Ergod. Th. & Dynam. Sys. (First published online 2019), page 1 of 47*
doi:10.1017/etds.2019.62

*Provisional—final page numbers to be inserted when paper edition is published

A condition that implies full homotopical complexity of orbits for surface homeomorphisms

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(*Received* 15 *January* 2019 *and accepted in revised form* 24 *July* 2019)

Abstract. We consider closed orientable surfaces S of genus $g > 1$ and homeomorphisms $f : S \to S$ isotopic to the identity. A set of hypotheses is presented, called a fully essential system of curves $\mathscr{C}$ and it is shown that under these hypotheses, the natural lift of f to the universal cover of S (the Poincaré disk $\mathbb{D}$), denoted by $\widetilde{f}$, has complicated and rich dynamics. In this context, we generalize results that hold for homeomorphisms of the torus isotopic to the identity when their rotation sets contain zero in the interior. In particular, for $C^{1+\epsilon}$ diffeomorphisms, we show the existence of rotational horseshoes having non-trivial displacements in every homotopical direction. As a consequence, we found that the homological rotation set of such an f is a compact convex subset of $\mathbb{R}^{2g}$ with maximal dimension and all points in its interior are realized by compact f-invariant sets and by periodic orbits in the rational case. Also, f has uniformly bounded displacement with respect to rotation vectors in the boundary of the rotation set. This implies, in case where f is area preserving, that the rotation vector of Lebesgue measure belongs to the interior of the rotation set.

Key words: low dimensional dynamics, topological dynamics, rotation sets, rotational horseshoes, pseudo-Anosov maps
2010 Mathematics Subject Classification: 37E30, 37E45 (Primary); 37D25, 37C25 (Secondary)

1. *Introduction*

1.1. *Preliminaries.* The main motivation for this work is to generalize some results that hold for homeomorphisms and diffeomorphisms of the torus isotopic to the identity to homeomorphisms and diffeomorphisms of closed surfaces of higher genus (for us, higher genus means larger than one) which are also isotopic to the identity.

In the study of torus homeomorphisms, a useful concept inherited from Poincaré's work on circle homeomorphisms is that of rotation number, or, in the two-dimensional case, rotation vectors. Actually, in the two-dimensional setting, one usually does not have a single rotation vector, but a rotation set, which is most precisely defined as follows. Given a homeomorphism $f:\mathbb{T}^2\to\mathbb{T}^2$ isotopic to the identity and a lift of f to $\mathbb{R}^2$, $\widetilde{f}:\mathbb{R}^2\to\mathbb{R}^2$, the Misiurewicz–Ziemian rotation set $\rho(\widetilde{f})$ is defined as (see [**24**])

$$\rho(\widetilde{f})=\bigcap_{i\geq 1}\overline{\bigcup_{n\geq i}\left\{\frac{\widetilde{f}^n(\widetilde{p})-\widetilde{p}}{n}:\widetilde{p}\in\mathbb{R}^2\right\}}. \tag{1}$$

This set is a compact convex subset of $\mathbb{R}^2$ (see [**24**]), and it was proved in [**10**, **25**] that all points in its interior are realized by compact f-invariant subsets of $\mathbb{T}^2$, which can be chosen as periodic orbits in the rational case. By saying that some vector $v\in\rho(\widetilde{f})$ is realized by a compact f-invariant set, we mean that there exists a compact f-invariant subset $K\subset\mathbb{T}^2$ such that, for all $p\in K$ and any $\widetilde{p}\in\pi^{-1}(p)$, where $\pi:\mathbb{R}^2\to\mathbb{T}^2$ is the associated covering map,

$$\lim_{n\to\infty}\frac{\widetilde{f}^n(\widetilde{p})-\widetilde{p}}{n}=v. \tag{2}$$

Moreover, the above limit, whenever it exists, is called the rotation vector of the point p, denoted $\rho(p)$.

Before presenting the results in the torus that we want to generalize to other surfaces, we need a definition.

Definition. (Topologically transverse intersections) If M is a surface, $f:M\to M$ is a C^1 diffeomorphism and $p,q\in M$ are f-periodic saddle points, then we say that $W^u(p)$ has a topologically transverse intersection with $W^s(q)$ (and write $W^u(p)\pitchfork W^s(q)$), whenever there exists a point $r\in W^s(q)\cap W^u(p)$ (r, clearly, can be chosen arbitrarily close to q or to p) and an open ball B centered at r such that $B\backslash\alpha=B_1\cup B_2$, where α is the connected component of $W^s(q)\cap B$ which contains r and has the following property. There exists a closed connected arc $\beta\subset W^u(p)$ such that $\beta\subset B$, $r\in\beta$ and $\beta\backslash r$ has two connected components, one contained in $B_1\cup\alpha$ and the other contained in $B_2\cup\alpha$, such that $\beta\cap B_1\neq\emptyset$ and $\beta\cap B_2\neq\emptyset$. Clearly, a C^1-transverse intersection is topologically transverse. Note that as $\beta\cap\alpha$ may contain a connected arc containing r, the ball B may not be chosen arbitrarily small.

Remark. The consequence of a topologically transverse intersection which is more relevant to us is a C^0 λ-lemma: if $W^u(p)$ has a topologically transverse intersection with $W^s(q)$, then $W^u(p)$ C^0-accumulates on $W^u(q)$.

In [**1**] it is proved that if $(0,0)\in\operatorname{int}(\rho(\widetilde{f}))$ and f is a $C^{1+\epsilon}$-diffeomorphism for some $\epsilon>0$, then $\widetilde{f}$ has a hyperbolic periodic saddle point $\widetilde{p}\in\mathbb{R}^2$ such that

$$W^u(\widetilde{p})\pitchfork W^s(\widetilde{p})+(a,b), \tag{3}$$

for all $(a,b)\in\mathbb{Z}^2$ ($W^u(\widetilde{p})$ is the unstable manifold of $\widetilde{p}$ and $W^s(\widetilde{p})$ is its stable manifold). Note that, as $\widetilde{p}$ is a periodic point for $\widetilde{f}$, the same holds for all integer translations of $\widetilde{p}$ and, moreover, for any integer vector (a,b), $W^{u,s}(\widetilde{p}+(a,b))=W^{u,s}(\widetilde{p})+(a,b)$.

In the area-preserving case, this result implies the following.

- $\overline{W^u(\widetilde{p})} = \overline{W^s(\widetilde{p})}$ is a $\widetilde{f}$-invariant equivariant closed connected subset of $\mathbb{R}^2$ and there exists $M = M(f) > 0$ such that any connected component $\widetilde{D}$ of $(\overline{W^u(\widetilde{p})})^c$ is an open topological disk whose diameter is less than M and $D \overset{\text{def.}}{=} \pi(\widetilde{D})$ is a f-periodic disk. Moreover, for any f-periodic disk $D \subset \mathbb{T}^2$, $\pi^{-1}(D) \subset (\overline{W^u(\widetilde{p})})^c$.
- For any $\rho = (s/q, r/q) \in \mathrm{int}(\rho(\widetilde{f})) \cap \mathbb{Q}^2$, if we consider the map $\widetilde{f}^q(\bullet) - (s, r)$, then there exists a point $\widetilde{p}_\rho$ that is a hyperbolic periodic saddle point for $\widetilde{f}^q(\bullet) - (s, r)$ whose stable and unstable manifolds have similar intersections to those in (3) and
$$\overline{W^u(\widetilde{p}_\rho)} = \overline{W^s(\widetilde{p}_\rho)} = \overline{W^u(\widetilde{p})} = \overline{W^s(\widetilde{p})}.$$
So, the above set is the same for all rational vectors in the interior of the rotation set. We denote it by $R.I.(\widetilde{f})$ (region of instability of $\widetilde{f}$) and a similar definition can be considered in the torus: $R.I.(f) \overset{\text{def.}}{=} \pi(\overline{W^u(\widetilde{p})}) = \overline{W^u(p)}$, where $p = \pi(\widetilde{p})$ is f-periodic. Every f-periodic open disk in $\mathbb{T}^2$ is contained in a connected component of the complement of $R.I.(f)$ and every such connected component is a f-periodic open disk, whose diameter when lifted to the plane is smaller than M.
- Every open ball centered at a point of $R.I.(f)$ has points with all rational rotation vectors contained in the interior of $\rho(\widetilde{f})$.
- If f is transitive, then $\widetilde{f}$ is topologically mixing in the plane. This follows easily from the fact that if f is transitive, then $R.I.(f) = \mathbb{T}^2$ and $R.I.(\widetilde{f}) = \overline{W^u(\widetilde{p})} = \overline{W^s(\widetilde{p})} = \mathbb{R}^2$.

As we have already said, the above results were obtained in [**1**] under a $C^{1+\epsilon}$ condition. In [**13**, **20**], some analogous results were proved for homeomorphisms, by completely different methods, but the conclusions of some are weaker.

What about surfaces of higher genus?

In this setting, starting with the definition of rotation set, things are more involved. If S is a closed orientable surface of genus $g > 1$, the definition of rotation set needs to take into account the fact that $\pi_1(S)$, the fundamental group of S and $H_1(S, \mathbb{Z})$, the first integer homology group of S, are different: the first is almost a free group with $2g$ generators. There is only one relation satisfied by the generators. While the second is $\mathbb{Z}^{2g}$.

Possibly the most immediate consequence of this is the fact that in order to define a rotation set for surfaces of higher genus, if one does not want it to be too complicated but wants it to have some properties similar to what happens in the torus, a homological definition must be considered. In the following, we present the definition of a homological rotation set and a homological rotation vector as they appeared in [**21**]. The idea of using homology in order to define rotation vectors goes back to the work of Schwartzman [**28**].

1.2. *Rotation vectors and rotation sets.* Let S be a closed orientable surface of genus $g > 1$ and let $I : [0, 1] \times S \to S$ be an isotopy from the identity map to a homeomorphism $f : S \to S$.

For α a loop in S (a closed curve), $[\alpha] \in H_1(S, \mathbb{Z}) \subset H_1(S, \mathbb{R})$ is its homology class. Recall that $H_1(S, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ and $H_1(S, \mathbb{R}) \simeq \mathbb{R}^{2g}$. We will also consider $H_1(S, \mathbb{R})$ endowed with the stable norm as in [**12**], which has the property that $\|[\gamma]\| \leq l(\gamma)$ for any rectifiable loop γ, where $l(\gamma)$ is the length of the loop.

For any fixed base point $b \in S$, $\mathcal{A}_b = \{\gamma_p : p \in S\}$ is a family of rectifiable paths such that γ_p joins b to p and the length of γ_p is bounded by a uniform constant $C_{\mathcal{A}_b}$.

For any point $p \in S$, we want to construct a path in S from p to $f^n(p)$ and then form a loop by adding γ_p and $\gamma_{f^n(p)}$. Consider the path I_p joining p to $f(p)$ given by $t \mapsto I(t, p)$. Also, for each $n \in \mathbb{N}$, define the path I_p^n joining p to $f^n(p)$ by

$$I_p^n = I_p * I_{f(p)} * \cdots * I_{f^{n-1}(p)},$$

where $\beta * \delta$ is the concatenation of the path β with the path δ.

For each $p \in S$, let α_p^n be the closed loop based at b formed by the concatenation of γ_p, the path I_p^n in S from p to $f^n(p)$ and $\gamma_{f^n(p)}$ traversed backwards, that is

$$\alpha_p^n = \gamma_p * I_p^n * \gamma_{f^n(p)}^{-1}.$$

We can now define the homological displacement function of p as

$$\Psi_f(p) = [\alpha_p].$$

For the function $\Psi_f : S \to H_1(S, \mathbb{R})$, we abbreviate its Birkhoff sums as

$$\Psi_f^n(p) = \sum_{k=0}^{n-1} \Psi_f(f^k(p)).$$

Note that, since α_p^n is homotopic to $\alpha_p * \alpha_{f(p)} * \cdots * \alpha_{f^{n-1}(p)}$,

$$[\alpha_p^n] = \sum_{k=0}^{n-1} [\alpha_{f^k(p)}] = \sum_{k=0}^{n-1} \Psi_f(f^k(p)) = \Psi_f^n(p).$$

Also, the path I_p^n can be replaced by any path joining p to $f^n(p)$ and homotopic with fixed endpoints to I_p^n. This implies that Ψ_f depends only on f, on the choice of $\mathcal{A}_b$ and on the homotopy class of the isotopy I. In particular, Ψ_f is bounded. Indeed, as S is compact, $\sup\{d_{\mathbb{D}}(\widetilde{q}, \widetilde{f}(\widetilde{q})) : \widetilde{q} \in \mathbb{D}\} = C_{\max\_f} < \infty$, and if we replace the path I_p by the projection of the geodesic segment in $\mathbb{D}$ joining $\widetilde{p} \in \pi^{-1}(p)$ to $\widetilde{f}(\widetilde{p})$, as the length of this path is smaller than $C_{\max\_f}$, then $\|\Psi_f\| \leq 2C_{\mathcal{A}_b} + C_{\max\_f}$.

As we just said, Ψ_f depends on the choice of the basepoint b and the family $\mathcal{A}_b$. However, given another basepoint $b' \in S$ and a family $\mathcal{A}'_{b'} = \{\gamma_p : p \in S\}$ of rectifiable paths whose lengths are uniformly bounded by $C_{\mathcal{A}'_{b'}}$ such that γ'_p joins b' to p, defining α'^n_p analogously, one has

$$[\alpha'^n_p] = [\gamma'_p * I_p^n * \gamma'^{-1}_{f^n(p)}] = [\alpha_p^n * \delta_p^n] = [\delta_p^n] + \Psi_f^n(p), \tag{4}$$

where $\delta_p^n = \gamma_{f^n(p)} * \gamma'^{-1}_{f^n(p)} * \gamma'_p * \gamma_p^{-1}$. Indeed, the loop α'^n_p is freely homotopic to $I_p^n * \delta_p^n$. In particular, if $\Psi'_f(p) = [\alpha'_p]$, then

$$\|\Psi_f^n(p) - \Psi'^n_f(p)\| \leq 2C_{\mathcal{A}_b} + 2C_{\mathcal{A}'_{b'}}. \tag{5}$$

Finally, if the limit

$$\rho(f, p) = \lim_{n\to\infty} \frac{1}{n} \Psi_f^n(p) \in H_1(S, \mathbb{R}) \tag{6}$$

exists, we say that p has a well-defined (homological) rotation vector.

After all this, we are ready to present the definition of the (homological) rotation set of f, which is analogous to the definition for the torus [**24**]. The Misiurewicz–Ziemian rotation set of f over S is defined as the set $\rho_{mz}(f)$ consisting of all limits of the form

$$v = \lim_{k\to\infty} \frac{1}{n_k} \Psi_f^{n_k}(p_k) \in H_1(S, \mathbb{R}),$$

where $p_k \in S$ and $n_k \to \infty$. By (5), the rotation set depends only on f, but not on the choice of the isotopy, the basepoint b or the arcs γ_p. This definition coincides with

$$\rho_{mz}(f) = \bigcap_{m\geq 0} \overline{\bigcup_{n\geq m} \left\{ \frac{\Psi_f^n(p)}{n} : p \in S \right\}}.$$

In particular, since Ψ_f is bounded, the rotation set is compact.

