Dynamics of homeomorphisms of the torus homotopic to Dehn twists

Salvador Addas-Zanata, Fábio A. Tal and Bráulio A. Garcia

Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, Cidade Universitária,
05508-090 São Paulo, SP, Brazil

Abstract

In this paper we consider homeomorphisms f of the torus homotopic to Dehn twists. We prove that if f is area preserving and it has a lift $\widehat{f}$ to the cylinder with zero flux, then either f is an annulus homeomorphism, or there are points in the cylinder with positive vertical velocity and others with negative vertical velocity, for iterates of $\widehat{f}$. This solves a version of Boyland's conjecture to this setting.

Key words: vertical rotation set, Dehn twists, omega limits, area-preservation

e-mail: sazanata@ime.usp.br, fabiotal@ime.usp.br and braulio@ime.usp.br

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1 Introduction and main results

In this paper we study homeomorphisms f of the torus homotopic to Dehn twists. These homotopy classes are in some way simpler to analyze than the identity case. One of the reasons for this is the fact that there is no sense in defining a two dimensional rotation set for torus maps homotopic to Dehn twists, instead a vertical rotation set is defined, see expression (1).

Many important conjectures for homotopic to the identity maps have their analogs in this setting. For instance, how is the rotation interval of a minimal Dehn twist homeomorphism? Does the set of minimal Dehn twist C^r-diffeomorphisms ($r \geq 2$) have no interior? If f is a Dehn twist homeomorphism which preserves area and has zero Lebesgue measure vertical rotation number, is it true that either f is an annulus homeomorphism or the vertical rotation interval has no empty interior? Before continuing, we need some definitions.

Definitions:

1. Let $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus and let $p : \mathbb{R}^2 \longrightarrow \mathrm{T}^2$ and $\pi : \mathbb{R}^2 \longrightarrow S^1 \times \mathbb{R}$ be the associated covering maps. Coordinates are denoted as $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$, $(\widehat{x}, \widehat{y}) \in S^1 \times \mathbb{R}$ and $(x, y) \in \mathrm{T}^2$.

2. Let $DT(\mathrm{T}^2)$ be the set of homeomorphisms of the torus homotopic to a Dehn twist $(x, y) \longrightarrow (x + ky \bmod 1, y \bmod 1)$, for some $k \in \mathbb{Z}^*$, and let $DT(S^1 \times \mathbb{R})$ and $DT(\mathbb{R}^2)$ be the sets of lifts of elements from $DT(\mathrm{T}^2)$ to the cylinder and plane. Homeomorphisms from $DT(\mathrm{T}^2)$ are denoted f and their lifts to the vertical cylinder and plane are respectively denoted $\widehat{f}$ and $\widetilde{f}$.

3. Let $p_{1,2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the standard projections; $p_1(\tilde{x}, \tilde{y}) = \tilde{x}$ and $p_2(\tilde{x}, \tilde{y}) = \tilde{y}$. Projection on the cylinder are also denoted by p_1 and p_2.

4. Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, the so called vertical

rotation set can be defined as follows, see [11]:

$$\rho_V(\widehat{f}) = \bigcap_{i \geq 0} \overline{\bigcup_{n \geq i} \left\{ \frac{p_2 \circ \widehat{f}^n(\widehat{z}) - p_2(\widehat{z})}{n} : \forall \widehat{z} \in S^1 \times \mathbb{R} \right\}} \qquad (1)$$

This set is a closed interval (maybe a single point, but never empty) and it was proved in [1] and [3] (and much earlier in [7], although the first author discovered this only recently) that all numbers in its interior are realized by compact f-invariant subset of T^2, which are periodic orbits in the rational case.

So, given $f \in DT(\mathrm{T}^2)$ and $\widehat{f} \in DT(S^1 \times \mathbb{R})$, as we said above, one wants to know, under which conditions can f be minimal? It is not difficult to see that in this case the vertical rotation interval must be a single point, otherwise there would be infinitely many periodic orbits. But more can be said.

Theorem 1 *: Given $f \in DT(\mathrm{T}^2)$ and $\widehat{f} \in DT(S^1 \times \mathbb{R})$, suppose that f is minimal. Then, $\rho_V(\widehat{f}) = \{\alpha\}$ for some irrational number α.*

So, if f is a C^r diffeomorphism, for some $r \geq 2$, is there a natural perturbation that destroys the minimality? As the extreme points of $\rho_V(\widehat{f})$ vary continuously with $\widehat{f} \in DT(S^1 \times \mathbb{R})$ (see [8]), a way to attack this problem is by showing that irrational extremes are not stable under perturbations. This was done in [2] for twist mappings on the torus.

