 \to \mathbb{R}$ is given by $\phi(x, y) = p_2 \circ \widehat{f}(\widehat{x}, \widehat{y}) - \widehat{y}$, for any $(\widehat{x}, \widehat{y}) \in S^1 \times \mathbb{R}$ such that $\pi^{-1}(\widehat{x}, \widehat{y}) \subset p^{-1}(x, y)$.

So, given $f \in DT(\mathrm{T}^2)$ and $\widehat{f} \in DT(S^1 \times \mathbb{R})$, as we said above, one wants to know, under which conditions can f be minimal? It is not difficult to see that in this case the vertical rotation interval must be a single point, otherwise there would be infinitely many periodic orbits. But more can be said.

Theorem 1 *: Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, suppose that f is minimal. Then, $\rho_V(\widehat{f}) = \{\alpha\}$ for some irrational number α.*

So, if f is a C^r diffeomorphism, for some $r \geq 2$, is there a natural perturbation that destroys minimality? As the extreme points of $\rho_V(\widehat{f})$ vary continuously with $\widehat{f} \in DT(S^1 \times \mathbb{R})$ (see [7]), a way to attack this problem is by showing that irrational extremes are not stable under perturbations. This was done in [2] for twist mappings on the torus.

The main problem addressed in this paper is in a way, complementary to the above. Suppose for instance that $\rho_V(\widehat{f})$ contains a single rational number p/q. What can we say about the dynamics of f? And if f preserves area and the center of gravity, that is Lebesgue measure has zero vertical rotation number, what can we say about its vertical rotation interval? When it is not reduced to zero, is zero always an interior point? This is the so called Boyland's Conjecture.

Below we state our main results:

Theorem 2 *: Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, if $\rho_V(\widehat{f}) = \{p/q\}$, for some rational p/q, then there exists a compact connected set $K \subset S^1 \times \mathbb{R}$, invariant under $\widehat{f}^q - (0,p)$, which separates the ends of the cylinder. So, all points have uniformly bounded orbits under the action of $\widehat{f}^q - (0,p)$.*

Note that no area preservation hypothesis appear in our theorem. The following corollary is almost immediate:

Corollary 1 *: Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, suppose that $\rho_V(\widehat{f}) = [a, p/q]$, for some rational p/q and some real a smaller than p/q. Then there exists $M > 0$ such that for all $\widehat{z} \in S^1 \times \mathbb{R}$, $p_2 \circ \widehat{f}^n(\widehat{z}) - p_2(\widehat{z}) - np/q < M$, for all integers $n > 0$.*

The next result gives an explicit criteria which implies non-degenerate vertical rotation sets and thus by a result analogous to the one in [11], implies positive topological entropy (see for instance [6] and [1]).

Theorem 3 *: Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, there exists $M > 0$ (which can be explicitly computed) such that if for some points $\widehat{z}_1, \widehat{z}_2 \in$*

$S^1 \times \mathbb{R}$, *we have* $p_2 \circ \widehat{f}^{n_1}(\widehat{z}_1) - p_2(\widehat{z}_1) < -M$ *and* $p_2 \circ \widehat{f}^{n_2}(\widehat{z}_2) - p_2(\widehat{z}_2) > M$, *for certain positive integers* n_1 *and* n_2, *then* 0 *is an interior point of* $\rho_V(\widehat{f})$.

The next result gives a positive answer for Boyland's conjecture in this setting:

Corollary 2 *: Given an area-preserving* $f \in DT(\mathrm{T}^2)$ *and a lift* $\widehat{f} \in DT(S^1 \times \mathbb{R})$ *with zero Lebesgue measure vertical rotation number, then either* $\rho_V(\widehat{f})$ *is reduced to* 0, *or* 0 *is an interior point of* $\rho_V(\widehat{f})$.

This paper is organized as follows. In the second section we present some background results we use, with references and a few proofs and in the third section we prove our main results. From now on we assume, without loss of generality, that any $f \in DT(\mathrm{T}^2)$ we consider, is homotopic to a Dehn twist $(x, y) \longrightarrow (x + k_{Dehn} y \bmod 1, y \bmod 1)$ with $k_{Dehn} > 0$.

2 Basic Tools

2.1 Brick Decompositions of the plane

We define a brick decomposition of the plane as follows:

$$\mathbb{R}^2 = \bigcup_{i=0}^{\infty} D_i,$$

where each $D_i \in Brick\_Decomposition$ is the closure of a connected simply connected open set, such that ∂D_i is a polygonal simple curve and $interior(D_i) \cap interior(D_j) = \emptyset$, for $i \neq j$. Moreover, the decomposition is locally finite, that is, $\bigcup_{i=0}^{\infty} \partial D_i$ is a graph whose vertices have three edges adjacent to them and the number of elements of the decomposition contained in any compact subset of the plane is finite.

Given an orientation preserving homeomorphism of the plane $\widetilde{h}$, we say that the brick decomposition is free, if all its bricks are free, that is, $\widetilde{h}(D_i) \cap D_i = \emptyset$, for all $i \in \mathbb{N}$. Given two bricks, D and E,we say that there is a chain connecting them, if there are bricks

$$D = D_0,\ D_1, D_2, ..., D_{n-1}, D_n = E$$

such that $\widetilde{h}(D_i) \cap D_{i+1} \neq \emptyset$, for $i = 0, 1, ..., n-1$. If $D = E$, the chain is said to be closed.

In the following we will present a version of a theorem of J. Franks [8] due to Le Roux and Guillou, see [9], page 39:

Lemma 1 *: The existence of a closed chain of free closed bricks implies that there exists a simple closed curve* $\gamma \subset \mathbb{R}^2$*, such that*

$$index(\gamma, \widetilde{h}) = degree(\gamma, \frac{\widetilde{h}(z) - z}{\left\| \widetilde{h}(z) - z \right\|}) = 1.$$

This result is a clever application of Brouwer's lemma on translation arcs.