Note that, using a computation similar to (4), if one chooses a rectifiable arc β joining $f^n(p)$ to p,

$$[I_p^n * \beta] = [\gamma_p^{-1} * \alpha_p^n * \gamma_{f^n(p)} * \beta] = \Psi_f^n(p) + [\gamma_{f^n(p)} * \beta * \gamma_p^{-1}]. \tag{7}$$

Thus, $\|I_p^n * \beta - \Psi_f^n(p)\| \leq 2C_{\mathcal{A}_b} + l(\beta)$. As a consequence, an alternate but equivalent definition of rotation vectors and rotation sets is obtained by considering all limits of the form

$$v = \lim_{k\to\infty} \frac{1}{n_k} [I_{p_k}^{n_k} * \beta_k],$$

where $p_k \in S$, $n_k \to \infty$ and β_k are rectifiable arcs joining $f^{n_k}(p_k)$ to p_k such that $l(\beta_k) < \infty$.

Moreover, it is possible to choose the arcs γ_p in the definition of Ψ_f so that the map $p \mapsto \Psi_f$ is not only bounded, but also Borel measurable [**11**].

This is important if one wants to define rotation vectors of invariant measures. Let $\mathcal{M}(f)$ be the set of all f-invariant Borel probability measures. The rotation vector of the measure $\mu \in \mathcal{M}(f)$ is defined as

$$\rho_m(f, \mu) = \int \Psi_f \, d\mu \in H_1(S, \mathbb{R}).$$

By the Birkhoff ergodic theorem, for μ-almost every point $p \in S$ the limit $\rho(f, p) = \lim_{n\to\infty} (1/n)\Psi_f^n(p)$ exists and $\rho_m(f, \mu) = \int \rho(f, p)\, d\mu$. Moreover, if μ is an ergodic measure, then $\rho(f, p) = \rho_m(f, \mu)$ for μ-almost every point p.

Due to these facts and (5), the rotation vector of a measure is also independent of any choices made in the definitions. Denote by $\rho_m(f)$ the rotation set of invariant measures, that is, $\rho_m(f) = \bigcup_{\mu\in\mathcal{M}(f)} \rho_m(f, \mu)$ and denote by $\rho_{\text{erg}}(f)$ the corresponding set for ergodic measures. Then [**24**, proof of Theorem 2.4], without modifications, implies that

$$\rho_m(f) = \operatorname{Conv}(\rho_{\text{erg}}(f)) = \operatorname{Conv}(\rho_{mz}(f)).$$

In particular, every extremal point of the convex hull of $\rho_{mz}(f)$ is the rotation vector of some ergodic measure and, therefore, it is the rotation vector of some recurrent point.

The main problems with this definition of rotation set are the following.

- Although it is compact, it does not need to be convex.

- It is not true that vectors in the interior of the rotation set are always realized by invariant sets; in certain cases they are not. An example was communicated to us by Passeggi [**27**].
- It is also not known whether, when zero is in the interior of the rotation set, a result analogous to (3) holds, not even in the Abelian cover of S (see definition below).

Definition. (Abelian cover) Let S be a closed orientable surface of genus $g > 1$. The Abelian cover of S is a covering space for S, for which the group of deck transformations is the integer homology group of S.

1.3. *A more precise motivation and statements of the main results.* The main objective of this work is to give conditions which imply complicated and rich dynamics in the universal cover of S, analogous to what happens for a homeomorphism of the torus isotopic to the identity when its rotation set contains (0, 0) in its interior.

This type of problem has already been studied for surfaces of higher genus by Boyland in [**4**]. But, in that paper, he considered the Abelian cover of S instead of the universal cover. As far as we know, this is the only published result on this kind of problem. Boyland considered homeomorphisms $f : S \to S$ of a special type, which are very important for our work: f is isotopic to the identity as a homeomorphism of S, but it is pseudo-Anosov relative to a finite f-invariant set $K \subset S$ (see [**9**]). He presented some conditions equivalent to f having a transitive lift to the Abelian cover of S.

The hypotheses of our main results will imply, in particular, that if a $C^{1+\epsilon}$ diffeomorphism $f : S \to S$ isotopic to the identity satisfies these hypotheses, then analogous results to those in [**1**] hold.

As a by-product of these results, we obtain that in the $C^{1+\epsilon}$ setting, the homological rotation set is a compact convex subset of $\mathbb{R}^{2g}$ which is $2g$-dimensional: it is equal to the rotation set of the f-invariant Borel probability measures and all rational points in its interior are realized by periodic orbits. Non-rational points in the interior of the rotation set are also realized by compact f-invariant sets.

We are indebted to Alejandro Passeggi, who pointed out this consequence of Theorem 2 to us.

Moreover, as a corollary of the ideas used in this last result, we can extend the main theorems from [**2**] to our setting. This is done in Theorems 4 and 5.

In what follows, we precisely present the main results of this paper. Assume that S is a closed orientable surface of genus $g > 1$ and $\pi : \widetilde{S} \to S$ is its universal covering map. We may identify the universal cover $\widetilde{S}$ with the Poincaré disk $\mathbb{D}$ and denote by $\mathrm{Deck}(\pi)$ the groups of deck transformations of S. Consider $f : S \to S$, which is a homeomorphism isotopic to the identity, and let $\widetilde{f} : \mathbb{D} \to \mathbb{D}$ be the endpoint of the lift of the isotopy from Id to f which starts at $\mathrm{Id} : \mathbb{D} \to \mathbb{D}$. We call $\widetilde{f}$ the natural lift of f.

Definition 1.3. (Fully essential system of curves $\mathscr{C}$) We say that $f : S \to S$ is a homeomorphism with a fully essential system of curves $\mathscr{C} = \bigcup_{i=1}^{k} \gamma_i$ if the following conditions are satisfied:

(1) there exist different oriented closed geodesics $\gamma_1, \ldots, \gamma_k$ in S, $k \geq 1$, such that $(\bigcup_{i=1}^{k} \gamma_i)^c$ only has non-essential connected components;

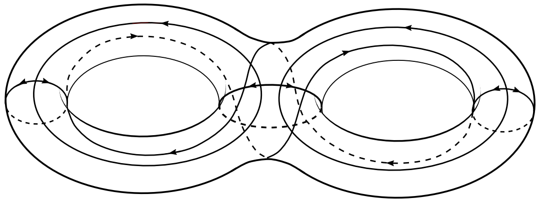

FIGURE 1. An example where not all geodesics appear twice.

(2) for each $i \in \{1, \ldots, k\}$, there is a f-periodic point p_i such that its trajectory under the isotopy is a closed curve freely homotopic to γ_i with the correct orientation;

(3) for every open intervals I, $F \subset \partial\mathbb{D}$, there exists an oriented simple arc $\widetilde{\alpha} \subset \pi^{-1}(\mathscr{C})$ formed by the concatenation of a finite number of oriented subarcs of extended lifts of geodesics in $\mathscr{C}$ and such that the initial point of $\widetilde{\alpha}$ is contained in I and the final point belongs to F.

Remarks.

- No matter how large g (the genus) is, it is possible to construct examples having a fully essential system of curves with $k = 2$. Although the fundamental group has $2g$ generators, the number of geodesics may be much smaller.
- The third condition above is a little tricky to check. A much easier one, which implies it, is the following: for each $i \in \{1, \ldots, k\}$, there are f-periodic points p_i^- and p_i^+ such that their trajectories under the isotopy are closed curves freely homotopic to γ_i, or concatenations of γ_i, with both possible orientations. In order to see that this implies the third condition, use Propositions 6 and 7. Nevertheless, we present this more general condition because what we really need about the fully essential system of curves is the property that, when considered as a connected subset of S, its complement only has disks as connected components and $\mathscr{C}$ contains oriented closed curves (the orientations are inherited by the orientations of the $\gamma_{i's}$) whose homotopy classes generate $\pi_1(S)$ as a semi-group. This is achieved, for instance, when $\mathscr{C}$ contains a generator for $\pi_1(S)$, with each curve appearing twice, and with both possible orientations (as explained above), or, more generally, with any set of curves. This more general situation is the one we describe in the third condition above. See the example in Figure 1, which shows a situation in which we can find generators for $\pi_1(S)$ as a semi-group in a fully essential system of curves but some curves do not appear twice with different orientations.

Now we present the main theorems in the order in which we prove them in the paper. The exception is the first one, which we only sketch here, because its precise statement is more technical. The formal statement can be found in §3.

THEOREM 1. (Informal statement) *Let $f : S \to S$ be a homeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$ and let $\widetilde{f}$ be its natural lift. Then there exists a real number $c = c(f) \geq 0$ such that the $\widetilde{f}$-iterates of an open c-neighborhood of any fundamental domain $\widetilde{Q} \subset \mathbb{D}$ of S accumulate on all translates of the c-neighborhood of $\widetilde{Q}$ under deck transformations and thus on the whole boundary of $\mathbb{D}$.*

THEOREM 2. *For some* $\epsilon > 0$ *let* $f : S \to S$ *be a* $C^{1+\epsilon}$ *diffeomorphism isotopic to the identity with a fully essential system of curves* $\mathscr{C}$ *and let* $\widetilde{f}$ *be its natural lift. Then there exists a contractible hyperbolic* f*-periodic saddle point* $p \in S$ *such that, for any* $\widetilde{p} \in \pi^{-1}(p)$ *and for every* $g \in \mathrm{Deck}(\pi)$,

$$W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p})).$$

Remark. A point $p \in S$ being contractible means that all $\widetilde{p} \in \pi^{-1}(p)$ are $\widetilde{f}$-periodic.

To prove this result we have to work with a pseudo-Anosov map ϕ isotopic to f relative to the finite invariant set of periodic points associated with the fully essential system of curves $\mathscr{C}$. Using several properties of the stable and unstable foliations of this map, it is possible to prove a result similar to Theorem 2 for ϕ, and then, using a theorem of Boyland [**3**] (see also [**14**]) and other technical results on Pesin theory [**7**, **18**], we can finally prove the theorem for the original map f. This procedure is similar to what was done in [**1**].

The main part of this paper is proving Theorem 2 for relative pseudo-Anosov maps and this is done in Lemma 13.

We would like to point out that the conclusion of Theorem 1 clearly implies the existence of a fully essential system of curves. In other words, Theorems 1 and 2 are both 'if and only if' statements.

The next results are consequences of Theorem 2, exactly as in [**1**]. They all share the same hypotheses: suppose that, for some $\epsilon > 0$, $f : S \to S$ is a $C^{1+\epsilon}$ area-preserving diffeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$.

COROLLARY 1. *If* f *is transitive, then* f *cannot have a periodic open disk. In the general case, there exists* $M = M(f) > 0$ *such that if* $D \subset S$ *is a* f*-periodic open disk, then for any connected component* $\widetilde{D}$ *of* $\pi^{-1}(D)$, $\mathrm{diam}(\widetilde{D}) < M$ *in the metric* $d_{\mathbb{D}}$, *the lift of the hyperbolic metric* d *in* S.

In [**21**], it is proved that in the case where f is just an area-preserving homeomorphism of S and the fixed point set is inessential, then all f-invariant open disks have diameter bounded by some constant $M > 0$. If, moreover, for all $n > 0$, the set of n-periodic points is inessential, then, for each $n > 0$, the set of n-periodic open disks has bounded diameter. But the bound may not be uniform with the period. In our situation, with much stronger hypotheses, Corollary 1 gives a uniform bound.

COROLLARY 2. *There exists a contractible hyperbolic* f*-periodic saddle point* $p \in S$ *(the one from Theorem 2) such that* $R.I.(f) \overset{\text{def.}}{=} \overline{W^u(p)} = \overline{W^s(p)}$, *is compact,* f*-invariant and all connected components of the complement of* $R.I.(f)$ *are* f*-periodic disks. Moreover, for all* $\widetilde{p} \in \pi^{-1}(p)$, $R.I.(\widetilde{f}) \overset{\text{def.}}{=} \pi^{-1}(R.I.(f)) = \overline{W^s(\widetilde{p})} = \overline{W^u(\widetilde{p})}$ *is a connected, closed,* $\widetilde{f}$*-invariant, equivariant subset of* $\mathbb{D}$.

COROLLARY 3. *If* f *is transitive, then there exists a contractible hyperbolic* f*-periodic saddle point* $p \in S$ *(the one from Theorem 2) such that* $\overline{W^u(p)} = \overline{W^s(p)} = S$ *and, for any* $\widetilde{p} \in \pi^{-1}(p)$, $\overline{W^u(\widetilde{p})} = \overline{W^s(\widetilde{p})} = \mathbb{D}$, *something that implies that* $\widetilde{f}$ *is topologically mixing.*

Finally, in the third theorem, we study the homological rotation set $\rho_{mz}(f)$.

THEOREM 3. *Let $f : S \to S$ be a $C^{1+\epsilon}$ diffeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$. Then the (homological) rotation set $\rho_{mz}(f)$ is a $2g$-dimensional compact convex subset of $H_1(S, \mathbb{R}) \simeq \mathbb{R}^{2g}$. Moreover, if $v \in \mathrm{int}(\rho_{mz}(f))$, then there exists a compact set $K \subset S$ such that, for all $q \in K$, $\rho(f, q) = v$. In the case where v is a rational point, K can be chosen as a periodic orbit.*

The last two results generalize the main theorems of [**2**] to the context of this paper.

THEOREM 4. *Let $f : S \to S$ be a $C^{1+\epsilon}$ diffeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$. Then there exists $M(f) > 0$ such that, for any $\omega \in \partial\rho_{mz}(f)$, any hyperplane $\omega \in H \subset \mathbb{R}^{2g}$ that does not intersect* interior$(\rho_{mz}(f))$ *(H is called a supporting hyperplane), any $p \in S$ and $n > 0$,*

$$([\alpha_p^n] - n \cdot \omega) \cdot \overrightarrow{v_H} < M(f),$$

where $\overrightarrow{v_H}$ is the unitary normal to H, which points towards the connected component of H^c that does not intersect $\rho_{mz}(f)$.

THEOREM 5. *Let $f : S \to S$ be a $C^{1+\epsilon}$ area-preserving diffeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$. Then the rotation vector of Lebesgue measure belongs to* interior$(\rho_{mz}(f))$.

2. *Some background, auxiliary results and their proofs*

In this section, we present some important results that we will use, along with some definitions and a short digression on hyperbolic surfaces, Thurston classification of homeomorphisms of surfaces and a little of Pesin theory. We also prove some auxiliary results, which we will use in the following sections to prove Theorems 1–5.

2.1. *Properties of hyperbolic surfaces.*

Let S be a closed orientable surface of genus $g > 1$ and let $\pi : \widetilde{S} \to S$ be its universal covering map. As we said before, the universal cover $\widetilde{S}$ is identified with the Poincaré disk $\mathbb{D}$ endowed with the hyperbolic metric $d_{\mathbb{D}}$. Hence, we assume that $S = \mathbb{D}/\Gamma$, where Γ is a cocompact freely acting group of Moebius transformations. Any non-trivial deck transformation $g \in \mathrm{Deck}(\pi) = \Gamma$ is a hyperbolic isometry and extends to the 'boundary at infinity' $\partial\mathbb{D}$ as a homeomorphism which has exactly two fixed points: one attractor and one repeller. These fixed points are the endpoints of some g-invariant geodesic δ_g of $\mathbb{D}$, called the axis of g. For any point $\widetilde{p} \in \overline{\mathbb{D}}$, the sequence $g^n(\widetilde{p})$ converges to one endpoint of δ_g as $n \to -\infty$ and to the other one as $n \to \infty$. Any subarc of δ_g joining a point $\widetilde{p}$ to $g(\widetilde{p})$, when projected to S, becomes an essential loop γ_g, which is the unique geodesic in its free homotopy class.