The main problem addressed in this paper is in a way, complementary to the above. Suppose for instance that $\rho_V(\widehat{f})$ contains a single rational number p/q. What can we say about the dynamics of f? And if f preserves area and $\widehat{f}$ has zero flux, that is, for any homotopically non-trivial simple closed curve γ, the area above γ and below $\widehat{f}(\gamma)$ is equal to the area below γ and above $\widehat{f}(\gamma)$, what can we say about its vertical rotation interval? When it is not reduced to zero, is zero always an interior point? This is the so called Boyland's Conjecture, adapted to this setting. Clearly, the condition of having zero flux is equivalent to saying that the vertical rotation number of the Lebesgue measure is zero, that is

$$\int_{S^1\times[0,1]} \left[p_2 \circ \widehat{f}(\widehat{x},\widehat{y}) - \widehat{y} \right] d\widehat{x}d\widehat{y} = 0.$$

Below we state our main results:

Theorem 2 *: Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathrm{I\!R})$, if $\rho_V(\widehat{f}) = \{p/q\}$, for some rational p/q and $\widehat{f}^q - (0,p)$ does not have wandering points, then there exists a compact connected set $K \subset S^1 \times \mathrm{I\!R}$, invariant under $\widehat{f}^q - (0,p)$, which separates the ends of the cylinder. So, all points have uniformly bounded orbits under the action of $\widehat{f}^q - (0,p)$.*

As a corollary of this result and some consequences of Thurston's classification of surface homeomorphisms, we have the following:

Corollary 1 *: Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathrm{I\!R})$ such that $0 \in \rho_V(\widehat{f})$ and $\widehat{f}$ does not have wandering points, if the set $\bigcup_{n\geq 0} \widehat{f}^n(S^1 \times \{0\})$ is unbounded, then $\rho_V(\widehat{f})$ is a non-degenerate interval (it has interior) and f has positive topological entropy.*

The only thing in the above corollary that is not implied by theorem 2 is the fact that any Dehn twist homeomorphism with a non-degenerate vertical rotation set has positive topological entropy. The proof of this result can be found in [7] and [1] and it is completely inspired by the result for the homotopic to the identity case, see [10]. The next result gives a positive answer for Boyland's conjecture in this setting:

Theorem 3 *: Given an area-preserving $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathrm{I\!R})$ with zero flux, if $\rho_V(\widehat{f})$ is not reduced to 0, then 0 is an interior point of $\rho_V(\widehat{f})$.*

This paper is organized as follows. In the second section we present some results we use, with references and a few proofs. In the third section we prove our theorems. From now on we assume, without loss of generality, that any $f \in DT(\mathrm{T}^2)$ we consider, is homotopic to a Dehn twist $(x,y) \longrightarrow (x + ky \bmod 1, y \bmod 1)$ with $k > 0$.

2 Basic Tools

Here we present a theory developed in [4].

2.1 On the sets $\mathbf{B}_S^-$ and $\mathbf{B}_N^+$

Let us extend some constructions from [4] to this setting. For this, consider a homeomorphism $f \in DT(\mathrm{T}^2)$, a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$ and a lift of $\widehat{f}$ to the plane, denoted $\widetilde{f} \in DT(\mathbb{R}^2)$. Given a real number a, let

$$H_a = S^1 \times \{a\},$$

$$H_a^- = S^1 \times]-\infty, a] \text{ and } H_a^+ = S^1 \times [a, +\infty[.$$

We will also denote the sets H_0, H_0^- and H_0^+ simply by H, H^- and H^+ respectively. If we consider the closed sets,

$$B^- = \bigcap_{n \leq 0} \widehat{f}^n(H^-)$$

$$and$$

$$B^+ = \bigcap_{n \leq 0} \widehat{f}^n(H^+),$$

we get that they are both closed and positively $\widehat{f}$-invariant. For each of these sets, consider the following subsets: $B_S^- \subset B^-$ and $B_N^+ \subset B^+$, each of which consisting of exactly all unbounded connected components of respectively, B^- and B^+. The sets B_S^- and B_N^+ are always closed (see [4]), but in some cases may be empty. The next lemma tells us that under certain conditions, they really exist.