2.2 On the sets $\mathbf{B}_S^-$ and $\mathbf{B}_N^+$

Here we present a theory developed in [4] and extend some constructions to our new setting. For this, consider a homeomorphism $f \in DT(\mathrm{T}^2)$, a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$ and a lift of $\widehat{f}$ to the plane, denoted $\widetilde{f} \in DT(\mathbb{R}^2)$. Given a real number a, let

$$H_a = S^1 \times \{a\},$$

$$H_a^- = S^1 \times]-\infty, a] \text{ and } H_a^+ = S^1 \times [a, +\infty[.$$

We will also denote the sets H_0, H_0^- and H_0^+ simply by H, H^- and H^+ respectively. If we consider the closed sets,

$$B^- = \bigcap_{n \leq 0} \widehat{f}^n(H^-)$$

$$and$$

$$B^+ = \bigcap_{n \leq 0} \widehat{f}^n(H^+),$$

we get that they are both closed and positively $\widehat{f}$-invariant. For each of these sets, consider the following subsets: $B_S^- \subset B^-$ and $B_N^+ \subset B^+$, each of which consisting of exactly all unbounded connected components of respectively, B^- and B^+. The sets B_S^- and B_N^+ are always closed, but in some cases may be empty. The next lemma tells us that under certain conditions, they really exist.

Lemma 2 *: Suppose $0 \in \rho_V(\widehat{f})$. Then B_N^+ and B_S^- are not empty.*

Proof:

The proof of this result goes back to Le Calvez [10] and even Birkhoff [5].

First, suppose that $\bigcup_{n\geq 0} \widehat{f}^n(H)$ is unbounded both from above and from below. In this case, considering the set B_S^-, the only thing we have to prove is that, for all $a \leq -1$, there exists a first positive integer $n = n(a)$, such that

$$\widehat{f}^{-n}(H_a) \cap H \neq \emptyset \text{ and } n(a) \to \infty \text{ as } a \to -\infty. \tag{2}$$

Our assumption on $\bigcup_{n\geq 0} \widehat{f}^n(H)$ implies that $\widehat{f}^N(H) \cap H_a^- \neq \emptyset$ for some integer $N > 0$. If expression (2) does not hold for N, then $\widehat{f}^{-N}(H_a) \subset H^+ \subset H_a^+ + (0,1)$, which would imply that $0 \notin \rho_V(\widehat{f})$, a contradiction. So expression (2) is true and the proof continues, for instance as in lemma 6 of [4]. A similar argument holds for B_N^+ (in this case $a \geq 1$).

If for some integer $M_0 > 0$, $\widehat{f}^n(S^1 \times \{0\}) \subset S^1 \times [-M_0, +\infty[$ for all integers $n \geq 0$, then clearly $\widehat{f}^n(S^1 \times [M_0, +\infty[) \subset S^1 \times [0, +\infty[$ for all integers $n \geq 0$, so $B_N^+ \supset S^1 \times [M_0, +\infty[$ and thus, it is not empty.

To prove that B_S^- is also not empty, we have to work a little more.

Let $O^* = \bigcup_{n\geq 0} \widehat{f}^n(S^1 \times]0, +\infty[)$ and let O be the complement of the connected component of $(O^*)^c$ which contains the lower end of the cylinder. We claim that O^c is connected and the same holds for $\partial O \overset{def.}{=} K$. This follows if we consider the $North - South$ compactification of the cylinder and remember that it is a classical result, in the plane or sphere, that the frontier of any connected component of the complement of a compact connected subset is also connected. Clearly, $O^* \subset O$ (we just fill the holes), O contains the upper end of the cylinder and $\widehat{f}(O) \subset O$.

If $\bigcap_{n\leq 0} \widehat{f}^n(O^c) = \emptyset$, then $0 \notin \rho_V(\widehat{f})$. So $\bigcap_{n\leq 0} \widehat{f}^n(O^c) \neq \emptyset$ and as each connected component of this closed $\widehat{f}$-invariant set is bounded from above and unbounded, we get that for a sufficiently large integer $j \geq 0$, $\bigcap_{n\leq 0} \widehat{f}^n(O^c) - (0,j) \subset B_S^- \neq \emptyset$.

The remaining possibility can be treated in an analogous way. □

2.3 The ω-limit sets of $\mathbf{B}_S^-$ and $\mathbf{B}_N^+$

In this subsection we examine some properties of the set

$$\omega(B_S^-) \overset{def.}{=} \bigcap_{n=0}^{\infty} \overline{\bigcup_{i=n}^{\infty} \widehat{f}^i(B_S^-)}. \tag{3}$$

Due to the fact that $\widehat{f}(B_S^-) \subset B_S^- = \overline{B_S^-}$, we get that

$$\omega(B_S^-) = \bigcap_{n=0}^{\infty} \widehat{f}^n(B_S^-) = \bigcap_{n=-\infty}^{\infty} \widehat{f}^n(B_S^-). \tag{4}$$

Lemma 3 *: $\omega(B_S^-)$ is a closed, $\widehat{f}$-invariant set, whose connected components are all unbounded.*

Proof:

See lemma 7 of [4]. $\square$

So from (4), $\omega(B_S^-) \subset B_S^-$ and it is still possible that $\omega(B_S^-) = \emptyset$. The next lemma tells us that in this case, things are easier.

Lemma 4 *: Suppose $0 \in \rho_V(\widehat{f})$, which implies that B_S^- is not empty. If $\omega(B_S^-) = \emptyset$, then $\rho_V(\widehat{f}) \supset [-\epsilon, 0]$, for some $\epsilon > 0$.*

Proof:

See the proof of lemma 10 of [4] and the paragraph below it. $\square$

Now, if we consider the set B_S^- for $\widehat{f}^{-1}$, denoted $B_S^-(inv)$, we get the following:

Lemma 5 *: The sets $\omega(B_S^-)$ and $\omega(B_S^-(inv))$ are equal.*

Proof:

Let Γ be a connected component of $\omega(B_S^-)$. From the definition, $\widehat{f}^n(\Gamma) \subset H^-$ for all integers n. So $\Gamma \subset B_S^-(inv)$ and moreover, for each positive integer n, as $\widehat{f}^n(\Gamma)$ is contained in H^-, we get that $\Gamma \subset \widehat{f}^{-n}(B_S^-(inv))$, which means that $\Gamma \subset \omega(B_S^-(inv))$. Thus $\omega(B_S^-) \subset \omega(B_S^-(inv))$. The other inclusion is proved in an analogous way. $\square$

The following are important results on the structure of these sets.