Given an essential loop $\gamma : [0, 1] \to S$, an extended lift of γ is an arc $\widetilde{\gamma} : \mathbb{R} \to \mathbb{D}$ obtained by the concatenation of arcs that are the translation of a lift of γ by all iterates of some deck transformation. Two extended lifts of an essential loop coincide if and only if they share the same endpoints in $\partial\mathbb{D}$.

If h is a deck transformation that commutes with g, then the axis of g is equal to the axis of h, and the group of all deck transformations that commute with g is cyclic and generated by g if γ_g is in the free homotopy class of a simple loop.

Let $f: S \to S$ be a homeomorphism isotopic to the identity and let $I: [0, 1] \times S \to S$ be an isotopy from the identity map to f. The isotopy $\widetilde{I}: [0, 1] \times \mathbb{D} \to \mathbb{D}$ obtained by lifting I with basepoint $\mathrm{Id}: \mathbb{D} \to \mathbb{D}$ is called the natural lift of I. As we have already defined in §1.3, the map $\widetilde{f}: \mathbb{D} \to \mathbb{D}$, given by $\widetilde{f}(\widetilde{p}) = \widetilde{I}(1, \widetilde{p})$, is called the natural lift of f associated with the isotopy I. Natural lifts of a homeomorphism are characterized by the property of commuting with all deck transformations, and, moreover, $\widetilde{f}$ can be extended to a homeomorphism of $\overline{\mathbb{D}}$ as the identity on the 'boundary at infinity' $\partial\mathbb{D}$ (see [**8**]).

2.2. *On the fully essential system of curves $\mathscr{C}$.* In this subsection, we prove some properties for $\pi^{-1}(\mathscr{C})$, where $\mathscr{C}$ is a fully essential system of curves.

PROPOSITION 6. *The lift $\pi^{-1}(\mathscr{C})$ is a closed connected subset of $\mathbb{D}$ that accumulates all over $\partial\mathbb{D}$ with the Euclidean metric.*

Proof. First, observe that $\mathscr{C}$ is the union of a finite number of closed geodesics in S; therefore $\mathscr{C}$ is closed. Since $\pi: \mathbb{D} \to S$ is continuous, $\pi^{-1}(\mathscr{C})$ is closed. To see that $\pi^{-1}(\mathscr{C})$ is connected, we just observe that $S \backslash \mathscr{C}$ is a union of open topological disks, and therefore all connected components of $\mathbb{D} \backslash \pi^{-1}(\mathscr{C})$ are bounded topological open disks.

In order to prove that $\pi^{-1}(\mathscr{C})$ accumulates everywhere in $\partial\mathbb{D}$, we first note that, since, for all $\widetilde{z} \in \mathbb{D}$ and $g \in \mathrm{Deck}(\pi)$, $\pi(\widetilde{z}) = \pi(g(\widetilde{z}))$, $\pi^{-1}(\mathscr{C})$ is invariant under deck transformations. This and the fact that the subset $\{\widetilde{z} \in \partial\mathbb{D} : \widetilde{z}$ is fixed by some $g \in \mathrm{Deck}(\pi)\}$ is dense in $\partial\mathbb{D}$ with the Euclidean metric (see [**8**]) imply that $\pi^{-1}(\mathscr{C})$ accumulates all over $\partial\mathbb{D}$. □

In the next proposition, we consider the geodesics in $\mathscr{C}$ without their orientations.

PROPOSITION 7. *For every $\widetilde{p}, \widetilde{r} \in \pi^{-1}(\mathscr{C})$, there exists a path γ in $\pi^{-1}(\mathscr{C})$ joining these two points, which is contained in the union of a finite number of subarcs of extended lifts of geodesics in $\mathscr{C}$.*

Proof. Fix a point $\widetilde{p} \in \pi^{-1}(\mathscr{C})$ and let $\mathcal{P}_{\widetilde{p}}$ be the set of all points $\widetilde{q} \in \pi^{-1}(\mathscr{C})$ such that there exists a path joining $\widetilde{p}$ to $\widetilde{q}$ formed by subarcs of a finite number of extended lifts of geodesics in $\mathscr{C}$. We will show that $\mathcal{P}_{\widetilde{p}}$ is an open and closed subset of $\pi^{-1}(\mathscr{C})$.

Let $\widetilde{q}$ be a point in $\mathcal{P}_{\widetilde{p}}$. As the set $\mathscr{C}$ is equal to the union of a finite number of closed geodesics, there exists $\epsilon > 0$ small enough so that $B_\epsilon(\widetilde{q}) \cap \pi^{-1}(\mathscr{C})$ satisfies one of the possibilities in Figure 2.

In the first case, $\widetilde{q}$ belongs to just one extended lift of a geodesic in $\mathscr{C}$. If γ is the path joining $\widetilde{p}$ to $\widetilde{q}$ and it is formed by $k > 0$ subarcs of extended lifts of geodesics, it is clear that, for all points in $B_\epsilon(\widetilde{q}) \cap \pi^{-1}(\mathscr{C})$, there is a path γ' joining $\widetilde{p}$ to this point formed by the same number of subarcs of extended lifts of geodesics. In the second case, $\widetilde{q}$ belongs to the intersection of a finite number of extended lifts of geodesics and, again, if the path γ is formed by $k > 0$ subarcs, then, for all points in $B_\epsilon(\widetilde{q}) \cap \pi^{-1}(\mathscr{C})$, there is a path γ'

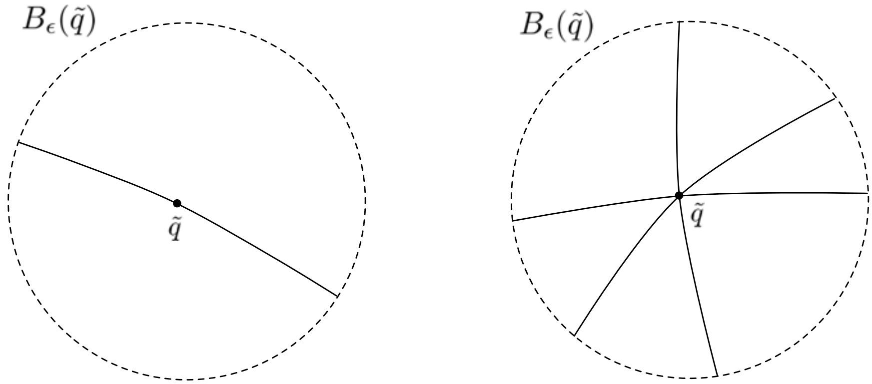

FIGURE 2. Possibilities for a neighborhood of $\widetilde{q}$.

joining $\widetilde{p}$ to this point formed by at most $k+1$ subarcs of extended lifts of geodesics. So $\mathcal{P}_{\widetilde{p}}$ is open.

We will now prove that $\mathcal{P}^c_{\widetilde{p}} = \pi^{-1}(\mathscr{C})\backslash\mathcal{P}_{\widetilde{p}}$ is open. Again, if $\widetilde{q}$ is a point in $\mathcal{P}^c_{\widetilde{p}}$, there exists $\epsilon > 0$ small enough such that $B_\epsilon(\widetilde{q}) \cap \pi^{-1}(\mathscr{C})$ satisfies one of the possibilities in Figure 2. In both cases, if $\widetilde{q}' \in B_\epsilon(\widetilde{q}) \cap \pi^{-1}(\mathscr{C}) \cap \mathcal{P}_{\widetilde{p}}$, then, by the same argument as above, there is a path γ' joining $\widetilde{p}$ to $\widetilde{q}$ with a finite number of subarcs of extended lifts of geodesics. But this is a contradiction because $\widetilde{q} \in \mathcal{P}^c_{\widetilde{p}}$, so all points in $B_\epsilon(\widetilde{q}) \cap \pi^{-1}(\mathscr{C})$ are points of $\mathcal{P}^c_{\widetilde{p}}$. Hence $\mathcal{P}^c_{\widetilde{p}}$ is open. Since $\mathcal{P}_{\widetilde{p}}$ is an open and closed subset of the connected set $\pi^{-1}(\mathscr{C})$, $\mathcal{P}_{\widetilde{p}} = \pi^{-1}(\mathscr{C})$. □

2.3. *Nielsen–Thurston classification of homeomorphisms of surfaces.* In this subsection, we present a brief overview of Thurston's classification of homeomorphisms of surfaces and prove a result analogous to [**23**, Theorem 1(i)].

2.3.1. *Some definitions and the classification theorem.* Let M be a compact, connected, orientable surface, possibly with boundary, and let $f : M \to M$ be a homeomorphism. There are two basic types of homeomorphisms which appear in the Nielsen–Thurston classification: the finite order homeomorphisms and the pseudo-Anosov ones.

A homeomorphism f is said to be of finite order if $f^n = \mathrm{Id}$ for some $n \in \mathbb{N}$. The least such n is called the order of f. Finite order homeomorphisms have zero topological entropy.

A homeomorphism f is said to be pseudo-Anosov if there is a real number $\lambda > 1$ and a pair of transverse measured foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ such that $f(\mathscr{F}^s) = \lambda^{-1}\mathscr{F}^s$ and $f(\mathscr{F}^u) = \lambda\mathscr{F}^u$. Pseudo-Anosov homeomorphisms are topologically transitive, have positive topological entropy and Markov partitions [**9**].

A homeomorphism f is said to be reducible by a system

$$C = \bigcup_{i=1}^{n} C_i$$

of disjoint simple closed curves $C_1, \ldots, C_n$, called reducing curves, if:

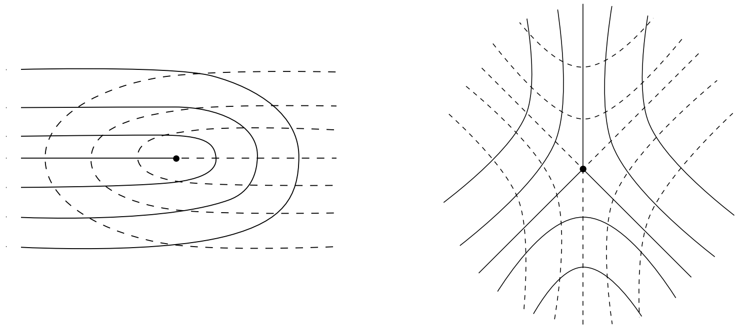

FIGURE 3. Examples of a 1-prong and a 3-prong singularity, respectively.

- for all i, C_i is not homotopic to a point, nor to a component of ∂M;
- for all $i \neq j$, C_i is not homotopic to C_j; and
- C is invariant under f.

THEOREM 8. (Nielsen–Thurston) *If the Euler characteristic* $\chi(M) < 0$*, then every homeomorphism* $f : M \to M$ *is isotopic to a homeomorphism* $\phi : M \to M$ *such that:*

(1) ϕ *is of finite order;*
(2) ϕ *is pseudo-Anosov; or*
(3) ϕ *is reducible by a system of curves* C*, and there exist disjoint open annular neighborhoods* U_i *of* C_i *such that*

$$U = \bigcup_i U_i$$

is ϕ*-invariant. Each component* S_i *of* $M \backslash U$ *is mapped to itself by some least positive iterate* n_i *of* ϕ*, and each* $\phi^{n_i}|_{S_i}$ *satisfies* (1) *or* (2)*. Each* U_i *is mapped to itself by some least positive iterate* m_i *of* ϕ *fixing the boundary components, and each* $\phi^{m_i}|_{U_i}$ *is a generalized twist.*

Homeomorphisms ϕ as in Theorem 8 are called Thurston canonical forms for f.

We say that $\phi : M \to M$ is pseudo-Anosov relative to a finite invariant set K if it satisfies all of the properties of a pseudo-Anosov homeomorphism except that the associated stable and unstable foliations may have 1-pronged singularities at points in K [**15**], see Figure 3. Equivalently, let N be the compact surface obtained from $M \backslash K$ by compactifying each puncture with a boundary circle and let $p : N \to M$ be the map that collapses these boundary circles to points. Then ϕ is pseudo-Anosov relative to K if and only if there is a pseudo-Anosov homeomorphism $\Phi : N \to N$ such that $\phi \circ p = p \circ \Phi$.

2.3.2. *The beginning of the work.* The following result is the first step towards the proof of the main theorems.

LEMMA 9. *Let* $f : S \to S$ *be a homeomorphism isotopic to the identity with a fully essential system of curves* $\mathscr{C}$ *and let* P *be the set of periodic points associated with the geodesics in* $\mathscr{C}$*. Then there exists an integer* $m_0 > 0$ *such that* f^{m_0} *is isotopic relative to* P *to a homeomorphism* $\phi : S \to S$*, which is pseudo-Anosov relative to* P.

Proof. Let f be a homeomorphism with a fully essential system of curves $\mathscr{C}$ and let P be the set of all periodic points associated with the geodesics in $\mathscr{C}$. We write $P = \{p_1, p_2, \ldots, p_k\}$. For each $1 \leq i \leq k$, there exists an integer $n_i > 0$ such that $f^{n_i}(p_i) = p_i$. Take $m_0 > 0$ to be an integer such that all points in P are fixed points for f^{m_0}.

We will follow the same ideas used by Llibre and MacKay in [**23**]. Let $\phi : S \to S$ be the Thurston canonical form associated to f^{m_0}. Of course, we are considering $f^{m_0} : S\backslash P \to S\backslash P$ and so ϕ is also a homeomorphism from $S\backslash P$ into itself. But it can be extended in a standard way to the set P (fixing everybody), giving a homeomorphism of S into itself that is also isotopic to the identity as a homeomorphism of S, which we still call ϕ.

Let us show that ϕ is pseudo-Anosov relative to P. First, note that ϕ can not be of finite order, since points in $\pi^{-1}(P)$ move in non-trivial homotopical directions. To be more precise, if ϕ had finite order, then, for some $N > 0$, $\phi^N \equiv \mathrm{Id}$. This implies that the natural lift of ϕ^N is also the identity. But there is at least one fixed point for ϕ, $p_1 \in P$, such that, for any $\widetilde{p}_1 \in \pi^{-1}(p_1)$, its trajectory under the natural lift $\widetilde{\phi}^N : \mathbb{D} \to \mathbb{D}$ follows a non-trivial deck transformation.

Now, suppose ϕ is reducible by a system of curves C. As in [**23**], we say that a simple closed curve γ on a surface of genus g with punctures is non-rotational if, after closing the punctures, γ is homotopically trivial. If γ is a non-rotational reducing curve, then it must surround at least two punctures. So, suppose γ surrounds p_i and p_j, $i \neq j$. Since γ is a reducing curve, $\phi^n(\gamma) = \gamma$, for some $n > 0$. This means that there exists $g \in \mathrm{Deck}(\pi)$ such that $\widetilde{\phi}^n(\widetilde{\gamma}) = g(\widetilde{\gamma})$, where $\widetilde{\gamma}$ is a lift of γ ($\widetilde{\gamma}$ is a simple closed curve in $\mathbb{D}$) surrounding $\widetilde{p}_i$ and $\widetilde{p}_j$, which are lifts of p_i and p_j, respectively. By induction, it follows that $\widetilde{\phi}^{mn}(\widetilde{\gamma}) = g^m(\widetilde{\gamma})$ encloses both $\widetilde{\phi}^{mn}(\widetilde{p}_i)$ and $\widetilde{\phi}^{mn}(\widetilde{p}_j)$ for all $m \in \mathbb{Z}$. But this is a contradiction because as $i \neq j$, $\lim_{l\to\infty} \widetilde{\phi}^l(\widetilde{p}_i)$ and $\lim_{l\to\infty} \widetilde{\phi}^l(\widetilde{p}_j)$ are different points of $\partial\mathbb{D}$.