Lemma 1 *: Suppose $0 \in \rho_V(\widehat{f})$ and for any given $M > 0$, there exists a positive integer i and a point $\hat{z} \in S^1 \times [0,1]$ such that $p_2 \circ \hat{f}^i(\hat{z}) > M$. Then $B_N^+ \cap H \neq \emptyset$.*

Proof:

The proof of this result goes back to Le Calvez [9] and even Birkhoff [6]. The only thing we have to prove in our particular situation is that, for all $a \geq 2$, there exists a first positive integer $n = n(a)$, such that

$$\widehat{f}^{-n}(H_a) \cap H \neq \emptyset \text{ and } n(a) \to \infty \text{ as } a \to \infty. \tag{2}$$

If this were not the case, then the lemma hypotheses would imply that for some integer $N > 0$, $\widehat{f}^{-N}(H_a) \subset H^- \subset H_a^- - (0,1)$, which would imply that $0 \notin \rho_V(\widehat{f})$, a contradiction. So expression (2) is true and the proof continues, for instance as in lemma 6 of [4]. □

Remark: A similar argument implies that if $0 \in \rho_V(\widehat{f})$ and for any given $M > 0$, there exists a positive integer i and a point $\hat{z} \in S^1 \times [0,1]$ such that $p_2 \circ \hat{f}^i(\hat{z}) < -M$, then $B_S^- \cap H \neq \emptyset$.

2.2 The ω-limit sets of $\mathbf{B}_S^-$ and $\mathbf{B}_N^+$

In this subsection we examine some properties of the set

$$\omega(B_S^-) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{i=n}^{\infty} \widehat{f}^i(B_S^-)}. \tag{3}$$

Lemma 2 *: $\omega(B_S^-)$ is a closed, $\widehat{f}$-invariant set, whose connected components are all unbounded.*

Proof:

See lemma 7 of [4]. □

Of course, since B_S^- is closed and positively $\widehat{f}$-invariant, we also have that $\omega(B_S^-) \subset B_S^-$. It is still possible that $\omega(B_S^-) = \emptyset$, and the next lemma tells us that in this case, things are easier.

Lemma 3 *: Suppose $0 \in \rho_V(\widehat{f})$ and B_S^- is not empty. If $\omega(B_S^-) = \emptyset$, then $\rho_V(\widehat{f}) \supset [0, -\epsilon]$, for some $\epsilon > 0$.*

Proof:

See lemma 10 of [4]. □

Now, if we consider the set B_S^- for $\widehat{f}^{-1}$, denoted $B_S^-(inv)$, we get the following:

Lemma 4 *: The sets* $\omega(B_S^-)$ *and* $\omega(B_S^-(inv))$ *are equal.*

Proof:

Let Γ be a connected component of $\omega(B_S^-)$. From the definition, $\widehat{f}^n(\Gamma) \subset H^-$ for all integers n. So $\Gamma \subset B_S^-(inv)$ and moreover, for each positive integer n, as $\widehat{f}^n(\Gamma)$ is contained in H^-, we get that $\Gamma \subset \widehat{f}^{-n}(B_S^-(inv))$, which means that $\Gamma \subset \omega(B_S^-(inv))$. Thus $\omega(B_S^-) \subset \omega(B_S^-(inv))$. The other inclusion is proved in an analogous way. $\square$

The following are very important results on the structure of these sets.

Lemma 5 *: For any connected component* $\widetilde{\Gamma}$ *of* $\pi^{-1}(\omega(B_S^-))$, $\widetilde{\Gamma}^c$ *is connected.*

Proof:

Take a connected component $\widetilde{\Gamma}$ of $\pi^{-1}(\omega(B_S^-))$. First note that $\widetilde{\Gamma}^c$ has one connected component, denoted O^+, which contains $\mathbb{R}\times]0,+\infty[$. So, if there is another one, denoted O_1, it must be contained in $\mathbb{R}\times]-\infty,0]$. As $\pi^{-1}(\omega(B_S^-))$ is also contained in $\mathbb{R}\times]-\infty,0]$ and is $\widetilde{f}$-invariant, we get that $\widetilde{f}^n(O_1) \subset \mathbb{R}\times]-\infty,0]$ for all integers n.Thus, $O_1 \cup \widetilde{\Gamma}$ is closed, connected and contained in $\pi^{-1}(B_S^-)$. It is not difficult to see that it is in fact contained in $\pi^{-1}(\omega(B_S^-))$, a contradiction with the choice of $\widetilde{\Gamma}$. $\square$

Lemma 6 *: The complement of* $\pi^{-1}(\omega(B_S^-))$ *is connected.*

Proof:

Analogous to the one above. $\square$

Clearly, similar results hold for B_N^+.

3 Proofs

3.1 Proof of theorem 1

Assume $f \in DT(\mathrm{T}^2)$ and $\widehat{f} \in DT(S^1\times\mathbb{R})$ are such that f is minimal and $\rho_V(\widehat{f})$ is rational. Without loss of generality we can assume that $\rho_V(\widehat{f}) = 0$, because if f is minimal, the same happens for all its iterates.