Lemma 6 *: Any connected component $\widetilde{\Gamma}$ of $\pi^{-1}(\omega(B_S^-))$ is unbounded, not necessarily in the $\widetilde{y}$-direction.*

Proof:

Let d be the metric on $S^1\times$ IR and let $\widetilde{d}$ be the lifted metric on the plane. Consider a point $\widetilde{P} \in \widetilde{\Gamma}$ and let $P = \pi(\widetilde{P})$. As $P \in \omega(B_S^-)$, there exists a connected component Γ of $\omega(B_S^-)$ that contains P. Since by lemma 3 Γ is unbounded, for every sufficiently large integer n there exists a simple continuous arc $\gamma_n \subset S^1\times$ IR such that:

- P is one endpoint of γ_n;
- γ_n is contained in $S^1 \times [-n, 0]$ and it intersects $S^1 \times \{-n\}$ only at its other endpoint;
- γ_n is contained in a $(1/n, d)$-neighborhood of Γ;

Now let $\widetilde{\gamma}_n$ be the connected component of $\pi^{-1}(\gamma_n)$ that contains $\widetilde{P}$. This arc $\widetilde{\gamma}_n$ is contained in a $(1/n, \widetilde{d})$-neighborhood of $\pi^{-1}(\Gamma) \subset \pi^{-1}(\omega(B_S^-))$ because the covering map is locally an isometry.

Now, embed the plane in the sphere $S^2 = \mathbb{R}^2 \sqcup \{\infty\}$ equipped with a metric D topologically equivalent to the metric $\widetilde{d}$ on the plane. Then there exists a subsequence $\widetilde{\gamma}_{n_i} \overset{i\to\infty}{\to} \Theta$ in the Hausdorff topology, for some compact connected set $\Theta \subset S^2$. Clearly, both ∞ and $\widetilde{P}$ belong to Θ. Furthermore, since $\pi^{-1}(\omega(B_S^-)) \cup \{\infty\}$ is a closed set and

$$\lim_{n\to\infty} \left(\sup_{\widetilde{z}\in\widetilde{\gamma}_n} \widetilde{d}(\widetilde{z}, \pi^{-1}(\omega(B_S^-))) \right) = 0,$$

we get that $\pi^{-1}(\omega(B_S^-)) \cup \{\infty\}$ contains Θ and the proof is over. $\Box$

Lemma 7 *: For any connected component $\widetilde{\Gamma}$ of $\pi^{-1}(\omega(B_S^-))$, $\widetilde{\Gamma}^c$ is connected.*

Proof:

Take a connected component $\widetilde{\Gamma}$ of $\pi^{-1}(\omega(B_S^-))$. First note that $\widetilde{\Gamma}^c$ has one connected component, denoted O^+, which contains $\mathbb{R}\times]0, +\infty[$. So, if there is another one, denoted O_1, it must be contained in $\mathbb{R}\times]-\infty,0]$. In the following we will prove that $\widetilde{f}^n(O_1) \subset \mathbb{R}\times]-\infty,0]$ for all integers n.

By contradiction, suppose that

$$\text{there is an integer } n_0 \text{ such that } \widetilde{f}^{n_0}(O_1) \text{ is not contained in } \mathbb{R}\times]-\infty,0]. \quad (5)$$

There exists a number $m_0 > 0$ such that if $\widetilde{y} > m_0$, then the point $\widetilde{f}^{-n_0}(\widetilde{x},\widetilde{y})$ has positive $\widetilde{y}$-coordinate, for all $\widetilde{x} \in \mathbb{R}$ (see (6)). So our hypothesis in (5) implies that $\widetilde{f}^{-n_0}(\mathbb{R}\times]0,\infty[) \cap \partial O_1 \neq \emptyset$, which means that $\widetilde{f}^{n_0}(\partial O_1)$ intersects $\mathbb{R}\times]0,\infty[$, a contradiction with the fact that

$$\widetilde{f}^{n_0}(\partial O_1) \subset \widetilde{f}^{n_0}(\widetilde{\Gamma}) \subset \pi^{-1}(\omega(B_S^-)) \subset \mathbb{R}\times]-\infty,0].$$

So (5) does not hold. To conclude, let Γ be the connected component of $\omega(B_S^-)$ that contains $\pi(\widetilde{\Gamma})$, which as we know by lemma 3 is unbounded. The set $O_1 \cup \widetilde{\Gamma}$ is connected as well as $\pi(O_1 \cup \widetilde{\Gamma}) \cup \Gamma = \pi(O_1) \cup \Gamma$ and the later is contained in $\omega(B_S^-)$ because $\widehat{f}^n(\pi(O_1)) \subset H^-$ for all integers n. It follows that $\pi(O_1) \cup \Gamma = \Gamma \subset \omega(B_S^-)$ and therefore $O_1 \cup \widetilde{\Gamma}$ is contained in $\pi^{-1}(\omega(B_S^-))$,a contradiction with the choice of $\widetilde{\Gamma}$. $\Box$

Clearly, similar results hold for B_N^+.

3 Proofs

3.1 Proof of theorem 1

Assume $f \in DT(\mathrm{T}^2)$ and its lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$ are such that f is minimal and $\rho_V(\widehat{f})$ is rational. Without loss of generality we can assume that $\rho_V(\widehat{f}) = 0$, because if f is minimal, the same happens for all its iterates. This follows from the fact that, if for some integer $q > 0$, f^q is not minimal, then it has a compact invariant minimal set $K \subset \mathrm{T}^2$, which by minimality, has empty interior. But then,

$$K \cup f(K) \cup ... \cup f^{q-1}(K)$$

is invariant under f and as K^c is open and dense, Baire 's property also implies that $K \cup f(K) \cup ... \cup f^{q-1}(K)$ has empty interior, a contradiction with the minimality of f.