In the case where γ is a rotational reducing curve, let $\widetilde{\gamma} \subset \mathbb{D}$ be an extended lift of γ. The curve $\widetilde{\gamma}$ has two distinct endpoints at the 'boundary at infinity' $\partial\mathbb{D}$, and $\mathbb{D}\backslash\widetilde{\gamma}$ has exactly two connected components. Since $\widetilde{\phi}|_{\partial\mathbb{D}} = \mathrm{Id}$, $\widetilde{\phi}(\widetilde{\gamma})$ has the same endpoints on $\partial\mathbb{D}$ as $\widetilde{\gamma}$. Since $S\backslash\mathscr{C}$ is a union of topological disks, there exists $g \in \mathrm{Deck}(\pi)$ associated with some geodesic γ_i in $\mathscr{C}$ such that the fixed points of g in $\partial\mathbb{D}$ separate the endpoints of $\widetilde{\gamma}$.

Finally, choose $\widetilde{p}_i \in \pi^{-1}(p_i)$ such that it belongs to one connected component of $\mathbb{D}\backslash\widetilde{\gamma}$ and $\lim_{n\to\infty} \widetilde{\phi}^n(\widetilde{p}_i)$ is in the 'boundary at infinity' of the other connected component. Since $\widetilde{\phi}(\widetilde{\gamma})$ and $\widetilde{\gamma}$ have the same endpoints in $\partial\mathbb{D}$ and $\phi^n(\gamma) = \gamma$, we have $\widetilde{\phi}^{mn}(\widetilde{\gamma}) = \widetilde{\gamma}$, for all $m > 0$. As $\widetilde{\phi}$ preserves orientation, this clearly implies a contradiction (see Figure 4). This shows that ϕ cannot be of finite order or reducible by a system of curves. So ϕ is pseudo-Anosov relative to P. □

2.4. *On Handel's fixed point theorem.*

2.4.1. *Preliminaries and a statement of Handel's theorem.* In [**16**], Michael Handel proved the existence of a fixed point for an orientation-preserving homeomorphism of the open unit disk that can be extended to the closed disk as the identity on the boundary, provided that, for certain points in the open disk, their α and ω-limit sets are single points

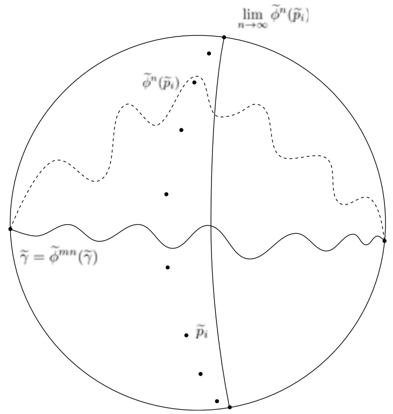

FIGURE 4. The final contradiction.

in the boundary of the disk, distributed with a certain cyclic order. Later, in [**22**], Patrice Le Calvez gave a different proof of this theorem based only on Brouwer theory and plane topology arguments. In Le Calvez's proof, the existence of the fixed point follows from the existence a simple closed curve contained in the open disk, whose topological index can be calculated and is equal to one.

THEOREM 10. (Handel's fixed point theorem, [**22**]) *Consider a homeomorphism $\widetilde{h} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ of the closed unit disk satisfying the following hypotheses.*

(1) *There exists $r \geq 3$ points $\widetilde{p}_1, \ldots, \widetilde{p}_r$ in $\mathbb{D}$ and $2r$ pairwise distinct points $\alpha_1, \omega_1, \ldots, \alpha_r, \omega_r$ on the boundary $\partial\mathbb{D}$ such that, for every $1 \leq i \leq r$,*
$$\lim_{n\to\infty} \widetilde{h}^{-n}(\widetilde{p}_i) = \alpha_i, \qquad \lim_{n\to\infty} \widetilde{h}^{n}(\widetilde{p}_i) = \omega_i.$$
(2) *The cyclic order on $\partial\mathbb{D}$ is, as represented on Figure 5,*
$$\alpha_1, \omega_r, \alpha_2, \omega_1, \alpha_3, \omega_2, \ldots, \alpha_r, \omega_{r-1}, \alpha_1.$$

Then there exists a fixed point free simple closed curve $\gamma \subset \mathbb{D}$ such that $\operatorname{ind}(\widetilde{h}, \gamma) = 1$.

Remember that, if $\widetilde{p}$ is an isolated fixed point of $\widetilde{h}$, the Poincaré–Lefschetz index of $\widetilde{h}$ at $\widetilde{p}$ is defined as
$$\operatorname{ind}(\widetilde{h}, \widetilde{p}) = \operatorname{ind}(\widetilde{h}, \gamma),$$
where γ is a (small) simple closed curve surrounding $\widetilde{p}$ and no other fixed point. The index of $\widetilde{h}$ at $\widetilde{p}$ does not depend of the choice of γ.

In the case where $\widetilde{h}$ has only isolated fixed points, if $\operatorname{int}(\gamma)$ is the bounded connected component of γ^c and $\operatorname{Fix}(\operatorname{int}(\gamma)) = \{\widetilde{p} \in \operatorname{int}(\gamma) : \widetilde{h}(\widetilde{p}) = \widetilde{p}\}$ then, by properties of the Poincaré–Lefschetz index,
$$\operatorname{ind}(\widetilde{h}, \gamma) = \sum_{\widetilde{p} \in \operatorname{Fix}(\operatorname{int}(\gamma))} \operatorname{ind}(\widetilde{h}, \widetilde{p}).$$

So, if $\widetilde{h} : \mathbb{D} \to \mathbb{D}$ is a homeomorphism with only isolated fixed points satisfying the hypothesis of Handel's theorem, as $\operatorname{ind}(\widetilde{h}, \gamma) = 1$, there exists a fixed point $\widetilde{p}' \in \operatorname{int}(\gamma)$ with $\operatorname{ind}(\widetilde{h}, \widetilde{p}') > 0$.

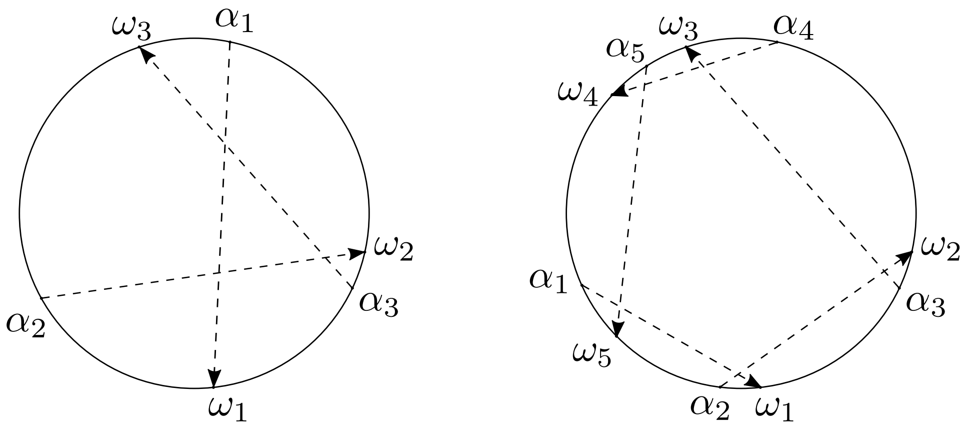

FIGURE 5. Cyclic order for Handel's fixed point theorem when $r = 3$ and $r = 5$.

2.4.2. *Existence of a hyperbolic $\widetilde{\phi}$-periodic point.* Remember that $\phi : S \to S$ is a homeomorphism which is pseudo-Anosov relative to P (see Lemma 9). As a map from S to itself, ϕ is a homeomorphism isotopic to the identity. The map $\widetilde{\phi} : \mathbb{D} \to \mathbb{D}$ is the natural lift of ϕ, the one which commutes with all deck transformations and extends as a homeomorphism of $\overline{\mathbb{D}}$, which is the identity on the 'boundary at infinity'.

In the next proposition, we prove that $\widetilde{\phi}$ has a hyperbolic periodic saddle point. When we say hyperbolic saddle in this context, we mean that the local dynamics at the point is obtained by gluing exactly four hyperbolic sectors, or, equivalently, the point is a regular point of the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$.

PROPOSITION 11. *The natural lift $\widetilde{\phi} : \mathbb{D} \to \mathbb{D}$ of the map ϕ from Lemma 9 has a hyperbolic periodic (saddle) point $\widetilde{p}$.*

Proof. In the first part of this proof, we want to find a well-oriented Jordan curve $\widetilde{\beta}$ contained in $\pi^{-1}(\mathscr{C})$. After finding such a curve, we consider interior$(\widetilde{\beta}) \cap \pi^{-1}(\mathscr{C})^c$. We will show that there is a connected component $\widetilde{U}$ of the previous open set, whose boundary is also a well-oriented Jordan curve. Finally, taking appropriate lifts of the periodic points associated with the geodesics in $\mathscr{C}$ which have extended lifts containing arcs in $\partial \widetilde{U}$, we get that the hypotheses of Handel's theorem are satisfied for them.

First, choose some unoriented geodesics $\alpha_1, \alpha_2, \ldots, \alpha_r$, for some $r \leq k$, such that, as a set, $\bigcup_{i=1}^r \alpha_i = \bigcup_{i=1}^k \gamma_i$, where $\mathscr{C} = \bigcup_{i=1}^k \gamma_i$ is the fully essential system of curves.

If, for every $1 \leq i \leq r$, there are two periodic points whose trajectories under the isotopy are closed curves freely homotopic to α_i, or concatenations of α_i, with both possible orientations, then any Jordan curve which is the boundary of a connected component of $\pi^{-1}(\mathscr{C})^c$ can be well oriented according to the orientations of the geodesics in $\mathscr{C}$. This is what we need.

So, assume that the above does not hold and choose some oriented extended lift $\widetilde{\gamma}_a$ of a geodesic γ_a in $\mathscr{C}$, for which there is no periodic point following it with the opposite orientation.

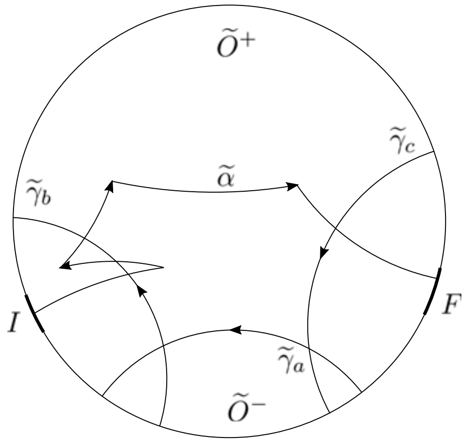

FIGURE 6. How to find the well-oriented Jordan curve $\widetilde{\beta}$.

Let g_a be a deck transformation which has $\widetilde{\gamma}_a$ as axis and translates points according to the orientation of $\widetilde{\gamma}_a$. Denote the two connected components of $\widetilde{\gamma}_a^c$ as $\widetilde{O}^+$ and $\widetilde{O}^-$. It is clearly possible to choose two oriented geodesics in $\mathscr{C}$, such that, for some extended lifts of them, one starts in $\partial\widetilde{O}^-\backslash\widetilde{\gamma}_a$ and ends in $\partial\widetilde{O}^+\backslash\widetilde{\gamma}_a$ and the other one goes in the opposite direction. Denote these lifts by $\widetilde{\gamma}_b$ and $\widetilde{\gamma}_c$. Iterating them under g_a, if necessary, we can suppose that they are disjoint and their relative position is as in Figure 6.

Still considering Figure 6, let $\widetilde{\alpha}$ be an oriented simple arc contained in $\pi^{-1}(\mathscr{C})$ which starts at some point in the open interval $I \subset \partial\mathbb{D}$ and ends at some point in the other open interval $F \subset \partial\mathbb{D}$. Remember that, as $\mathscr{C}$ is a fully essential system of curves, it is possible to choose such an arc $\widetilde{\alpha}$ formed by the concatenation of finitely many oriented subarcs of extended lifts of geodesics in $\pi^{-1}(\mathscr{C})$.

If $\widetilde{\alpha} \cap \widetilde{\gamma}_a$ has two or more points, then, clearly, $\widetilde{\alpha} \cup \widetilde{\gamma}_a$ contains a well-oriented Jordan curve $\widetilde{\beta}$. This follows from the choice of γ_a : there is no periodic point in S with a lift that follows $\widetilde{\gamma}_a$ in the opposite orientation.

If not, then it is still easy to find a well-oriented Jordan curve $\widetilde{\beta}$ contained in $\widetilde{\alpha} \cup \widetilde{\gamma}_a \cup \widetilde{\gamma}_b \cup \widetilde{\gamma}_c$.

Now, let us look at interior($\widetilde{\beta}$). If $\pi^{-1}(\mathscr{C})$ intersects interior($\widetilde{\beta}$), pick any extended lift $\widetilde{\eta} \subset \pi^{-1}(\mathscr{C})$ that intersects interior($\widetilde{\beta}$). The oriented arc $\widetilde{\eta}$ divides interior($\widetilde{\beta}$) into finitely many disks, at least one of them with a well-oriented boundary, still contained in $\pi^{-1}(\mathscr{C})$. Denote this boundary by $\widetilde{\beta}_1$, which, as we just said, is a well-oriented Jordan curve contained in $\pi^{-1}(\mathscr{C})$. If $\pi^{-1}(\mathscr{C})$ intersects interior($\widetilde{\beta}_1$), repeat the process and find a well-oriented Jordan curve $\widetilde{\beta}_2 \subset \pi^{-1}(\mathscr{C})$, and so on. As there are only finitely many extended lifts of geodesics in $\pi^{-1}(\mathscr{C})$ that intersect interior($\widetilde{\beta}$), after finitely many steps, we arrive at a well-oriented Jordan curve $\widetilde{\beta}_* \subset \pi^{-1}(\mathscr{C})$ such that $\pi^{-1}(\mathscr{C})$ does not intersect $\widetilde{U} = \text{interior}(\widetilde{\beta}_*)$.