As $\widehat{f}$ has no fixed points, lemma 2 of [3] implies that there exists a homotopically non trivial simple closed curve γ in the cylinder such that $\gamma \cap \widehat{f}(\gamma) = \emptyset$. Without loss of generality, we can suppose that $\widehat{f}(\gamma) \subset \gamma^-$, the connected component of γ^c which is below γ. Let $k > 0$ be an integer such that $\gamma - (0,k) \subset \gamma^-$. If for some $n > 0$, $\widehat{f}^n(\gamma) \subset (\gamma - (0,k))^-$, then 0 would not belong to $\rho_V(\widehat{f})$. So, for all $n > 0$, there exists a point $\widehat{z}_n$, above $\widehat{f}(\gamma)$ and below γ, such that

$$\{\widehat{z}_n, \widehat{f}(\widehat{z}_n), \widehat{f}^2(\widehat{z}_n), ..., \widehat{f}^n(\widehat{z}_n)\} \text{ is above } \gamma - (0,k).$$

Taking a subsequence if necessary, we can assume that $\widehat{z}_{n_i} \overset{i \to \infty}{\to} \widehat{z}^*$, a point in the closure of the region between $\widehat{f}(\gamma)$ and γ. Clearly, the positive orbit of $\widehat{z}^*$ is bounded in the cylinder and so its ω-limit set $\omega(\widehat{z}^*)$ is a compact $\widehat{f}$-invariant subset of the cylinder. Moreover, as any integer vertical translate of $\omega(\widehat{z}^*)$ is also $\widehat{f}$-invariant, if we pick a minimal $\widehat{f}$-invariant compact set K contained in $\omega(\widehat{z}^*)$, clearly, by minimality it satisfies $K \cap K + (0,n) = \emptyset$ for all $n \neq 0$ and $K = \partial K$, that is, K has no interior.

As f is minimal, when K is projected to the torus is must be the whole torus, a contradiction.

3.2 Proof of Theorem 2

Given $f \in DT(\mathrm{T}^2)$ and a lift $\widehat{f} \in DT(S^1 \times \mathbb{R})$, without any loss of generality we can assume that $\rho_V(\widehat{f}) = 0$. First, let us deal with the case when $\underset{n \geq 0}{\cup} \widehat{f}^n(H)$ is bounded. Clearly, this implies that $\underset{n \in \mathbb{Z}}{\cup} \widehat{f}^n(H)$ is bounded. Consider the open set

$$O = \underset{n \in \mathbb{Z}}{\cup} \widehat{f}^n(interior(H^-)).$$

It is easy to see that O is open, connected, $\widehat{f}(O) = O$ and for some sufficiently large $M > 0$, $O \subset H_M^-$. Also, there is only one boundary component of O which is a compact connected set contained in $S^1 \times [-1, M+1]$ that separates the ends of the cylinder. Denote this component by K^*. As $\widehat{f}(O) = O$, $\widehat{f}(K^*) = K^*$. So, in the remainder of our proof we will show that under our hypotheses, $\underset{n \geq 0}{\cup} \widehat{f}^n(H)$ is always bounded. As we did above, this proves the theorem.

If $\bigcup_{n\geq 0} \widehat{f}^n(H)$ is unbounded, without loss of generality, we can suppose that for every $M>0$, there exists a point $\widehat{z}\in S^1\times \mathbb{R}$ and an integer $n>0$ such that

$$p_2(\widehat{f}^n(\widehat{z}))-p_2(\widehat{z})>M.$$

Lemma 1 implies two things: $B_N^+\neq\emptyset$ and $B_S^-(inv)\neq\emptyset$. If $B_S^-=\emptyset$, then lemma 4 implies that $\omega(B_S^-(inv))=\emptyset$ and so lemma 3 implies that there exists $\epsilon>0$ such that $\rho_V(\widehat{f}^{-1})\supset[0,-\epsilon]$, which gives $\rho_V(\widehat{f})\supset[0,\epsilon]$. So, we can assume that $B_N^+\neq\emptyset$, $B_S^-\neq\emptyset$ and the same holds for their ω-limits, $\omega(B_N^+)\neq\emptyset$ and $\omega(B_S^-)\neq\emptyset$, otherwise the theorem is proved.