As $\widehat{f}$ has no fixed points, lemma 2 of [3] implies that there exists a homotopically non trivial simple closed curve γ in the cylinder such that $\gamma \cap \widehat{f}(\gamma) = \emptyset$. Without loss of generality, we can suppose that $\widehat{f}(\gamma) \subset \gamma^-$, the connected component of γ^c which is below γ. Let $k > 0$ be an integer such that $\gamma - (0,k) \subset \gamma^-$. If for some $n > 0$, $\widehat{f}^n(\gamma) \subset (\gamma - (0,k))^-$, then 0 would not belong to $\rho_V(\widehat{f})$. So, for all $n > 0$, there exists a point $\widehat{z}_n$, above $\widehat{f}(\gamma)$ and below γ, such that

$$\{\widehat{z}_n, \widehat{f}(\widehat{z}_n), \widehat{f}^2(\widehat{z}_n), ..., \widehat{f}^n(\widehat{z}_n)\} \text{ is above } \gamma - (0,k).$$

Taking a subsequence if necessary, we can assume that $\widehat{z}_{n_i} \stackrel{i\to\infty}{\to} \widehat{z}^*$, a point in the closure of the region between $\widehat{f}(\gamma)$ and γ. Clearly, the positive orbit of $\widehat{z}^*$ is bounded in the cylinder and so its ω-limit set $\omega(\widehat{z}^*)$ is a compact $\widehat{f}$-invariant subset of the cylinder. Moreover, as any integer vertical translate of $\omega(\widehat{z}^*)$ is also $\widehat{f}$-invariant, if we pick a minimal $\widehat{f}$-invariant compact set K contained in $\omega(\widehat{z}^*)$, clearly, by minimality it satisfies $K \cap K + (0,n) = \emptyset$ for all $n \neq 0$.

As f is minimal, when K is projected to the torus is must be the whole torus, a contradiction. $\Box$

3.2 Proof of theorem 2

Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, without any loss of generality we can assume that $\rho_V(\widehat{f}) = 0$.

Lemma 2 implies that $B_N^+ \neq \emptyset$ and $B_S^- \neq \emptyset$, and lemma 4 implies that the same holds for their ω-limits, $\omega(B_N^+) \neq \emptyset$ and $\omega(B_S^-) \neq \emptyset$.

In the following we will present two technical results. For each $\widehat{x} \in S^1$, consider the following functions, which as the next lemma shows, are well defined at all $\widehat{x} \in S^1$:

$$\mu(\widehat{x}) = \max\{\widehat{y} \in \mathbb{R} : (\widehat{x}, \widehat{y}) \in \omega(B_S^-)\}$$
$$\nu(\widehat{x}) = \min\{\widehat{y} \in \mathbb{R} : (\widehat{x}, \widehat{y}) \in \omega(B_N^+)\}$$

Lemma 8 *: There exists a constant $M_f > 0$ such that*

$$\sup_{\widehat{x},\widehat{y} \in S^1} |\mu(\widehat{x}) - \mu(\widehat{y})| \leq M_f \text{ and } \sup_{\widehat{x},\widehat{y} \in S^1} |\nu(\widehat{x}) - \nu(\widehat{y})| \leq M_f.$$

Proof:

The proof is analogous for both cases, so let us only consider the function μ. As $\omega(B_S^-)$ is closed and bounded from above, choose some $\widehat{x}_0 \in S^1$ such that $\{\widehat{x}_0\} \times]-\infty, 0] \cap \omega(B_S^-) \neq \emptyset$ and for some $\widehat{y}_0 \leq 0$, $(\widehat{x}_0, \widehat{y}_0)$ belongs to $\omega(B_S^-)$ and has maximal $\widehat{y}$-coordinate. Then $\mu(\widehat{x}_0) = \widehat{y}_0$ is well defined.

Note that as f is homotopic to a Dehn twist, for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$ there are constants $A_f > 0$ and $B_f > 0$ such that

$$\left| p_2 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) - \widetilde{y} \right| < A_f \text{ and } \left| p_1 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) - \widetilde{x} - k_{Dehn}\widetilde{y} \right| < B_f. \qquad (6)$$

So for any compact set $G \subset \mathbb{R}^2$ with

$$|p_2(G)| \overset{def.}{=} \max(p_2(G)) - \min(p_2(G)) \geq V_f \overset{def.}{=} \frac{(3 + 2B_f)}{k_{Dehn}}$$

and

$$|p_1(G)| \overset{def.}{=} \max(p_1(G)) - \min(p_1(G)) < 1,$$

we have:

$$\left| p_1(\widetilde{f}(G)) \right| > 2 \text{ and } p_2 \mid_{\widetilde{f}(G)} > \min(p_2(G)) - A_f.$$

Consider the intersection $\pi^{-1}(\omega(B_S^-)) \cap \mathbb{R} \times [\mu(\widehat{x}_0) - V_f, \mu(\widehat{x}_0)]$. If all vertical segments $Seg_{\widetilde{x}} = \{\widetilde{x}\} \times [\mu(\widehat{x}_0) - V_f, \mu(\widehat{x}_0)]$ intersect $\pi^{-1}(\omega(B_S^-))$, then for all $\widehat{x} \in S^1$, $\mu(\widehat{x}_0) - V_f \leq \mu(\widehat{x}) \leq 0$ and the proof is over. So, suppose that there exists a real number $\widetilde{x}^*$ such that $Seg_{\widetilde{x}^*}$ do not intersect $\pi^{-1}(\omega(B_S^-))$. This implies that for any integer n, $Seg_{\widetilde{x}^*} + (n, 0)$ do not intersect $\pi^{-1}(\omega(B_S^-))$. Let θ be the connected component of $\omega(B_S^-)$ containing $(\widehat{x}_0, \widehat{y}_0)$ and let Θ be a component of $\pi^{-1}(\theta)$. The set Θ is also a connected component of $\pi^{-1}(\omega(B_S^-))$, so by lemma 6 it is unbounded. It is now clear that Θ intersects the two horizontal boundaries of $[\widetilde{x}^* + n_\Theta, \widetilde{x}^* + n_\Theta + 1] \times [\mu(\widehat{x}_0) - V_f, \mu(\widehat{x}_0)]$ for some integer n_Θ, because it can not meet the open half plane $\{\widetilde{y} > \mu(\widehat{x}_0)\}$.

Thus, $\left| p_1(\widetilde{f}(\Theta)) \right| > 2$ and $p_2 \mid_{\widetilde{f}(\Theta)} > \mu(\widehat{x}_0) - V_f - A_f$. As $\omega(B_S^-)$ is invariant, $\pi\left(\widetilde{f}(\Theta)\right) \subset \omega(B_S^-)$ and so for any $\widehat{x} \in S^1$, $\mu(\widehat{x}_0) - V_f - A_f < \mu(\widehat{x}) \leq 0$.