Fix an oriented side $\widetilde{\rho}$ of $\partial\widetilde{U} = \widetilde{\beta}_*$, which is given by the intersection of a certain extended lift of a geodesic γ_{i_1} in $\mathscr{C}$ with $\widetilde{\beta}_*$. Denote this extended lift by $\widetilde{\gamma}_{i_1}$. Associated

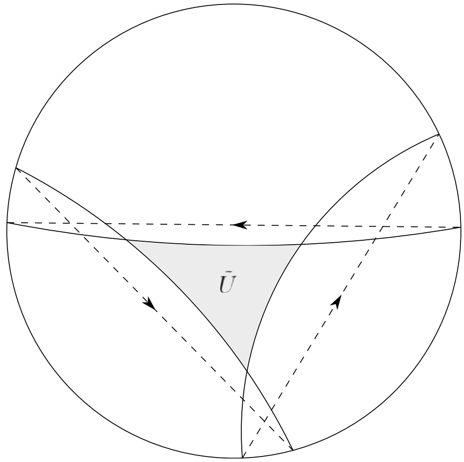

FIGURE 7. $\widetilde{U}$ and how some points move with respect to its boundary.

with $\widetilde{\gamma}_{i_1}$, we can find an appropriate lift $\widetilde{p}_{i_1}$ of p_{i_1} following $\widetilde{\gamma}_{i_1}$ with the correct orientation under iterates of $\widetilde{\phi}$.

If $\widetilde{\rho}$ and $\widetilde{\rho}'$ are two consecutive oriented sides of $\partial\widetilde{U}$, then the endpoints of the extended lift of the geodesic associated to $\widetilde{\rho}$ separate the endpoints of the extended lift of the geodesic associated to $\widetilde{\rho}'$. Putting all these observations together, we see that $\widetilde{\phi}$ satisfies the hypotheses of Handel's theorem (see Figure 7).

Since ϕ is pseudo-Anosov relative to a finite set P, for each period, it has only isolated periodic points, and the same holds for $\widetilde{\phi}$. This means, by Handel's theorem, that there exists a fixed point $\widetilde{p}_1$ of $\widetilde{\phi}$ such that

$$\mathrm{ind}(\widetilde{\phi}, \widetilde{p}_1) = \mathrm{ind}(\phi, \pi(\widetilde{p}_1)) > 0.$$

Observe that the same conclusion holds for $\widetilde{\phi}^m$, for any $m > 0$.

But, for some appropriate large $m_1 > 0$, the local dynamics at points in $\mathrm{Fix}(\phi)$ imply that

$$\mathrm{ind}(\phi^{m_1}, p) \leq 0 \quad \text{for all } p \in \mathrm{Fix}(\phi).$$

This happens because all points in $\mathrm{Fix}(\phi)$ with non-positive indexes are saddle-like (maybe with more than four sectors) with ϕ-invariant separatrices, and points with positive indexes are rotating saddles. So, for some $m_1 > 0$ sufficiently large, ϕ^{m_1} fixes the separatrices of all points in $\mathrm{Fix}(\phi)$, and thus they all have non-positive indexes with respect to ϕ^{m_1}. In particular, $\mathrm{ind}(\phi^{m_1}, \pi(\widetilde{p}_1)) < 0$.

Now let us look at ϕ^{m_1}. Again, as a consequence of Handel's theorem, there is a fixed point $\widetilde{p}_2$ of $\widetilde{\phi}^{m_1}$ with $\mathrm{ind}(\widetilde{\phi}^{m_1}, \widetilde{p}_2) = \mathrm{ind}(\phi^{m_1}, \pi(\widetilde{p}_2)) > 0$. In the same way as above for some sufficiently large $m_2 > 0$, the local dynamics at points in $\mathrm{Fix}(\phi^{m_1})$ imply that

$$\mathrm{ind}(\phi^{m_1 m_2}, p) \leq 0 \quad \text{for all } p \in \mathrm{Fix}(\phi^{m_1}),$$

and, in particular, $\mathrm{ind}(\phi^{m_1 m_2}, \pi(\widetilde{p}_2)) < 0$.

If we continue this process, we get a sequence of pairwise different points $\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3, \ldots$. In S, the points $\pi(\widetilde{p}_1), \pi(\widetilde{p}_2), \pi(\widetilde{p}_3), \ldots$ are also pairwise different.

So, at some j, the cardinality of $\{\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_j\}$ is larger than the number of singularities of the foliations $\mathscr{F}^u$, $\mathscr{F}^s$. This implies that, for some $\widetilde{\phi}$-periodic point $\widetilde{p}$, $\pi(\widetilde{p})$ does not coincide with a singularity of the foliations $\mathscr{F}^u$, $\mathscr{F}^s$. Hence $\widetilde{p}$ is a hyperbolic periodic saddle point for $\widetilde{\phi}$. □

2.5. *A first result towards the proof of Theorem 2 in the relative pseudo-Anosov case.* The stable and unstable foliations for ϕ lift to stable and unstable foliations for $\widetilde{\phi}$. If F^s_p is the stable leaf of $\mathscr{F}^s$ that contains a point $p \in S$, we will denote by $\widetilde{F}^s_{\widetilde{p}}$ the lift of F^s_p that contains a point $\widetilde{p} \in \pi^{-1}(p)$. We do the same for unstable leaves of $\mathscr{F}^u$. Now we will state some definitions and properties of pseudo-Anosov maps relative to finite invariant sets, which will be useful in the proof of the next lemma.

Let $p \in S$ be a fixed point of ϕ. As we already said, the dynamics of a sufficiently large iterate of ϕ in a neighborhood of p can be obtained by gluing finitely many invariant hyperbolic sectors together. In each sector, the dynamics are locally like the dynamics in the first quadrant of the map $(x, y) \mapsto (\lambda_1 x, \lambda_2 y)$, for some real numbers $0 < \lambda_2 < 1 < \lambda_1$.

We define the stable set of p as the set $W^s(p)$ of points z in S such that $\phi^n(z) \to p$ when $n \to \infty$, and we define the unstable set of p as the set $W^u(p)$ of points z in S such that $\phi^{-n}(z) \to p$ when $n \to \infty$. If p is a regular point of the foliations $\mathscr{F}^s$, $\mathscr{F}^u$, then $W^u(p)$ is the union of two branches; the same is true for $W^s(p)$. This is the situation in which we called the point a hyperbolic saddle point in the previous proposition. In the case where p is a singular point of the foliations, p is a k-prong singularity (for $k = 1$ or some $k \geq 3$), which implies that $W^u(p)$ is the union of k branches; the same is true for $W^s(p)$. In this singular case, each branch is actually a leaf of the proper foliation, which emanates from the singularity, while, in the regular case, each leaf gives two branches. In both the regular and the singular cases, the branches are either invariant or rotated around p under iterates of ϕ (and are thus ϕ^n-invariant for some $n > 0$).

In the case where $p' \in S$ is a ϕ-periodic point, if $n_{p'}$ is the least period of p', then it is a fixed point of $\phi^{n_{p'}}$, so we define the stable and unstable sets of p' accordingly, using $\phi^{n_{p'}}$ instead of ϕ.

LEMMA 12. *Let $\widetilde{\phi}$ be the natural lift of ϕ. Then there exists $\widetilde{p} \in \mathbb{D}$ a $\widetilde{\phi}$-hyperbolic periodic saddle point and deck transformations g_1, g_2 such that $g_1 \circ g_2 \neq g_2 \circ g_1$ and*

$$\widetilde{F}^{u+}_{\widetilde{p}} \pitchfork \widetilde{F}^{s+}_{g_i(\widetilde{p})}, \quad i \in \{1, 2\},$$

where $W^u(\widetilde{p}) = \widetilde{F}^{u+}_{\widetilde{p}} \cup \widetilde{F}^{u-}_{\widetilde{p}}$, $W^s(\widetilde{p}) = \widetilde{F}^{s+}_{\widetilde{p}} \cup \widetilde{F}^{s-}_{\widetilde{p}}$ and $\widetilde{F}^{u+}_{\widetilde{p}}$, $\widetilde{F}^{u-}_{\widetilde{p}}$, $\widetilde{F}^{s+}_{\widetilde{p}}$, $\widetilde{F}^{s-}_{\widetilde{p}}$ are the four branches at $\widetilde{p}$.

Proof. Let $\widetilde{p} \in \mathbb{D}$ be the $\widetilde{\phi}$-periodic point given in Proposition 11. So, $p = \pi(\widetilde{p})$ is a hyperbolic ϕ-periodic saddle point. Without loss of generality, considering an iterate of ϕ, if necessary, we will assume that each point in $K = \{p\} \cup P$ is fixed and, moreover, that each stable or unstable branch at a point in K is also invariant under ϕ.

The map ϕ is pseudo-Anosov relative to P. In particular, any stable leaf $F^s \in \mathscr{F}^s$ intersects all unstable leaves $F^u \in \mathscr{F}^u$ C^1-transversely and *vice-versa*. Let F^u_p be the unstable leaf at the point p (as p is regular, $F^u_p = W^u(p)$) and let $F^s_{*p'}$ be a stable leaf at some point $p' \in P = \{p_1, \ldots, p_k\}$. The point p' may be singular or regular. From what we said above, $F^u_p \pitchfork F^s_{*p'}$. So, there exists an unstable branch at p, denoted by F^{u+}_p, and an unstable branch at p', denoted by $F^{u'}_{*p'}$, such that F^{u+}_p accumulates on $F^{u'}_{*p'}$ and $F^{u'}_{*p'} \pitchfork W^s(p)$. Let F^{s+}_p be a stable branch at p such that $F^{u'}_{*p'} \pitchfork F^{s+}_p$. Lifting everything to the universal cover, having fixed some $\widetilde{p} \in \pi^{-1}(p)$, there exist deck transformations $g' \neq \mathrm{Id}$ and h such that

$$\widetilde{F}^{u+}_{\widetilde{p}} \pitchfork \widetilde{F}^{s+}_{(g')^n h(\widetilde{p})}, \tag{8}$$

for all sufficiently large $n > 0$. This follows from the fact that, having fixed some $\widetilde{p} \in \pi^{-1}(p)$, there exist a $\widetilde{p}' \in \pi^{-1}(p')$ and deck transformations $g' \neq \mathrm{Id}$ and h such that $\widetilde{\phi}(\widetilde{p}') = g'(\widetilde{p}')$, $\widetilde{F}^{u+}_{\widetilde{p}} \pitchfork \widetilde{F}^s_{*\widetilde{p}'}$ and $\widetilde{F}^{u'}_{*\widetilde{p}'} \pitchfork \widetilde{F}^{s+}_{h(\widetilde{p})}$.

Let $g_1 = (g')^n h$ for some $n > 0$ such that (8) holds. Now consider $\widetilde{\theta}$ to be a path in $\mathbb{D}$ constructed as follows: $\widetilde{\theta} = \widetilde{\theta}' * \widetilde{\theta}''$, where $\widetilde{\theta}'$ is a compact subarc of $\widetilde{F}^{u+}_{\widetilde{p}}$ starting at $\widetilde{p}$ and ending at a point in $\widetilde{F}^{u+}_{\widetilde{p}} \cap \widetilde{F}^{s+}_{g_1(\widetilde{p})}$, and $\widetilde{\theta}''$ is a compact subarc of $\widetilde{F}^{s+}_{g_1(\widetilde{p})}$ starting at the endpoint of $\widetilde{\theta}'$ and ending at $g_1(\widetilde{p})$.

Let ω_1 be the fixed point in $\partial\mathbb{D}$ of g_1 such that $\lim_{n\to\infty} g_1^n(\widetilde{q}) = \omega_1$ for all $\widetilde{q} \in \overline{\mathbb{D}}$, and let α_1 be the other fixed point.

Define

$$\Theta = \bigcup_{i\in\mathbb{Z}} g_1^i(\widetilde{\theta}).$$

By construction, Θ is a path connected subset of $\mathbb{D}$, joining α_1 to ω_1. Since $S \backslash \mathscr{C}$ is a union of open topological disks, there exists an oriented geodesic γ^* in $\mathscr{C}$ and $m \in \mathrm{Deck}(\pi)$ such that the projection of the oriented axis of m in S is γ^* and the fixed points of m in $\partial\mathbb{D}$ separate the endpoints ω_1 and α_1 of Θ. This follows from Propositions 6 and 7.

Now consider the fixed points ω_m and α_m of m in $\partial\mathbb{D}$ such that $\lim_{n\to\infty} m^n(\widetilde{q}) = \omega_m$ and $\lim_{n\to-\infty} m^n(\widetilde{q}) = \alpha_m$, for all $\widetilde{q} \in \overline{\mathbb{D}}$. We know that the axis of m is an oriented extended lift of γ^*, so ω_m and α_m are coherent with the orientation of γ^*. Let $n_0 > 0$ be a sufficiently large integer such that $m^{n_0}(\omega_1)$ and $m^{n_0}(\alpha_1)$ are close to ω_m and $\Theta \cap m^{n_0}(\Theta) = \emptyset$. This is possible because Θ accumulates on ω_m under positive iterates of m.

Then

$$\Theta = \bigcup_{i\in\mathbb{Z}} g_1^i(\widetilde{\theta}) \Rightarrow m^{n_0}(\Theta) = \bigcup_{i\in\mathbb{Z}} m^{n_0} g_1^i(\widetilde{\theta}).$$

As $\widetilde{\phi}$ commutes with all deck transformations, $\widetilde{\theta}' \subset \widetilde{F}^{u+}_{\widetilde{p}}$ and $\widetilde{\phi}(\widetilde{F}^{u+}_{\widetilde{p}}) = \widetilde{F}^{u+}_{\widetilde{p}}$, we get that, for all $n > 0$ and $t \in \mathrm{Deck}(\pi)$, $t(\widetilde{\theta}') \subset \widetilde{\phi}^n(t(\widetilde{\theta}'))$. Similarly, since $\widetilde{\theta}'' \subset \widetilde{F}^{s+}_{g_1(\widetilde{p})}$, $\widetilde{\phi}^n(t(\widetilde{\theta}'')) \subset t(\widetilde{\theta}'')$, for all $n > 0$ and $t \in \mathrm{Deck}(\pi)$.

The hypotheses on $\mathscr{C}$ imply that there is a point $\widetilde{p}_m \in \pi^{-1}(P)$ such that $\widetilde{\phi}(\widetilde{p}_m) = m(\widetilde{p}_m)$ and $\widetilde{p}_m$ is in the connected component of $\mathbb{D}\backslash\Theta$ that contains α_m in its boundary. As $m^{n_0}(\Theta)$ is in the other connected component of $\mathbb{D}\backslash\Theta$, $\lim_{n\to\infty} \widetilde{\phi}^n(\widetilde{p}_m) = \omega_m$ and $\widetilde{\phi}|_{\partial\mathbb{D}} = \mathrm{Id}$, we get that, for a sufficiently large $n' > 0$, there must exists two integers i', i'' such that

$$\widetilde{\phi}^{n'}(g_1^{i'}(\widetilde{\theta}')) \pitchfork m^{n_0} g_1^{i''}(\widetilde{\theta}'').$$

In particular, $\widetilde{F}^{u+}_{g_1^{i'}(\widetilde{p})} \pitchfork \widetilde{F}^{s+}_{m^{n_0} g_1^{i''}(\widetilde{p})}$, and so, $\widetilde{F}^{u+}_{\widetilde{p}} \pitchfork \widetilde{F}^{s+}_{(g_1^{-i'} m^{n_0} g_1^{i''})(\widetilde{p})}$ (see Figure 8).

Finally, let $g_2 = g_1^{-i'} m^{n_0} g_1^{i''}$. We will show that g_1 and g_2 do not commute. If $g_1 \circ g_2 = g_2 \circ g_1$, then there exists $l \in \mathrm{Deck}(\pi)$ and integers k_1, k_2 such that $g_1 = l^{k_1}$ and $g_2 = l^{k_2}$.