In the following we will present two technical lemmas. For each $\widehat{x}\in S^1$, consider the following functions:

$$\mu(\widehat{x})=\max\{\widehat{y}\in\mathbb{R}:(\widehat{x},\widehat{y})\in\omega(B_S^-)\}$$
$$\nu(\widehat{x})=\min\{\widehat{y}\in\mathbb{R}:(\widehat{x},\widehat{y})\in\omega(B_N^+)\}$$

Lemma 7 *: There exists a constant $M_f>0$ such that*

$$\max_{\widehat{x},\widehat{y}\in S^1}|\mu(\widehat{x})-\mu(\widehat{y})|<M_f \text{ and } \max_{\widehat{x},\widehat{y}\in S^1}|\nu(\widehat{x})-\nu(\widehat{y})|<M_f.$$

Proof:

The proof is analogous for both cases, so let us only consider the function μ. As $\omega(B_S^-)$ is closed, for any fixed $\widehat{x}_0\in S^1$ the point $(\widehat{x}_0,\mu(\widehat{x}_0))$ belongs to $\omega(B_S^-)$. Let Γ be the connected component of $\omega(B_S^-)$ that contains $(\widehat{x}_0,\mu(\widehat{x}_0))$ and let us look at the closed set $\pi^{-1}(\Gamma)=\bigcup_{n\in\mathbb{Z}}\widetilde{\Gamma}+(n,0)$, for some closed connected set $\widetilde{\Gamma}$ such that $\pi(\widetilde{\Gamma})=\Gamma$.

As f is homotopic to a Dehn twist, there are numbers $V_f>0$ and $A_f>0$ such that for any compact connected set $G\subset\mathbb{R}^2$ with

$$|p_2(G)|\overset{def.}{=}\max(p_2(G))-\min(p_2(G))>V_f$$

and

$$|p_1(G)|\overset{def.}{=}\max(p_1(G))-\min(p_1(G))<1,$$

we have:

$$\left|p_1(\widetilde{f}(G))\right|>2 \text{ and } p_2\mid_{\widetilde{f}(G)}>\min(p_2(G))-A_f.$$

Consider the intersection $\pi^{-1}(\Gamma)\cap\mathbb{R}\times[\mu(\widehat{x}_0)-V_f,\mu(\widehat{x}_0)]$. Every connected component Θ of this intersection meets at least one of the lines $\mathbb{R}\times\{\mu(\widehat{x}_0)-V_f\}$ or $\mathbb{R}\times\{\mu(\widehat{x}_0)\}$. If all vertical segments $Seg_{\widetilde{x}}=\{\widetilde{x}\}\times[\mu(\widehat{x}_0)-V_f,\mu(\widehat{x}_0)]$ intersect $\pi^{-1}(\Gamma)$, then for all $\widehat{x}\in S^1$, $\mu(\widehat{x}_0)-V_f<\mu(\widehat{x})<0$ and the proof is over. So, suppose that there exists a real number $\widetilde{x}^*$ such that $Seg_{\widetilde{x}^*}$ do not intersect $\pi^{-1}(\Gamma)$. This implies that for any integer n, $Seg_{\widetilde{x}^*}+(n,0)$ do not intersect $\pi^{-1}(\Gamma)$. So, there is one (closed) connected component Θ of $\pi^{-1}(\Gamma)\cap\mathbb{R}\times[\mu(\widehat{x}_0)-V_f,\mu(\widehat{x}_0)]$ contained in $[\widetilde{x}^*,\widetilde{x}^*+1]\times[\mu(\widehat{x}_0)-V_f,\mu(\widehat{x}_0)]$ (therefore Θ is compact) which intersects both horizontal boundaries. Thus, $\left|p_1(\widetilde{f}(\Theta))\right|>2$ and $p_2\mid_{\widetilde{f}(\Theta)}>\mu(\widehat{x}_0)-V_f-A_f$. As $\omega(B_S^-)$ is invariant, $\pi\left(\widetilde{f}(\Theta)\right)\subset\omega(B_S^-)$ and so for any $\widehat{x}\in S^1$, $\mu(\widehat{x}_0)-V_f-A_f<\mu(\widehat{x})<0$.

Thus, if we choose $M_f=V_f+A_f$, we are done. □

Now, pick a real number $M_{Dehn}>0$ such that for any $\widetilde{x}\in\mathbb{R}$,

$$\begin{gathered}\widetilde{f}(\{\widetilde{x}\}\times[M_{Dehn},+\infty[)\cap(\{\widetilde{x}\}\times[M_{Dehn},+\infty[)=\\=\widetilde{f}(\{\widetilde{x}\}\times]-\infty,-M_{Dehn}])\cap(\{\widetilde{x}\}\times]-\infty,-M_{Dehn}])=\emptyset.\end{gathered}$$