The above argument implies that if we choose $M_f = V_f + A_f$, then we are done. $\square$

Now, let us define the number

$$M_{Dehn} = \frac{2 + B_f}{k_{Dehn}} > 0. \tag{7}$$

A simple computation shows that for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$ with $\widetilde{y} > M_{Dehn}$, we have

$$p_1 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) > \widetilde{x} + 2 \text{ and } p_1 \circ \widetilde{f}(\widetilde{x}, -\widetilde{y}) < \widetilde{x} - 2.$$

The construction performed below is analogous for both $\omega(B_N^+)$ and $\omega(B_S^-)$. The details will be presented for $\omega(B_S^-)$. First, note that for every $\widehat{x} \in S^1$, $\mu(\widehat{x}) + \left(-\max_{\widehat{z} \in S^1} \mu(\widehat{z}) + M_f \right) + M_{Dehn} \geq M_{Dehn}$. This means that if we define the following positive integer number $n_{trans} \overset{def.}{=} \left\lfloor -\max_{\widehat{z} \in S^1} \mu(\widehat{z}) + M_f + M_{Dehn} \right\rfloor + 1$ ($\lfloor a \rfloor$ is the integer part of a), then the set

$$\omega(B_S^-)_{trans} \overset{def.}{=} \omega(B_S^-) + (0, n_{trans}) \tag{8}$$

has, for every $\widehat{x} \in S^1$, a point of the form $(\widehat{x}, \widehat{y})$, with $\widehat{y} > M_{Dehn}$. In other words, the function μ_{trans} associated with $\omega(B_S^-)_{trans}$ satisfies $\mu_{trans}(\widehat{x}) \overset{def.}{=} \mu(\widehat{x}) + n_{trans} > M_{Dehn}$, for all $\widehat{x} \in S^1$.

Now, for a fixed $\widetilde{x} \in \mathbb{R}$, consider the semi-line $\{\widetilde{x}\} \times [M_{Dehn}, +\infty[$. When we intersect it with

$$\widetilde{\omega(B_S^-)}_{trans} \overset{def.}{=} \pi^{-1}\left(\omega(B_S^-)_{trans}\right)$$

we get that $\{\widetilde{x}\} \times]\mu_{trans}(\pi(\widetilde{x})), +\infty[\cap \widetilde{\omega(B_S^-)}_{trans} = \emptyset$ (note that $\widetilde{\omega(B_S^-)}_{trans}$ is also a $\widetilde{f}$-invariant set).

Let $v = \{\widetilde{x}\} \times]\mu_{trans}(\pi(\widetilde{x})), +\infty[$ and let Θ be the connected component of $\widetilde{\omega(B_S^-)}_{trans}$ that contains $(\widetilde{x}, \mu_{trans}(\pi(\widetilde{x})))$.

Lemma 9 *: The following holds: $\Theta \cup v$ is a closed connected set, $(\Theta \cup v)^c$ has two open connected components, one of which is positively invariant and $\widetilde{f}^n(v) \cap v = \emptyset$ for all integers $n \neq 0$.*

Proof:

The fact that $\Theta \cup v$ is closed and connected is obvious. As Θ is a connected component of $\omega(\widetilde{B_S^-})_{trans}$, it is unbounded and limited from above in the $\widetilde{y}$-direction.

By the Jordan separation theorem, we get that $(\Theta \cup v)^c$ has at least two connected components, O_L and O_R, defined as follows: For any point $\widetilde{P} \in v$, there exists $\delta > 0$ such that $B_\delta(\widetilde{P}) \cap \Theta = \emptyset$. Moreover, $B_\delta(\widetilde{P}) \backslash v$ has exactly 2 connected components, one to the left of v, contained in O_L and the other one to the right of v, contained in O_R. So their closures, $\overline{O_L}$ and $\overline{O_R}$ both contain v. Now, suppose $(\Theta \cup v)^c$ has another connected component, denoted O^*. Clearly ∂O^* do not intersect v because all points sufficiently close to a point in v and, not in v, are contained in $O_L \cup O_R$. So, $\partial O^* \subset \Theta$ and O^* is then a connected component of Θ^c bounded from above in the $\widetilde{y}$-direction. And this contradicts lemma 7. So, $(\Theta \cup v)^c = O_L \cup O_R$.

Note that $\widetilde{f}(v) \cap v = \widetilde{f}(v) \cap \Theta = \widetilde{f}^{-1}(v) \cap \Theta = \emptyset$. The paragraph after definition (7) implies that $\widetilde{f}(v) \subset O_R$. In the following we will show that $\widetilde{f}(O_R) \subset O_R$.

There are 2 possibilities:

1. $\tilde{f}(\Theta) \neq \Theta \Rightarrow \tilde{f}(\Theta) \cap \Theta = \emptyset$, because Θ is a connected component of an invariant set;
2. $\tilde{f}(\Theta) = \Theta$;

Assume first that $\tilde{f}(\Theta) \cap \Theta = \emptyset$. Then

$$\widetilde{f}(\Theta \cup v) \cap (\Theta \cup v) = \emptyset.$$

Since $\widetilde{f}(v) \subset O_R$ and $\widetilde{f}(\Theta \cup v)$ is connected, we get that $\widetilde{f}(\Theta \cup v) \subset O_R$, so $O_L \cup \Theta \cup v$ is contained either in $\widetilde{f}(O_L)$ or $\widetilde{f}(O_R)$. It can not be contained in $\widetilde{f}(O_R)$ because a point of the form $(-a, a)$ for a sufficiently large $a > 0$ is contained in O_L and $\widetilde{f}^{-1}(-a, a)$ is also contained in O_L, see (6). Thus, $O_L \cup \Theta \cup v \subset \widetilde{f}(O_L)$, which implies that, $\widetilde{f}(O_R) \subset O_R$.

Now suppose $\tilde{f}(\Theta) = \Theta$. This implies that $O_L \cup v \subset (\widetilde{f}(v \cup \Theta))^c$ because $\widetilde{f}(v) \subset O_R$ and $\widetilde{f}(\Theta) = \Theta$. So, by connectedness, $O_L \cup v$ is contained either in

$\widetilde{f}(O_R)$ or in $\widetilde{f}(O_L)$. As in the case $\widetilde{f}(\Theta)\cap\Theta=\emptyset$, one actually gets $O_L\cup v\subset\widetilde{f}(O_L)$ so

$$\widetilde{f}(O_R)\subset(\widetilde{f}(O_L))^c\subset(O_L\cup v)^c=O_R\cup\Theta$$

and since $\widetilde{f}(\Theta)=\Theta$, we finally get that $\widetilde{f}(O_R)\subset O_R$.