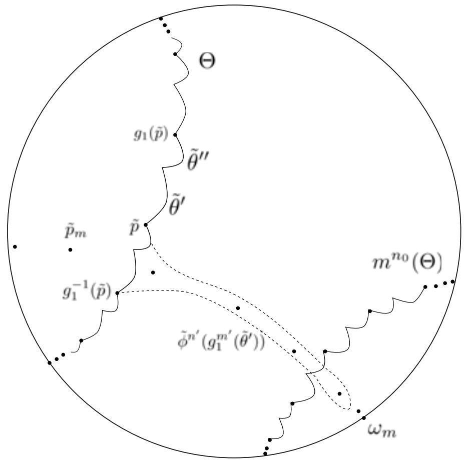

FIGURE 8. How to obtain g_2.

Thus

$$g_1^{-i'} m^{n_0} g_1^{i''} = l^{k_2} \Rightarrow l^{-i'k_1} m^{n_0} l^{i''k_1} = l^{k_2} \Rightarrow m^{n_0} = l^{k_2+k_1(i'-i'')}.$$

Since m^{n_0} and g_1 are iterates of the same deck transformation, the geodesics associated to the axes of m and g_1 are equal. But this is in contradiction with our choice of m. So, g_1 and g_2 do not commute. □

2.6. *Proof of Theorem 2 in a special case.* In this subsection, we prove Theorem 2 in case of relative pseudo-Anosov maps.

Remark 2.6. As $\phi : S \to S$ is pseudo-Anosov relative to a finite invariant set, if, for some leaves F^u of $\mathscr{F}^u$ and F^s of $\mathscr{F}^s$, there are connected components $\widetilde{F}^u$ of $\pi^{-1}(F^u)$ and $\widetilde{F}^s$ of $\pi^{-1}(F^s)$ which have non-empty intersection (not at a lift of a singularity of the foliations), then they intersect in a C^1-transverse way. In the proof of the next lemma, we will not make use of this fact because, when proving Theorem 2, at some point we say that the proof continues as the proof of the next lemma. So, in the proof of Lemma 13, although intersections between stable and unstable leaves, either in S or in $\mathbb{D}$, are always C^1-transverse, we will not use this fact.

Moreover, as we said in the introduction, the main feature of topologically transverse intersections is the fact that a C^0-version of the so called λ-lemma (see [**26**]) holds: if M is a surface, $f : M \to M$ is a C^1 diffeomorphism, $p, q \in M$ are f-periodic saddle points and $W^u(p)$ has a topologically transverse intersection with $W^s(q)$, then $W^u(p)$ C^0-accumulates on $W^u(q)$, and, in particular, $\overline{W^u(p)} \supset \overline{W^u(q)}$. So if $p_1, p_2, p_3 \in M$ are hyperbolic f-periodic saddle points, $W^u(p_1)$ has a topologically transverse intersection with $W^s(p_2)$ and $W^u(p_2)$ has a topologically transverse intersection with $W^s(p_3)$, then $W^u(p_1)$ has a topologically transverse intersection with $W^s(p_3)$.

LEMMA 13. (Theorem 2 in case of relative pseudo-Anosov maps) *Let $\widetilde{\phi}$ be the natural lift of the map ϕ. Then there exists a contractible hyperbolic ϕ-periodic point $p \in S$, such* *that, for any $\widetilde{p} \in \pi^{-1}(p)$ and any given $g \in \mathrm{Deck}(\pi)$,*

$$W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p})).$$

Proof. Let $\widetilde{p}$ be the hyperbolic $\widetilde{\phi}$-periodic point from Lemma 12 and let $p = \pi(\widetilde{p})$. This lemma implies the existence of g_1 and g_2 in $\mathrm{Deck}(\pi)$ and also the existence of an unstable branch λ_u of $W^u(p)$ and a stable branch β_s of $W^s(p)$ such that, if $\widetilde{\lambda}_u$ is the connected component of $\pi^{-1}(\lambda_u)$ contained in $W^u(\widetilde{p})$ and $\widetilde{\beta}_s$ is the connected component of $\pi^{-1}(\beta_s)$ contained in $W^s(\widetilde{p})$, then

$$\widetilde{\lambda}_u \pitchfork g_i(\widetilde{\beta}_s), \quad i \in \{1, 2\}.$$

Without loss of generality, as we did in Lemma 12, considering an iterate of $\widetilde{\phi}$ if, necessary, we will assume that $\widetilde{\phi}(\widetilde{p}) = \widetilde{p}$, $\widetilde{\phi}(\widetilde{\lambda}_u) = \widetilde{\lambda}_u$ and $\widetilde{\phi}(\widetilde{\beta}_s) = \widetilde{\beta}_s$.

Since $\widetilde{\phi}$ is the natural lift of ϕ, every point of the form $h(\widetilde{p})$ with $h \in \mathrm{Deck}(\pi)$ is fixed under $\widetilde{\phi}$. Moreover, if we consider the stable set of the point $h(\widetilde{p})$ with respect to $\widetilde{\phi}$, then

$$W^s(h(\widetilde{p})) = h(W^s(\widetilde{p})),$$

and the same is true for the unstable set of $\widetilde{p}$.

Consider the point $p \in S$. Choose $\epsilon > 0$ small enough so that $B_\epsilon(\widetilde{p}) \cap \pi^{-1}(p) = \widetilde{p}$, where $B_\epsilon(\widetilde{p}) = \{\widetilde{q} \in \mathbb{D} | d_{\mathbb{D}}(\widetilde{p}, \widetilde{q}) < \epsilon\}$. Observe that, since every point on the fiber of p is of the form $h(\widetilde{p})$ for some $h \in \mathrm{Deck}(\pi)$, and h is an isometry, $B_\epsilon(h(\widetilde{p})) \cap \pi^{-1}(p) = h(\widetilde{p})$. □

PROPOSITION 14. *There exists a path connected set θ in $\mathbb{D}$, containing $\widetilde{p}$, that geometrically is the concatenation of two curves in $\overline{\mathbb{D}}$, one joining $\widetilde{p}$ to ω_{g_1} and the other joining $\widetilde{p}$ to ω_{g_2}, where ω_{g_1} and ω_{g_2} are the (different) attractive fixed points at infinity of g_1 and g_2, respectively. Moreover, if θ avoids some curve ϕ and, for some $n > 0$, if $\widetilde{f}^n(\theta)$ has a topologically transverse intersection with ϕ, then $W^u(\widetilde{p})$ has a topologically transverse intersection with ϕ.*

Proof. From the fact that $\widetilde{\lambda}_u \pitchfork g_1(\widetilde{\beta}_s)$, we can construct a path η_1 in $\mathbb{D}$ joining $\widetilde{p}$ to $g_1(\widetilde{p})$ exactly as in the previous lemma: η_1 starts at $\widetilde{p}$, consists of a compact connected piece of $\widetilde{\lambda}_u$ until it reaches $g_1(\widetilde{\beta}_s)$ and then it continues as a compact connected piece of $g_1(\widetilde{\beta}_s)$ until it reaches $g_1(\widetilde{p})$. It is clear that we can choose the piece that belongs to $g_1(\widetilde{\beta}_s)$ to be totally contained in $B_\epsilon(g_1(\widetilde{p}))$. Analogously, we construct a path η_2 in $\mathbb{D}$ joining $\widetilde{p}$ to $g_2(\widetilde{p})$. Let $\theta \subset \mathbb{D}$ be the path connected set obtained as (see Figure 9)

$$\theta = \left(\bigcup_{i \geq 0} g_1^i(\eta_1)\right) \cup \left(\bigcup_{j \geq 0} g_2^j(\eta_2)\right). \tag{9}$$

Clearly, θ is the the concatenation of two curves in $\overline{\mathbb{D}}$, one joining $\widetilde{p}$ to ω_{g_1} and the other joining $\widetilde{p}$ to ω_{g_2}. The fact that g_1 and g_2 do not commute implies that the fixed points at infinity of these deck transformations are all different, so, in particular, $\omega_{g_1} \neq \omega_{g_2}$. The last part of the proposition follows from the C^0-version of the λ-lemma that holds for topologically transverse intersections. □

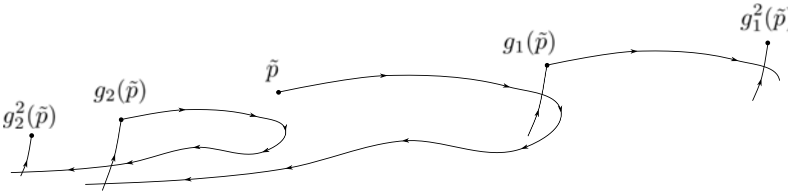

FIGURE 9. The construction of the path connected set θ.

In order to prove the Lemma 13, we show that, for every $g \in \mathrm{Deck}(\pi)$,

$$\widetilde{\lambda}_u \pitchfork g(\widetilde{\beta}_s) \Rightarrow W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p})). \tag{10}$$

Fix $g \in \mathrm{Deck}(\pi)$ with $g \neq \mathrm{Id}$. The case where $g = \mathrm{Id}$ will be considered at the end. As $\widetilde{\lambda}_u \pitchfork g_1(\widetilde{\beta}_s)$, we get that

$$g_1^{-1}(\widetilde{\lambda}_u) \pitchfork \widetilde{\beta}_s \text{ and so } gg_1^{-1}(\widetilde{\lambda}_u) \pitchfork g(\widetilde{\beta}_s)$$
$$\text{can be rewritten as}$$
$$gg_1^{-1}g^{-1}(g(\widetilde{\lambda}_u)) \pitchfork g(\widetilde{\beta}_s).$$

Notice that an analogous statement holds for g_2. Using this, let us construct a path connected set θ' containing $g(\widetilde{p})$ in a similar way to θ.

PROPOSITION 15. *There exists a path connected set θ' in $\mathbb{D}$, containing $g(\widetilde{p})$, such that, geometrically, θ' is the concatenation of two curves in $\overline{\mathbb{D}}$, one joining $g(\widetilde{p})$ to $g(\alpha_{g_1})$ and the other joining $g(\widetilde{p})$ to $g(\alpha_{g_2})$, where α_{g_1} and α_{g_2} are the different repulsive fixed points at infinity of g_1 and g_2, respectively. Moreover, if θ' avoids some curve ϕ and if, for some $n > 0$, $\widetilde{f}^{-n}(\theta')$ has a topologically transverse intersection with ϕ, then $W^s(g(\widetilde{p}))$ has a topologically transverse intersection with ϕ.*

Proof. An important simple observation here is the fact that, for a fixed point of $\widetilde{\phi}$, its stable set with respect to $\widetilde{\phi}$ coincides with its unstable set with respect to $\widetilde{\phi}^{-1}$. This duality allows us to construct the set θ' in the same way as θ, but using the point $g(\widetilde{p})$, the deck transformations $gg_1^{-1}g^{-1}$, $gg_2^{-1}g^{-1}$ and the map $\widetilde{\phi}^{-1}$. Hence we construct a path η_1' joining $g(\widetilde{p})$ to $gg_1^{-1}g(g(\widetilde{p}))$ such that η_1' starts at $g(\widetilde{p})$, consists of a compact connected piece of $g(\widetilde{\beta}_s)$ until it reaches $gg_1^{-1}g^{-1}(g(\widetilde{\lambda}_u))$ and then it continues as a compact connected piece of $gg_1^{-1}g^{-1}(g(\widetilde{\lambda}_u)) \cap B_\epsilon(gg_1^{-1}g^{-1}(g(\widetilde{p})))$ until it reaches $gg_1^{-1}g^{-1}(g(\widetilde{p}))$. Constructing η_2' analogously, we define

$$\theta' = \left(\bigcup_{i \geq 0} gg_1^{-i}g^{-1}(\eta_1')\right) \cup \left(\bigcup_{j \geq 0} gg_2^{-j}g^{-1}(\eta_2')\right).$$

Similarly to θ, the curve θ' is given by the concatenation of two curves in $\overline{\mathbb{D}}$, one joining $g(\widetilde{p})$ to $g(\alpha_{g_1})$ and the other joining $g(\widetilde{p})$ to $g(\alpha_{g_2})$, where α_{g_1} and α_{g_2} are the repulsive fixed points at infinity of g_1 and g_2, respectively. As in the previous proposition, the last part follows from the C^0-version of the λ-lemma. $\square$

The sets θ and θ' share similar properties.

Properties 2.6 of θ and θ'.

- For $i \in \{1, 2\}$ and for all $m > 0$, all points of the form $g_i^m(\widetilde{p})$ belong to θ and $\widetilde{\lambda}_u \pitchfork g_i^m(\widetilde{\beta}_s)$. Remember that $\widetilde{\lambda}_u$ is a branch of $W^u(\widetilde{p})$ and $g_i^m(\widetilde{\beta}_s)$ is a branch of $W^s(g_i^m(\widetilde{p}))$.
- For $i \in \{1, 2\}$ and for all $m > 0$, all points of the form $gg_i^{-m}g^{-1}(g(\widetilde{p}))$ belong to θ' and $gg_i^{-m}g^{-1}(g(\widetilde{\lambda}_u)) \pitchfork g(\widetilde{\beta}_s)$. And in this case, $gg_i^{-m}g^{-1}(g(\widetilde{\lambda}_u))$ is a branch of $W^u(gg_i^{-m}g^{-1}(g(\widetilde{p})))$ and $g(\widetilde{\beta}_s)$ is a branch of $W^s(g(\widetilde{p}))$.

If θ and θ' have a topologically transverse intersection, then the lemma is proved.

Indeed, if there is such an intersection, then at least one of the following four possibilities holds.

- There exists $j' \in \{1, 2\}$ and $m' > 0$ such that $g_{j'}^{m'}(\widetilde{p}) = g(\widetilde{p})$.
- There exists $j' \in \{1, 2\}$ and $m' > 0$ such that $gg_{j'}^{-m'}g^{-1}(g(\widetilde{p})) = \widetilde{p}$.
- There exists $j', j'' \in \{1, 2\}$ and $m', m'' > 0$ such that $g_{j'}^{m'}(\widetilde{p}) = gg_{j''}^{-m''}g^{-1}(g(\widetilde{p}))$.

In the three possibilities above, using Properties 2.6, we get that (10) holds. The last possibility is the following.

- There exists $j', j'' \in \{1, 2\}$ and $m', m'' \geq 0$ such that some compact piece of $\theta \cap W^u(g_{j'}^{m'}(\widetilde{p}))$ has a topologically transverse intersection with some compact piece of $\theta' \cap W^s(gg_{j''}^{-m''}g^{-1}(g(\widetilde{p})))$. This happens because, for $i \in \{1, 2\}$ and all $n \geq 0$,
$$\begin{gathered} \theta \cap W^s(g_i^n(\widetilde{p})) \subset B_\epsilon(g_i^n(\widetilde{p})) \quad \text{and} \\ \theta' \cap W^u(gg_i^{-n}g^{-1}(g(\widetilde{p}))) \subset B_\epsilon(gg_i^{-n}g^{-1}(g(\widetilde{p}))) \end{gathered}$$
and all of these balls are disjoint. So, by the $C^0\lambda$-lemma mentioned in Remark 2.6, the proof is complete.