The construction performed below is analogous for both $\omega(B_N^+)$ and $\omega(B_S^-)$. The details will be presented for $\omega(B_S^-)$. First, note that for every $\widehat{x}\in S^1$, $\mu(\widehat{x})+\left(-\sup_{\widehat{z}\in S^1}\mu(\widehat{z})+M_f\right)+M_{Dehn}>M_{Dehn}$. This means that the set

$$\omega(B_S^-)_{trans}\overset{def.}{=}\omega(B_S^-)+\left(0,\left\lfloor-\sup_{\widehat{z}\in S^1}\mu(\widehat{z})\right\rfloor+1+M_f+M_{Dehn}\right)\tag{4}$$

has, for every $\widehat{x}\in S^1$, a point of the form $(\widehat{x},\widehat{y})$, with $\widehat{y}>M_{Dehn}$. In other words, the function μ_{trans} associated with $\omega(B_S^-)_{trans}$ satisfies $\mu_{trans}(\widehat{x})>M_{Dehn}$, for all $\widehat{x}\in S^1$.

Now, for a fixed $\widetilde{x}\in\mathbb{R}$, consider the semi-line $\{\widetilde{x}\}\times[M_{Dehn},+\infty[$. When we intersect it with

$$\widetilde{\omega(B_S^-)}_{trans}\overset{def.}{=}\pi^{-1}\left(\omega(B_S^-)_{trans}\right)$$

we get that $\{\widetilde{x}\}\times]\mu_{trans}(\pi(\widetilde{x})),+\infty[\cap\widetilde{\omega(B_S^-)}_{trans}=\emptyset$.

Let $v=\{\widetilde{x}\}\times]\mu_{trans}(\pi(\widetilde{x})),+\infty[$ and let Θ be the connected component of $\widetilde{\omega(B_S^-)}_{trans}$ that contains $(\widetilde{x},\mu_{trans}(\pi(\widetilde{x})))$.

Lemma 8 *: The following holds: $\Theta \cup v$ is a closed connected set, $p_2(\Theta \cup v) = \mathbb{R}$, $(\Theta \cup v)^c$ has two open connected components, one of which is positively invariant and $\widetilde{f}^n(v) \cap v = \emptyset$ for all integers $n \neq 0$.*

Proof:

The fact that $\Theta \cup v$ is closed and connected is obvious. As $\pi(\Theta)$ is a connected component of $\omega(B_S^-)_{trans}$, it is unbounded and limited from above. So $p_2(\Theta \cup v) = \mathbb{R}$.

Clearly, $(\Theta \cup v)^c$ has at least two connected components, O_L and O_R, defined as follows: For any point $\widetilde{P} \in v$, there exists $\delta > 0$ such that $B_\delta(\widetilde{P}) \cap \Theta = \emptyset$. Moreover, $B_\delta(\widetilde{P}) \backslash v$ has exactly 2 connected components, one to the left of v, contained in O_L and the other one to the right of v, contained in O_R. So their closures, $\overline{O_L}$ and $\overline{O_R}$ both contain v. Now, suppose $(\Theta \cup v)^c$ has another connected component, denoted O^*. Clearly ∂O^* do not intersect v because all points sufficiently close to v and, not in v, are contained in $O_L \cup O_R$. So, $\partial O^* \subset \Theta$ and as Θ is contained in $\mathbb{R} \times]-\infty, \left\lfloor - \sup_{\widehat{z} \in S^1} \mu(\widehat{z}) \right\rfloor + 1 + M_f + M_{Dehn}]$, this contradicts lemma 5. So, $(\Theta \cup v)^c = O_L \cup O_R$.

Note that $\widetilde{f}(v) \cap v = \widetilde{f}(v) \cap \Theta = \widetilde{f}^{-1}(v) \cap \Theta = \emptyset$. Our hypothesis that the $k \in \mathbb{Z}^*$ which defines the homotopy class is positive implies that $\widetilde{f}(v) \subset O_R$. In the following we will show that $\widetilde{f}(O_R) \subset O_R$.

There are 2 possibilities:

1. $\tilde{f}(\Theta) \neq \Theta \Rightarrow \tilde{f}(\Theta) \cap \Theta = \emptyset$, because Θ is a connected component of an invariant set;

2. $\tilde{f}(\Theta) = \Theta$;

Assume first that $\tilde{f}(\Theta) \cap \Theta = \emptyset$. Then

$$\widetilde{f}(\Theta \cup v) \cap (\Theta \cup v) = \emptyset.$$

Since $\widetilde{f}(v) \subset O_R$ and $\widetilde{f}(\Theta \cup v)$ is connected, we get that $\widetilde{f}(\Theta \cup v) \subset O_R$, so $\widetilde{f}(O_R) \subset O_R$.