In order to finish the proof, note that, as $\widetilde{f}(v)\cap v=\emptyset$, for any $n\geq 2$, $\widetilde{f}^n(v)\subset\widetilde{f}(O_R)$, which do not intersect v. So $\widetilde{f}^n(v)\cap v=\emptyset$. This finishes the proof of our lemma. $\Box$

Remarks:

- as $\mu_{trans}(\pi(\widetilde{x}))<M_f+M_{Dehn}+2$ for all $\widetilde{x}\in\mathbb{R}$, we get that $\widetilde{f}^n(\{\widetilde{x}\}\times[M_f+M_{Dehn}+2,+\infty[)\cap\{\widetilde{x}\}\times[M_f+M_{Dehn}+2,+\infty[=\emptyset$ for all integers $n>0$.
- an analogous argument applied to $\omega(B_N^+)$ implies that for any $\widetilde{x}\in\mathbb{R}$, if $w=\{\widetilde{x}\}\times]-\infty,\nu(\pi(\widetilde{x}))-\left\lfloor\inf_{\widehat{z}\in S^1}\nu(\widehat{z})+M_f+M_{Dehn}\right\rfloor-1[$, then $\widetilde{f}^n(w)\cap w=\emptyset$ for all integers $n>0$. So as in the above remark, $\nu_{trans}(\pi(\widetilde{x}))>-2-M_f-M_{Dehn}$ for all $\widetilde{x}\in\mathbb{R}$, which implies that $\widetilde{f}^n(\{\widetilde{x}\}\times]-\infty,-M_f-M_{Dehn}-2[)\cap\{\widetilde{x}\}\times]-\infty,-M_f-M_{Dehn}-2[=\emptyset$ for all integers $n>0$.

Summarizing, there exists a real number $M'>0$ such that for all $\widetilde{x}\in\mathbb{R}$, $\widetilde{f}^n(\{\widetilde{x}\}\times[M',+\infty[)\cap\{\widetilde{x}\}\times[M',+\infty[=\emptyset$ and $\widetilde{f}^n(\{\widetilde{x}\}\times]-\infty,-M'])\cap\{\widetilde{x}\}\times]-\infty,-M']=\emptyset$ for all integers $n>0$ and

$$M'\overset{def.}{=}M_f+M_{Dehn}+2=\frac{5+3B_f}{k_{Dehn}}+A_f+2 \tag{9}$$

Now let us suppose by contradiction that there exists a point $\widehat{z}$ in the cylinder and an integer $n_0>0$ such that

$$\left|p_2(\widehat{f}^{n_0}(\widehat{z}))-p_2(\widehat{z})\right|>2M'+8.$$

Without loss of generality, we can assume that $p_2(\widehat{z})<-M'-3$ and $p_2(\widehat{f}^{n_0}(\widehat{z}))>M'+3$.

Let us also consider the fixed point free mapping of the plane

$$\widetilde{g}(\bullet)=\widetilde{f}^{n_0}(\bullet)-(0,1).$$

To see that it is actually fixed point free, note that if $\widetilde{g}$ has a fixed point, then $1/n_0 \in \rho_V(\widehat{f})$, a contradiction. Now, note that for all $\widetilde{x} \in \mathbb{R}$, $\widetilde{g}(\{\widetilde{x}\} \times [M'+2,+\infty[) \cap \{\widetilde{x}\} \times [M'+2,+\infty[= \emptyset$ and $\widetilde{g}(\{\widetilde{x}\} \times]-\infty,-M'-2]) \cap \{\widetilde{x}\} \times]-\infty,-M'-2] = \emptyset$. Moreover, using the fact that $\widetilde{g}$ is also the lift of a torus homeomorphism homotopic to a Dehn twist and a compacity argument, one can prove that there exists an integer $N > 0$, such that for all integers n, the sets

$$\begin{array}{c} F_n^- = [n/N,(n+1)/N] \times]-\infty,-M'-2] \\ \text{and} \\ F_n^+ = [n/N,(n+1)/N] \times [M'+2,\infty[\end{array} \tag{10}$$

are free under $\widetilde{g}$, that is, $\widetilde{g}(F_n^{+or-}) \cap F_n^{+or-} = \emptyset$, for all integers n. Moreover, the fact that $k_{Dehn} > 0$ (see the end of section 1) implies that there exists an integer $K_{crit} > 0$, such that for all integers n

$$\begin{array}{c} \widetilde{g}(F_n^+) \cap F_m^+ \neq \emptyset, \text{ for all } m \geq n + K_{crit} \\ \text{and} \\ \widetilde{g}(F_n^-) \cap F_m^- \neq \emptyset, \text{ for all } m \leq n - K_{crit}. \end{array}$$

These will be important bricks in a special brick decomposition of the plane in $\widetilde{g}$-free sets we will construct, which will be invariant under integer horizontal translations $(\widetilde{x},\widetilde{y}) \to (\widetilde{x}+1,\widetilde{y})$.

Clearly, such a construction is possible, because as $\widetilde{g}(\widetilde{x}+1,\widetilde{y}) = \widetilde{g}(\widetilde{x},\widetilde{y}) + (1,0)$, we just have to decompose $S^1 \times [-M'-2, M'+2]$ into a union of bricks with sufficiently small diameter, so that their pre-images under π are $\widetilde{g}$-free.

To conclude our proof, we will show that this brick decomposition has a closed brick chain, a contradiction with the fact that $\widetilde{g}$ is fixed point free, see lemma 1. This idea was already used in the proof of theorem 4 of [3].