Hence let us suppose that θ and θ' do not have topologically transverse intersections. Our goal is to show that, in this case, using the fully essential system of curves $\mathscr{C}$ and the periodic points associated with the geodesics, we can force a topologically transverse intersection between θ' and a path connected set $\theta_0 \in \mathbb{D}$ that has the same properties and is obtained from θ.

PROPOSITION 16. *$W^u(\widetilde{p})$ has a topologically transverse intersection with θ'.*

Proof. As we said above, we are assuming that θ and θ' do not have topologically transverse intersections, otherwise the proposition is proved. The set $\mathbb{D}\backslash\theta$ has two unbounded connected components U'_θ and U''_θ; the closure of one of them contains θ'. We will assume that $\theta' \subset \text{closure}(U''_\theta)$. The boundary at infinity of U'_θ is equal to a segment of $\partial\mathbb{D}$ delimited by ω_{g_1} and ω_{g_2} that will be denoted λ'_θ. Similarly, the boundary at infinity of U''_θ is equal to a segment of $\partial\mathbb{D}$ delimited by ω_{g_1} and ω_{g_2} that will be denoted byλ''_θ. In the same way, $\mathbb{D}\backslash\theta'$ has two unbounded connected components $U'_{\theta'}$ and $U''_{\theta'}$. We will assume that $\theta \subset \text{closure}(U'_{\theta'})$ and call $\lambda'_{\theta'}, \lambda''_{\theta'}$ the segments of $\partial\mathbb{D}$ delimited by $g(\alpha_{g_1})$ and $g(\alpha_{g_2})$ that are equal to the boundary at infinity of $U'_{\theta'}$ and $U''_{\theta'}$, respectively. Then $\lambda'_\theta \subseteq \lambda'_{\theta'}$ and $\lambda''_{\theta'} \subseteq \lambda''_\theta$ (see Figure 10).

Let $C(\lambda'_\theta, \lambda''_{\theta'})$ be the set of oriented simple arcs in $\pi^{-1}(\mathscr{C})$ joining a point in the interior of λ'_θ to a point in the interior $\lambda''_{\theta'}$ and formed by finitely many oriented subarcs of extended

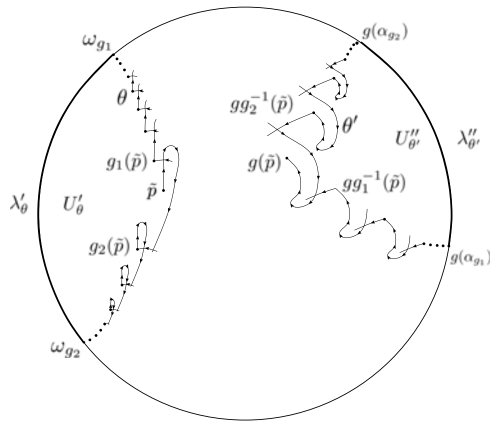

FIGURE 10. The case when θ and θ' do not have topologically transverse intersections.

lifts of geodesics in $\mathscr{C}$. The definition of a fully essential system of curves $\mathscr{C}$ implies that $C(\lambda'_\theta, \lambda''_{\theta'}) \neq \emptyset$. For every β in $C(\lambda'_\theta, \lambda''_{\theta'})$, we can write

$$\beta = \beta_1 * \beta_2 * \cdots * \beta_l,$$

where each β_i, $i \in \{1, 2, \ldots, l\}$ is an oriented subarc of an extended lift of one geodesic in $\mathscr{C}$. We will prove that $W^u(\widetilde{p}) \pitchfork \theta'$ and so $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$ by induction on $k = \min\{l \in \mathbb{N} | \beta \in C(\lambda'_\theta, \lambda''_{\theta'}), \beta = \beta_1 * \beta_2 * \cdots * \beta_l\}$.

The existence of a fully essential system of curves $\mathscr{C}$ (see Definition 1.3) implies that, for any geodesic $\gamma_i \in \mathscr{C}$ and any $\widetilde{\gamma_i}$ extended lift of that geodesic, there exists a point $\widetilde{p_i}$ in $\mathbb{D}$ and $h_i \in \mathrm{Deck}(\pi)$ such that $\widetilde{\gamma_i}$ is an oriented curve from α_{h_i} to ω_{h_i}, respectively, the repulsive and the attractive fixed points of h_i on $\partial\mathbb{D}$, $h_i(\widetilde{\gamma_i}) = \widetilde{\gamma_i}$, $\widetilde{\phi}(\widetilde{p_i}) = h_i(\widetilde{p_i})$ and

$$\lim_{n\to\infty} \widetilde{\phi}^n(\widetilde{p_i}) = \omega_i \quad \text{and} \quad \lim_{n\to\infty} \widetilde{\phi}^{-n}(\widetilde{p_i}) = \alpha_i.$$

First step. $k = 1$. In this case, there exist $\beta_1 \in C(\lambda'_\theta, \lambda''_{\theta'})$ and $\widetilde{\gamma_1}$ an extended lift of a geodesic in $\mathscr{C}$ with $\beta_1 = \widetilde{\gamma_1}$. It is clear that the orientation on $\widetilde{\gamma_1}$ is from λ'_θ to $\lambda''_{\theta'}$. Associated to the extended lift $\widetilde{\gamma_1}$, there is a point $\widetilde{p_1}$ such that, for some $h_1 \in \mathrm{Deck}(\pi)$ with $h_1(\widetilde{\gamma_1}) = \widetilde{\gamma_1}$,

$$\widetilde{\phi}(\widetilde{p_1}) = h_1(\widetilde{p_1}),$$

where

$$\lim_{n\to\infty} \widetilde{\phi}^{-n}(\widetilde{p_1}) = \lim_{n\to\infty} h_1^{-n}(\widetilde{p_1}) = \alpha_{h_1} \quad \text{and} \quad \lim_{n\to\infty} \widetilde{\phi}^{n}(\widetilde{p_1}) = \lim_{n\to\infty} h_1^{n}(\widetilde{p_1}) = \omega_{h_1}.$$

Note that α_{h_1} is the point at infinity of $\widetilde{\gamma_1}$ in interior(λ'_θ) and ω_{h_1} is the point at infinity of $\widetilde{\gamma_1}$ in interior$(\lambda''_{\theta'})$. The point $\widetilde{p_1}$ can be chosen as close as we want (in the Euclidean distance) to the point α_{h_1}, something that forces $\widetilde{p_1}$ to belong to U'_θ. Since $\widetilde{\phi}|_{\partial\mathbb{D}} \equiv \mathrm{Id}$, for all $n > 0$, $\widetilde{\phi}^n(\theta)$ is a path connected set in $\overline{\mathbb{D}}$ joining the points $\omega_{g_1}, \omega_{g_2} \in \partial\mathbb{D}$. As $\widetilde{p_1} \in U'_\theta$ and $\widetilde{\phi}$ preserves orientation, we get that, for sufficiently large $n > 0$,

$$\widetilde{\phi}^n(U'_\theta) \cap U''_{\theta'} \neq \emptyset,$$

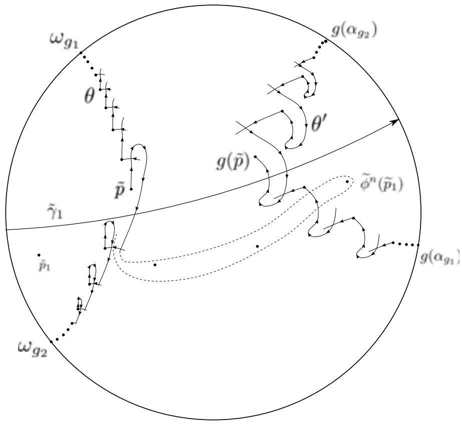

FIGURE 11. Intersection between $\widetilde{\phi}^n(\theta)$ and θ'.

which implies that

$$\widetilde{\phi}^n(\theta) \pitchfork \theta'.$$

(see Figure 11). So, by Propositions 14 and 15,

$$W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p})).$$

Second step. $k = 2$ (main idea behind the general case). If $k = 2$, there exist $\beta \in C(\lambda'_\theta, \lambda''_{\theta'})$, $\beta = \beta_1 * \beta_2$ and $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ extended lifts of geodesics in $\mathscr{C}$ such that $\beta_1 \subset \widetilde{\gamma}_1$ and $\beta_2 \subset \widetilde{\gamma}_2$. Moreover, in the same way as in the previous case, for $i \in \{1, 2\}$ there exists a point $\widetilde{p}_i \in \mathbb{D}$ and $h_i \in \mathrm{Deck}(\pi)$ that leave $\widetilde{\gamma}_i$ invariant such that

$$\widetilde{\phi}(\widetilde{p}_i) = h_i(\widetilde{p}_i).$$

The points $\alpha_{h_1}, \omega_{h_1}$ separate the points $\alpha_{h_2}, \omega_{h_2}$ at $\partial\mathbb{D}$, α_{h_1} is in the interior of λ'_θ and ω_{h_2} is in the interior of $\lambda''_{\theta'}$ (see Figure 12). Let us consider a sufficiently large $m_1 > 0$ in a way that $h_1^{m_1}(\theta)$ is close (in the Euclidean distance) to the point ω_{h_1} and $\theta \cap h_1^{m_1}(\theta) = \emptyset$. In particular, the points $h_1^{m_1}(\omega_{g_1})$ and $h_1^{m_1}(\omega_{g_2})$ are very close to ω_{h_1}.

Exactly as when $k = 1$, $\widetilde{p}_1$ can be chosen sufficiently close to α_{h_1} in the Euclidean distance, in a way that $\widetilde{p}_1 \in U'_\theta$, and then for a sufficiently large $n > 0$, $\widetilde{\phi}^n(\widetilde{p}_1)$ is very close to ω_{h_1}, so that it implies that $\widetilde{\phi}^n(\theta) \pitchfork h_1^{m_1}(\theta)$.

But then, as in the previous case, there exists $i', i'' \in \{1, 2\}$ and $j', j'' > 0$ such that

$$\begin{cases} W^u(g_{i'}^{j'}(\widetilde{p})) \pitchfork W^s(h_1^{m_1} g_{i''}^{j''}(\widetilde{p})), \\ \text{which implies, again by the } C^0\lambda\text{-lemma, that} \\ W^u(\widetilde{p}) \pitchfork W^s(h_1^{m_1} g_{i''}^{j''}(\widetilde{p})). \end{cases} \tag{11}$$

Note that, since $k = 2$, α_{h_2} is not in the interior of λ'_θ, otherwise there would be a path in $C(\lambda'_\theta, \lambda''_{\theta'})$ with size one, which contradicts the fact that $k = 2$. So, we can always choose $i_0 \in \{1, 2\}$ and construct a new path connected set θ_{h_1} using η_{i_0} of expression (9) and an analogous construction obtained from expression (11). Before going into details, we emphasize that the choice of i_0 is very important because α_{h_2} is not in the interior of

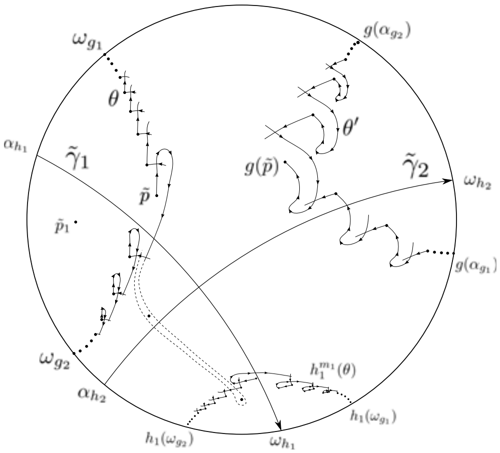

FIGURE 12. Intersection between $\widetilde{\phi}^n(\theta)$ and $h_1^{m_1}(\theta)$.

λ'_θ, but it could be one of its endpoints. So, if α_{h_2} is one of the endpoints of λ'_θ, g_{i_0} must be chosen to be associated with the other endpoint of λ'_θ.

In order to construct θ_{h_1}, first consider the set η_{i_0} of expression (9) associated to g_{i_0}, chosen as before. Since expression (11) holds, there exists a path η joining $\widetilde{p}$ to $h_1^{m_1} g_{i''}^{j''} h_1^{-m_1}(h_1^{m_1}(\widetilde{p}))$ as follows: η starts at $\widetilde{p}$, consists of a compact connected piece of $\widetilde{\lambda}_u$ until it reaches $h_1^{m_1} g_{i''}^{j''} h_1^{-m_1}(h_1^{m_1}(\widetilde{\beta}_s))$ and then it continues as a compact connected piece of $h_1^{m_1} g_{i''}^{j''} h_1^{-m_1}(h_1^{m_1}(\widetilde{\beta}_s))$ until it reaches $h_1^{m_1} g_{i''}^{j''} h_1^{-m_1}(h_1^{m_1}(\widetilde{p}))$. As before, we choose the arc in η contained in $h_1^{m_1} g_{i''}^{j''} h_1^{-m_1}(h_1^{m_1}(\widetilde{\beta}_s))$ to be very small; it is contained in $B_\epsilon(h_1^{m_1} g_{i''}^{j''} h_1^{-m_1}(h_1^{m_1}(\widetilde{p})))$. Note that $i'' \in \{1, 2\}$ was defined before expression (11). Finally, pick the $\eta_{i''}$ as in expression (9).

Then, define

$$\theta_{h_1} = \left(\bigcup_{i \geq 0} g_{i_0}^i(\eta_{i_0}) \right) \cup \eta \cup \left(\bigcup_{j \geq j''} h_1^{m_1} g_{i''}^j h_1^{-m_1}(h_1^{m_1}(\eta_{i''})) \right). \tag{12}$$

The new path connected set θ_{h_1} has similar properties to θ, and, as was explained for θ and θ', it can be understood as the concatenation of two curves in $\mathbb{D}$, one joining $\widetilde{p}$ to $\omega_{g_{i_0}}$ and the other joining $\widetilde{p}$ to $h_1^{m_1}(\omega_{g_{i''}})$. If $\theta_{h_1} \pitchfork \theta'$, then $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$. And in the case where there is no such topologically transverse intersection, as when α_{h_2} is in the interior of $\lambda'_{\theta_{h_1}}$ and ω_{h_2} is in the interior of $\lambda''_{\theta'}$, we are reduced to the previous case.

Hence, arguing exactly as when $k = 1$, we conclude that $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$.

Third step. (The induction). Suppose the result is true when

$$\min\{l \in \mathbb{N} | \beta \in C(\lambda'_\theta, \lambda''_{\theta'}), \beta = \beta_1 * \beta_2 * \cdots * \beta_l\} = 1, 2, \ldots, k-1$$

and let us prove that it holds for k. We can assume that $k \geq 3$. Fix $\beta \in C(\lambda'_\theta, \lambda''_{\theta'})$ with $\beta = \beta_1 * \beta_2 * \beta_3 * \cdots * \beta_k$. Let $\widetilde{\gamma}_i$, $0 \leq i \leq k$, be the extended lifts of the geodesics in $\mathscr{C}$ such that $\beta_i \subset \widetilde{\gamma}_i$. For each $i \in \{1, \ldots, k\}$, as the $\beta_{i'}$ are oriented, there exists a point

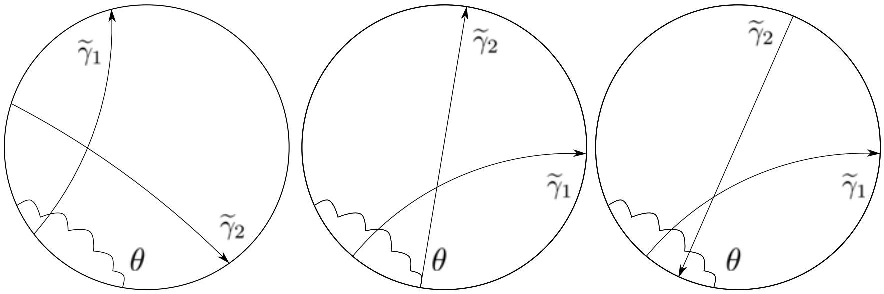

FIGURE 13. Position of the first two extended lifts when $k \geq 3$.