Now suppose $\tilde{f}(\Theta)=\Theta$. This implies that $\tilde{f}(O_R)$ is a connected component of $\left(\Theta\cup\tilde{f}(v)\right)^c$. If $\widetilde{f}(O_R)$ is not contained in O_R, then $\widetilde{f}(O_R)\cap O_L\neq\emptyset$ and thus $\widetilde{f}(v)\cap O_L\neq\emptyset$, a contradiction. So $\widetilde{f}(O_R)\subset O_R$.

In order to finish the proof, note that, as $\widetilde{f}(v)\cap v=\emptyset$, for any $n\geq 2$, $\widetilde{f^n}(v)\subset\widetilde{f}(O_R)$, which do not intersect v. So $\widetilde{f^n}(v)\cap v=\emptyset$. This finishes the proof of our lemma. □

Remarks:

- as $\mu_{trans}(\pi(\widetilde{x}))<M_f+M_{Dehn}+2$ for all $\widetilde{x}\in\mathbb{R}$, we get that $\widetilde{f^n}(\{\widetilde{x}\}\times[M_f+M_{Dehn}+2,+\infty[)\cap\{\widetilde{x}\}\times[M_f+M_{Dehn}+2,+\infty[=\emptyset$ for all integers $n>0$.

- an analogous argument applied to $\omega(B_N^+)$ implies that for any $\widetilde{x}\in\mathbb{R}$, if $w=\{\widetilde{x}\}\times]-\infty,\nu(\pi(\widetilde{x}))-\left\lfloor\inf_{\widehat{z}\in S^1}\nu(\widehat{z})\right\rfloor-1-M_f-M_{Dehn}[$, then $\widetilde{f^n}(w)\cap w=\emptyset$ for all integers $n>0$. So as in the above remark, $\nu_{trans}(\pi(\widetilde{x}))>-2-M_f-M_{Dehn}$ for all $\widetilde{x}\in\mathbb{R}$, which implies that $\widetilde{f^n}(\{\widetilde{x}\}\times]-\infty,-M_f-M_{Dehn}-2[)\cap\{\widetilde{x}\}\times]-\infty,-M_f-M_{Dehn}-2[=\emptyset$ for all integers $n>0$.

Now, let us choose a point $\widehat{z}$ in the cylinder which satisfies the following assumptions:

1. $p_2(\widehat{z})<-M_f-M_{Dehn}-10$ and for some $n_0>0$, $p_2(\widehat{f^{n_0}}(\widehat{z}))-p_2(\widehat{z})>2(M_{Dehn}+M_f)+20$. This clearly implies that $\widehat{z}$ and $\widehat{z}+(0,1)$ do not belong to the closed set $\omega(B_S^-)_{trans}\cup\omega(B_N^+)_{trans}$ because each of them is $\widehat{f}$-invariant.

2. As $\omega(B_S^-)_{trans}\cup\omega(B_N^+)_{trans}$ is closed, there exists a sufficiently small $0<\epsilon_1<1/2$ such that $B_{\epsilon_1}(\widehat{z})\cup(B_{\epsilon_1}(\widehat{z})+(0,1))$ do not intersect $\omega(B_S^-)_{trans}\cup\omega(B_N^+)_{trans}$ and for some integer $n_1>0$, $\widehat{f}^{n_1}(B_{\epsilon_1}(\widehat{z}))\cap B_{\epsilon_1}(\widehat{z})\neq\emptyset$, because we assumed that $\widehat{f}$ has no wandering points.

3. Fix a vertical $V=\{\widetilde{x}\}\times\mathbb{R}$ in the plane such that $|\widetilde{x}\ mod\ 1-p_1(\widehat{z})|>2\epsilon_1$.

There exists some integer i_1 (maybe more than one), such that for any $\widetilde{z} \in \pi^{-1}(\widehat{z})$,

$$\widetilde{f}^{n_1}(B_{\epsilon_1}(\widetilde{z})) \cap (B_{\epsilon_1}(\widetilde{z}) + (i_1, 0)) \neq \emptyset. \tag{5}$$