Consider a point $\widetilde{z} \in \pi^{-1}(\widehat{z})$ and a brick $F_{i_0}^-$ that contains $\widetilde{z}$. From our choices,

$$\widetilde{g}(F_{i_0}^-) \cap F_{i_1}^+ \neq \emptyset, \text{ for some integer } i_1.$$

As $\rho_V(\widehat{f}) = \{0\}$, let us choose a point $\widehat{w} \in S^1 \times]M'+2,+\infty[$ such that

$$p_2(\widehat{g}^n(\widehat{w})) \overset{n\to\infty}{\to} -\infty,$$

where $\widehat{g}(\bullet) \overset{def.}{=} \widehat{f}^{n_0}(\bullet) - (0,1)$ (as $\rho_V(\widehat{g}) = \{-1\}$, all points in $S^1 \times \mathbb{R}$ satisfy the above condition). So, we can choose a point $\widetilde{w} \in F^+_{i_2}$, for some integer i_2, such that:

- $i_2 > i_1 + K_{crit}$, so $\widetilde{g}(F^+_{i_1}) \cap F^+_{i_2} \neq \emptyset$;
- $\widetilde{g}^{n_2}(\widetilde{w}) \in F^-_{i_3}$, for some integers $n_2 > 0$ and $i_3 > i_0 + K_{crit}$;

As $\widetilde{g}(F^-_{i_3}) \cap F^-_{i_0} \neq \emptyset$, we get there exists a closed brick chain starting at $F^-_{i_0}$. As we said, this is a contradiction because $\widetilde{g}$ is fixed point free. Thus $\widehat{f}^n(S^1 \times \{0\}) \subset S^1 \times [-8 - 2M', 2M' + 8]$ for all integers $n > 0$. In order to conclude the proof, let K be the only connected component of the frontier of

$$\bigcap_{n\geq 0} \widehat{f}^n(closure(\bigcup_{m\geq 0} \widehat{f}^m(S^1\times]0, +\infty[)))$$

which does not bound a disc. Then K is a compact connected set that separates the ends of the cylinder, $\widehat{f}(K+(0,l)) = K+(0,l)$, for all integers l and $|p_2(K)| \leq 4M' + 20$. □

3.3 Proof of corollary 1

Without loss of generality, by considering $\widehat{f}^q - (0,p)$, we can suppose that $\rho_V(\widehat{f}) = [a, 0]$, for some $a < 0$. As in the proof of theorem 2, lemma 2 implies that $B^+_N \neq \emptyset$, $B^-_S \neq \emptyset$ and $B^+_N(inv) \neq \emptyset$, $B^-_S(inv) \neq \emptyset$. If for instance $\omega(B^-_S) = \emptyset$, then lemma 5 implies that $\omega(B^-_S(inv)) = \emptyset$ and so lemma 4 implies that there exists $\epsilon > 0$ such that $\rho_V(\widehat{f}^{-1}) \supset [-\epsilon, 0]$, which gives $\rho_V(\widehat{f}) \supset [0, \epsilon]$, a contradiction. So, we can assume that $\omega(B^+_N) \neq \emptyset$ and $\omega(B^-_S) \neq \emptyset$. If we suppose that for every $M > 0$, there exists a point $\widehat{z} \in S^1 \times \mathbb{R}$ and an integer $n > 0$ such that

$$p_2(\widehat{f}^n(\widehat{z})) - p_2(\widehat{z}) > M,$$

then following exactly the same ideas used in theorem 2, we arrive at a contradiction which proves the corollary. □

3.4 Proof of theorem 3

As in theorem 2, let us fix a $\widetilde{f} \in DT(\mathbb{R}^2)$, which is a lift of $\widehat{f}$. First, we will show that if

$$M \geq M_0 \overset{def.}{=} (20 + 2B_f)/k_{Dehn} + 10 \text{ (see (6))},$$

then $\widehat{f}$ has a fixed point. In case $\widehat{f}$ is fixed point free, lemma 2 of [3] tells us that there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that $\widehat{f}(\gamma) \cap \gamma = \emptyset$ and $\gamma \subset S^1 \times [-m_D, m_D]$, where $m_D > 0$ is the smallest real number that satisfies

$$\begin{gathered} \widetilde{f}(\{\widetilde{x}\} \times [m_D, +\infty[) \subset [\widetilde{x} + 10, +\infty[\times \mathbb{R} \\ and \\ \widetilde{f}(\{\widetilde{x}\} \times [-\infty, -m_D]) \subset] - \infty, \widetilde{x} - 10] \times \mathbb{R}, \end{gathered} \tag{11}$$

for all $\widetilde{x} \in \mathbb{R}$. A simple computation shows that if we take m_D equal $(10 + B_f)/k_{Dehn}$, then (11) is satisfied.

So, as $M \geq 2m_D + 10$, the theorem hypotheses imply that $\widehat{f}$ has a fixed point. Thus $0 \in \rho_V(\widehat{f})$ and lemma 2 implies that $B_N^+ \neq \emptyset$, $B_S^- \neq \emptyset$ and the same holds for the inverse of $\widehat{f}$, namely, $B_S^-(inv) \neq \emptyset$ and $B_N^+(inv) \neq \emptyset$. If $\omega(B_N^+) = \emptyset$, then lemma 4 implies that there exists $\delta > 0$ such that $\rho_V(\widehat{f}) \supset [0, \delta]$. Also, from lemma 5 we get that $\omega(B_N^+(inv)) = \emptyset$ and so again by lemma 4, there exists $\epsilon > 0$ such that $\rho_V(\widehat{f}^{-1}) \supset [0, \epsilon]$, which gives $\rho_V(\widehat{f}) \supset [-\epsilon, \delta]$ and the theorem is proved. So, again we can suppose that $\omega(B_S^-) \neq \emptyset$ and $\omega(B_N^+) \neq \emptyset$.

If $\rho_V(\widehat{f}) = [a, 0]$ for some $a \leq 0$, then if

$$M \geq M_1 \overset{def.}{=} 2M' + 8 = \frac{10 + 6B_f}{k_{Dehn}} + 2A_f + 12,$$

by the same argument used to prove theorem 2, we arrive at a contradiction. The same happens in the other possibility, that is, if $\rho_V(\widehat{f}) = [0, b]$, for some $b > 0$.

So, it is enough to choose

$$M = \max\{M_0, M_1\} \leq \frac{20 + 6B_f}{k_{Dehn}} + 2A_f + 12 \text{ to finish the proof. } \square$$

3.5 Proof of Corollary 2

Let us start by showing that there are two possibilities:

1) $\bigcup_{n\geq 0} \widehat{f}^n(H)$ is bounded and this means that $\rho_V(\widehat{f}) = \{0\}$;

2) $\bigcup_{n\geq 0} \widehat{f}^n(H)$ is unbounded from above and from below;

In order to understand that the above are the only possible cases, suppose for instance that $\bigcup_{n\geq 0} \widehat{f}^n(H)$ is unbounded and contained in H_a^+ for some real number $a < 0$.

As in lemma 2, let $O^* = \bigcup_{n\geq 0} \widehat{f}^n(S^1 \times]0, +\infty[)$ and let O be the complement of the connected component of $(O^*)^c$ which contains the lower end of the cylinder. As in that lemma, $\partial O \overset{def.}{=} K$ is a compact connected set that separates the ends of the cylinder. Clearly, $O^* \subset O$ (we just fill the holes), $H_1^+ \subset O \subset H_a^+$, O is an open set homeomorphic to the cylinder and $\widehat{f}(O) \subset O$.

Let us state a simple result, but before we present a definition:

Definition : If γ is a homotopically non trivial simple closed curve in $S^1 \times \mathbb{R}$, then $\gamma^c \overset{def.}{=} \gamma^{-o} \cup \gamma^{+o}$, where $\gamma^{-o(+o)}$ is the open connected component of γ^c which contains the lower (upper) end of the cylinder. We define $\gamma^- \overset{def.}{=} closure(\gamma^{-o}) = \gamma^{-o} \cup \gamma$ and the same for γ^+.

Proposition 1 *: Given an area-preserving $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$ with zero Lebesgue measure vertical rotation number, for any $b \in \mathbb{R}$ the following equality holds (in this case $\widehat{f}$ is said to be exact):*

$$Leb(H_b^+ \cap (\widehat{f}(H_b))^-) = Leb(H_b^- \cap (\widehat{f}(H_b))^+),$$

where for any measurable set D, $Leb(D) \overset{def.}{=}$ Lebesgue measure of D.

Proof:

If we remember (6), we get that there exists an integer $N > 0$ such that, for any given $b \in \mathbb{R}$, $\widehat{f}(H_b) \cap (H_{b+N} \cup H_{b-N}) = \emptyset$. So, consider the finite annulus $\Omega \overset{def.}{=} S^1 \times [b, b+N]$. As it is a finite union of fundamental domains of the torus, we get that

$$\int_\Omega \left[p_2 \circ \widehat{f}(\widehat{x}, \widehat{y}) - \widehat{y}\right] d\widehat{x}d\widehat{y} = 0 \text{ (this follows from } \rho_V(Leb) = 0). \tag{12}$$

Note that we can write

$$\Omega = \left(\widehat{f}(\Omega)\cap\Omega\right)\cup\left(H_b^+\cap(\widehat{f}(H_b))^{-o}\right)\cup\left(H_b^-\cap(\widehat{f}(H_b))^{+o}+(0,N)\right)$$

and

$$\widehat{f}(\Omega) = \left(\widehat{f}(\Omega)\cap\Omega\right)\cup\left(H_b^{+o}\cap(\widehat{f}(H_b))^{-}+(0,N)\right)\cup\left(H_b^{-o}\cap(\widehat{f}(H_b))^{+}\right),$$

where the unions are disjoint. Expression (12) together with the preservation of area imply that the $\widehat{y}$-coordinate of the geometric center of Ω and of $\widehat{f}(\Omega)$ are equal. So, let us compute them (for a measurable set Π in the cylinder, we denote the $\widehat{y}$-coordinate of its geometric center by $\widehat{y}_{G.C.(\Pi)}$):

$$\widehat{y}_{G.C.(\Omega)} = \begin{bmatrix} \widehat{y}_{G.C.(\widehat{f}(\Omega)\cap\Omega)}.Leb(\widehat{f}(\Omega)\cap\Omega)+ \\ +\widehat{y}_{G.C.(H_b^+\cap(\widehat{f}(H_b))^-)}.Leb(H_b^+\cap(\widehat{f}(H_b))^-)+ \\ +\left(\widehat{y}_{G.C.(H_b^-\cap(\widehat{f}(H_b))+)}+N\right).Leb(H_b^-\cap(\widehat{f}(H_b))^+) \end{bmatrix} / Leb(\Omega)$$

$$\widehat{y}_{G.C.(\widehat{f}(\Omega))} = \begin{bmatrix} \widehat{y}_{G.C.(\widehat{f}(\Omega)\cap\Omega)}.Leb(\widehat{f}(\Omega)\cap\Omega)+ \\ +\left(\widehat{y}_{G.C.(H_b^+\cap(\widehat{f}(H_b))^-)}+N\right).Leb(H_b^+\cap(\widehat{f}(H_b))^-)+ \\ +\widehat{y}_{G.C.(H_b^-\cap(\widehat{f}(H_b))+)}.Leb(H_b^-\cap(\widehat{f}(H_b))^+) \end{bmatrix} / Leb(\widehat{f}(\Omega))$$

As $Leb(\widehat{f}(\Omega)) = Leb(\Omega)$ and $\widehat{y}_{G.C.(\widehat{f}(\Omega))} = \widehat{y}_{G.C.(\Omega)}$, we get that

$$N.Leb(H_b^+\cap(\widehat{f}(H_b))^-) = N.Leb(H_b^-\cap(\widehat{f}(H_b))^+),$$

which proves the proposition (note that we used the fact that $Leb(H_b)=0$). $\square$

Now let us choose $c\in\mathbb{R}$ such that $\{K\cup\widehat{f}(K)\}\subset interior(H_c^-\cap(\widehat{f}(H_c))^-)$. From the preservation of Lebesgue measure and the above proposition, we get that

$$Leb(O\cap H_c^-) = Leb(\widehat{f}(O)\cap(\widehat{f}(H_c))^-) = Leb(\widehat{f}(O)\cap H_c^-).$$

The choice of c, together with the fact that $\widehat{f}(O)\subset O$, implies that $closure(O) = closure(\widehat{f}(O)) = \widehat{f}(closure(O))$. So $\partial(closure(O))$ separates the ends of the cylinder and is $\widehat{f}$-invariant. But this means that all orbits are uniformly bounded, a contradiction with our hypothesis that $\bigcup_{n\geq 0} \widehat{f}^n(H)$ is unbounded. So, either 1) or 2) from the beginning of the proof of the corollary can happen.

And in possibility 2) we can apply theorem 3 to conclude the proof. □

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