Now we perform the same construction as in lemma 8 with the vertical V. In other words, we find two vertical semi-lines contained in V,

$$v = \{\widetilde{x}\} \times]\mu_{trans}(\pi(\widetilde{x})), +\infty[\text{ and } w = \{\widetilde{x}\} \times] - \infty, \nu_{trans}(\pi(\widetilde{x}))[,$$

such that for some connected components Θ of $\omega(\widetilde{B_S^-})_{trans}$ and Γ of $\omega(\widetilde{B_N^+})_{trans}$ we have $v \cup \Theta$ is connected and the same holds for $w \cup \Gamma$. Let us first consider $\Theta \cup v$ and the two open connected components O_L and O_R of $(\Theta \cup v)^c$. Our choice of $\widehat{z}$ and V implies that $\pi^{-1}(B_{\epsilon_1}(\widehat{z}))$ do not intersect $V \cup \omega(\widetilde{B_S^-})_{trans}$. So, consider a point $\widetilde{z}^* \in \pi^{-1}(\widehat{z})$ such that $\widetilde{z}^* \in O_R$ and for any integer $n \geq 1$, $\widetilde{z}^* - (n, 0) \in O_L$. The reason why such a point exists is the following:

Proposition 1 *: If $\widehat{w} \notin \omega(B_S^-)_{trans} \cup \pi(v)$, then there exists a point $\widetilde{w} \in \pi^{-1}(\widehat{w})$ such that $\widetilde{w} \in O_R$ and for every integer $i > 0$, $\widetilde{w} - (i, 0) \in O_L$.*

Proof:

As the complement of $\omega(\widetilde{B_S^-})_{trans}$ is connected (see lemma 6), given $\widetilde{w} \in \pi^{-1}(\widehat{w})$, there exists a simple continuous arc γ which avoids $\omega(\widetilde{B_S^-})_{trans}$ and connects $\widetilde{w}$ with some point $\widetilde{w}_0$ in $\mathbb{R} \times \{M_f + M_{Dehn} + 2\}$. For any integer n, if $|n|$ is sufficiently large, then $\gamma + (n, 0)$ avoids V and thus connects $\widetilde{w} + (n, 0)$ to $\widetilde{w}_0 + (n, 0)$ not intersecting $v \cup \omega(\widetilde{B_S^-})_{trans}$. So if n is sufficiently large, then $\widetilde{w} + (n, 0) \in O_R$ and if n is sufficiently small, then $\widetilde{w} + (n, 0) \in O_L$. $\square$

As $\widetilde{f}(O_R) \subset O_R$, we get that $\widetilde{f}^n(B_{\epsilon_1}(\widetilde{z}^*)) \subset O_R$ for all integers $n \geq 0$. Proposition 1 implies that any i_1 that satisfies expression (5) must be greater or equal to zero.

If we perform the analogous construction for $w \cup \Gamma$, we get that any i_1 that satisfies expression (5) must be also smaller or equal to zero. So the only possible value for i_1 is zero. Now we perform the same construction we did for $\widehat{z}$, for $\widehat{z} + (0, 1)$. In the same way, we get that

$$\widetilde{f}^{n_1}(B_{\epsilon_1}(\widetilde{z})+(0,1))\cap(B_{\epsilon_1}(\widetilde{z})+(i,1))\neq\emptyset$$

only holds for $i=0$. But, as f is homotopic to a Dehn twist $(x,y)\longrightarrow(x+ky \bmod 1, y \bmod 1)$, the fact that expression (5) holds for $i_1=0$, implies that

$$\widetilde{f}^{n_1}(B_{\epsilon_1}(\widetilde{z})+(0,1))\cap(B_{\epsilon_1}(\widetilde{z})+(k.n_1,1))\neq\emptyset,$$

a contradiction which proves our theorem.

3.3 Proof of Theorem 3

Given an area-preserving $f\in DT(\mathrm{T}^2)$ with an exact lift $\widehat{f}\in DT(S^1\times\mathbb{R})$, if $\widehat{f}$ has a wandering point, then 0 is an interior point of $\rho_V(\widehat{f})$, see the proof of lemma 1 of [5]. So in the rest of this proof, we can assume that $\widehat{f}$ has no wandering points. We have two possibilities:

1) $\underset{n\geq 0}{\cup}\widehat{f}^n(H)$ is bounded and as in theorem 2, this means that $\rho_V(\widehat{f})=0$;

2) $\underset{n\geq 0}{\cup}\widehat{f}^n(H)$ is unbounded in both directions.

In this case, Lemma 1 implies that : $B_N^+\neq\emptyset$, $B_S^-\neq\emptyset$, $B_N^+(inv)\neq\emptyset$ and $B_S^-(inv)\neq\emptyset$. If for instance $\omega(B_N^+)=\emptyset$, then $\omega(B_N^+(inv))=\emptyset$, which imply by lemmas 3 and 4 that 0 is an interior point of $\rho_V(\widehat{f})$. So we can assume that $\omega(B_N^+)\neq\emptyset$ and $\omega(B_S^-)\neq\emptyset$, otherwise the theorem is proved. Theorem 2 tells us that this situation can not happen, thus proving our result.

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