$\widetilde{p}_i \in \mathbb{D}$ and $h_i \in \mathrm{Deck}(\pi)$ that leaves $\widetilde{\gamma}_i$ invariant and moves points according to the orientation of $\widetilde{\gamma}_i$ such that

$$\widetilde{\phi}(\widetilde{p}_i) = h_i(\widetilde{p}_i).$$

We claim that the following facts are true: since $k \geq 3$ and there exists no path in $C(\lambda'_\theta, \lambda''_{\theta'})$ using less than k extended lifts of geodesics, if the first two extended lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ associated to β_1 and β_2 are considered, then their relative positions with respect to θ can only be as in one of the three possibilities of Figure 13.

The reason for this is the following: $\widetilde{\gamma}_2$ does not start inside θ, otherwise there would be a shorter path. So, it may start and end outside θ (this is the first case in Figure 13). Or it starts at one endpoint of θ and ends outside θ. It cannot end inside θ because, in this case, some $\widetilde{\gamma}_i$ for $i > 2$ would have to start inside the region bounded by $\widetilde{\gamma}_2$ and inside θ and its endpoint would have to be outside θ. And this also gives a shorter path, which is a contradiction. The third possibility is when $\widetilde{\gamma}_2$ starts outside θ and ends inside it or at one of its endpoints. These are the three cases in the figure.

In this way, choose an integer $m_1 > 0$ sufficiently large so that in the segment $[h_1^{m_1}(\omega_{g_1}), h_1^{m_1}(\omega_{g_2})]$ of $\partial\mathbb{D}$ delimited by $h_1^{m_1}(\omega_{g_1})$ and $h_1^{m_1}(\omega_{g_2})$, and containing the point ω_{h_1}, there are no other α' and ω'. Since there is a finite number of α' and ω', this is always possible.

Now proceed as when $k = 2$ and construct the path connected set θ_{h_1} in the same way as in (12). Some care must be taken with the choice of the endpoints at infinity of θ_{h_1}.

- If $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are as in the first case of Figure 13, we can choose any of the endpoints of θ as one endpoint of θ_{h_1}.
- If $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are as in the second case of Figure 13, we choose as one of the endpoints of θ_{h_1} the endpoint of θ that is contained in the segment $[\alpha_{h_1}, \omega_{h_1}]$ of $\partial\mathbb{D}$ which contains ω_{h_2}.
- If $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are as in the third case of Figure 13, we choose as one of the endpoints of θ_{h_1} the endpoint of θ that is contained in the segment $[\alpha_{h_1}, \omega_{h_1}]$ of $\partial\mathbb{D}$ which contains α_{h_2}.

Constructing θ_{h_1} in this way, it follows that $\lambda'_{\theta_{h_1}} \subseteq \lambda'_{\theta'}$ and α_{h_2} is in the interior of $\lambda'_{\theta_{h_1}}$. If $\theta_{h_1} \pitchfork \theta'$, then $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$. In case where there is no such intersection, let β'_2

be the subarc of $\widetilde{\gamma}_2$ joining α_{h_2} to the intersection point of $\widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$. Create $\beta_{\text{mod}} = \beta_2' * \beta_3 * \cdots * \beta_k$, which is an oriented arc joining a point in the interior of $\lambda'_{\theta_{h_1}}$ to a point in the interior of $\lambda''_{\theta'}$ formed by $k-1$ subarcs of extended lifts of geodesics in $\mathscr{C}$. This implies that $\min\{l \in \mathbb{N} | \beta \in C(\lambda'_{\theta_{h_1}}, \lambda''_{\theta'}), \beta = \beta_1 * \beta_2 * \cdots * \beta_l\} \leq k-1$. So using the induction hypothesis we conclude that $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$. This concludes the induction.

So, for all $g \in \text{Deck}(\pi)$, $g \neq \text{Id}$, $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$. In order to deal with $g = \text{Id}$, consider some $h, h^{-1} \in \text{Deck}(\pi)$, $h \neq \text{Id}$. Then $W^u(\widetilde{p}) \pitchfork W^s(h(\widetilde{p}))$ and $W^u(\widetilde{p}) \pitchfork W^s(h^{-1}(\widetilde{p}))$. As $\widetilde{\phi}$ commutes with all deck transformations, $W^u(h^{-1}(\widetilde{p})) \pitchfork W^s(\widetilde{p})$ and so, by the C^0 λ-lemma, $W^u(\widetilde{p}) \pitchfork W^s(\widetilde{p})$.

Actually, as we said at the beginning, we proved something a little bit stronger: for all $g \in \text{Deck}(\pi)$, $\widetilde{\lambda}_u \pitchfork g(\widetilde{\beta}_s)$. □

2.7. *The global shadowing.* The next result says that as the map ϕ is pseudo-Anosov relative to some finite invariant set, its complicated dynamics being, in some sense, inherited by f. It is related to Handel's global shadowing [**14**]; more precisely it appeared as Theorem 3.2 of Boyland's paper [**3**].

THEOREM 17. (Global shadowing) *If $f : S \to S$ is a homeomorphism of a closed orientable surface S isotopic to the identity, P is a finite f-invariant set and f is isotopic relative to P to some map $\phi : S \to S$ which is pseudo-Anosov relative to P, then there exists a compact f-invariant set $W \subset S$ and a continuous surjection $s : W \to S$ that is homotopic to the inclusion map $i : W \to S$ such that s semi-conjugates $f|_W$ to ϕ : that is, $s \circ f|_W = \phi \circ s$.*

Observe that, as $s : W \to S$ is homotopic to the inclusion map $i : W \to S$, s has a lift $\widetilde{s} : \pi^{-1}(W) \to \mathbb{D}$ such that

$$\widetilde{s} \circ \widetilde{f}|_{\pi^{-1}(W)} = \widetilde{\phi} \circ \widetilde{s},$$

where $\widetilde{\phi}$ and $\widetilde{f}$ are the natural lifts of ϕ and f, and $\sup\{d_{\mathbb{D}}(\widetilde{s}(\widetilde{q}), \widetilde{q}) | \widetilde{q} \in \pi^{-1}(W)\} < C_f$, for some constant $C_f > 0$.

2.8. *Special horseshoes for the pseudo-Anosov map ϕ.* In this subsection, we state a simple lemma used in the proofs of Theorems 1 and 2. The setting is the following: let $f : S \to S$ be a homeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$ and let P be the set of periodic points associated with the geodesics in $\mathscr{C}$. From Lemma 9, we know that there exists an integer $m_0 > 0$ such that f^{m_0} is isotopic relative to P to $\phi : S \to S$, a homeomorphism which is pseudo-Anosov relative to P and isotopic to the identity as a homeomorphism of S. From Lemma 13, there exists a contractible hyperbolic ϕ-periodic point $p \in S$ such that, for any $\widetilde{p} \in \pi^{-1}(p)$ and any given $g \in \text{Deck}(\pi)$, $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$. In $\mathbb{D}$, we are considering the natural lift of ϕ, denoted by $\widetilde{\phi}$. As we did previously, without loss of generality, we assume that p is fixed under ϕ and also that all four branches at p are ϕ-invariant.

LEMMA 18. *For any $g \in \text{Deck}(\pi)$ and any fundamental domain $\widetilde{Q} \subset \mathbb{D}$ of S such that $\widetilde{p} = \pi^{-1}(p) \cap \widetilde{Q}$ is in the interior of $\widetilde{Q}$, there exists arbitrarily small rectangles $\widetilde{R} \subset \widetilde{Q}$ such that:*

(1) interior($\widetilde{R}$) *contains* $\widetilde{p}$, *and two sides of* $\widetilde{R}$ *are very close to an arc* $\widetilde{\beta}$, $\widetilde{p} \in \widetilde{\beta} \subset W^s(\widetilde{p}) \cap R$ *and the two other sides of* $\widetilde{R}$ *have very small length; and*

(2) *for some* $N > 0$, $\widetilde{\phi}^N(\widetilde{R}) \cap \widetilde{R} \supset \widetilde{R}_0$ *and* $\widetilde{\phi}^N(\widetilde{R}) \cap g(\widetilde{R}) \supset \widetilde{R}_1$, *where* $R_0 = \pi(\widetilde{R}_0)$, $R_1 = \pi(\widetilde{R}_1)$ *are rectangles contained in* $R = \pi(\widetilde{R})$ *that have two sides contained in the sides of* R *which are very close to* $\beta = \pi(\widetilde{\beta})$ *and two sides contained in the interior of* R.

Proof. This is a standard result in hyperbolic dynamics, so we omit the proof and just present a figure. □

2.9. *The* $C^{1+\epsilon}$ *case: some background in Pesin theory.* In this subsection, assume that $f : S \to S$ is a $C^{1+\epsilon}$ diffeomorphism, for some $\epsilon > 0$. Recall that an f-invariant Borel probability measure μ is hyperbolic if all the Lyapunov exponents of f are non-zero at μ-almost every point (for instance, see the supplement of [**19**]). The following paragraphs were taken from [**7**]. They consist of an informal description of the theory of non-uniformly hyperbolic systems, together with some definitions and lemmas from [**7**].

Let μ be a non-atomic hyperbolic ergodic f-invariant Borel probability measure. Given $0 < \delta < 1$, there exists a compact Pesin set Λ_δ, with $\mu(\Lambda_\delta) > 1 - \delta$, having the following properties: for every $p \in \Lambda_\delta$, there exists an open neighborhood U_p, a compact neighborhood $V_p \subset U_p$ and a diffeomorphism $F : (-1, 1)^2 \to U_p$, with $F(0, 0) = p$ and $F([-1/10, 1/10]^2) = V_p$, such that the local unstable manifolds $W^u_{\text{loc}}(q)$ of all points q in $\Lambda_\delta \cap V_p$ are the images under F of graphs of the form $\{(x, F_2(x)) | x \in (-1, 1)\}$, where F_2 a function with small Lipschitz constant. Any two such local unstable manifolds are either disjoint or equal and they depend continuously on the point $q \in \Lambda_\delta \cap V_p$. Similarly, the local stable manifolds $W^s_{\text{loc}}(q)$ of points $q \in \Lambda_\delta \cap V_p$ are the images under F of graphs of the form $\{(F_1(y), y) | y \in (-1, 1)\}$, where F_1 a function with small Lipschitz constant. Any two such local stable manifolds are either disjoint or equal and they depend continuously on the point $q \in \Lambda_\delta \cap V_p$.

It follows that there exists a continuous product structure in $\Lambda_\delta \cap V_p$: given any $r, r' \in \Lambda_\delta \cap V_p$, the intersection $W^u_{\text{loc}}(r) \cap W^s_{\text{loc}}(r')$ is transversal and consists of exactly one point, which will be denoted by $[r, r']$. This intersection varies continuously with the two points and may not be in Λ_δ. Hence we can define maps $P^s_p : \Lambda_\delta \cap V_p \to W^s_{\text{loc}}(p)$ and $P^u_p : \Lambda_\delta \cap V_p \to W^u_{\text{loc}}(p)$ as $P^s_p(q) = [q, p]$ and $P^u_p(q) = [p, q]$.

Let $R^{\pm}$ denote the set of all points in S which are both forward and backward recurrent. By the Poincaré recurrence theorem, $\mu(R^{\pm}) = 1$.

Definition 2.9. (Accessible and inaccessible points) A point $p \in \Lambda_\delta \cap V_p \cap R^{\pm}$ is inaccessible if it is accumulated on both sides of $W^s_{\text{loc}}(p)$ by points in $P^s_p(\Lambda_\delta \cap V_p \cap R^{\pm})$ and also accumulated on both sides of $W^u_{\text{loc}}(p)$ by points in $P^u_p(\Lambda_\delta \cap V_p \cap R^{\pm})$. Otherwise, p is accessible.

After this definition, we can state two lemmas from [**7**] about accessible and inaccessible points and the relation between these points and hyperbolic periodic points close to them.

LEMMA 19. *Let $q \in \Lambda_\delta \cap V_p \cap R^\pm$ be an inaccessible point. Then there exist rectangles enclosing q that have sides along the invariant manifolds of hyperbolic periodic saddles in V_p and that have arbitrarily small diameter.*

A rectangle is a Jordan curve made up of alternating segments of stable and unstable manifolds, having two of each. The segments forming the boundary are its sides and the intersection points of the sides are the corners. A rectangle is said to enclose p if it is the boundary of an open topological disk containing p.

LEMMA 20. *The subset of accessible points in $\Lambda_\delta \cap V_p \cap R^\pm$ has zero μ measure.*

Another concept that will be a crucial hypothesis for us is positive topological entropy. In the following we describe why.

When the topological entropy $h(f|_K)$ is positive, for some compact f-invariant set K, by the variational principle, there exists a f-invariant Borel probability measure μ_0 with $\mathrm{supp}(\mu_0) \subset K$ and $h_{\mu_0}(f) > 0$. Using the ergodic decomposition of μ_0, we find an extremal point μ of the set of Borel probability f-invariant measures, such that $\mathrm{supp}(\mu)$ is also contained in K and $h_\mu(f) > 0$. Since the extremal points of this set are ergodic measures, μ is ergodic. The ergodicity and the positiveness of the entropy imply that μ has no atoms and applying the Ruelle inequality to f we see that μ has a positive Lyapunov exponent (see [**19**]). Working with f^{-1} and using the fact that $h_\mu(f^{-1}) = h_\mu(f) > 0$, we see that f^{-1} must also have a positive Lyapunov exponent with respect to μ, which is the negative of the negative Lyapunov exponent for f.

Hence, when K is a compact f-invariant set and the topological entropy of $f|_K$ is positive, there always exists an ergodic, non-atomic, invariant measure supported on K with non-zero Lyapunov exponents, one positive and one negative, with the measure having positive entropy: a hyperbolic measure.

The existence of this kind of measure will be important for us because of the following theorem, which can be proved by combining the main lemma and [**18**, Theorem 4.2].

THEOREM 21. *Let f be a $C^{1+\epsilon}$ (for some $\epsilon > 0$) diffeomorphism of a surface M and suppose that μ is an ergodic hyperbolic Borel probability f-invariant measure with $h_\mu(f) > 0$ and compact support. Then, for any $\alpha > 0$ and any $p \in \mathrm{supp}(\mu)$, there exists a hyperbolic periodic point $q \in B_\alpha(p)$ which has a transversal homoclinic intersection, and the whole orbit of q is contained in the α-neighborhood of* $\mathrm{supp}(\mu)$.

3. *Statement and proof of a C^0 result*

In this section, we fully state and prove Theorem 1.

THEOREM 1. (Precise statement) *Let $f : S \to S$ be a homeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$ and let $\widetilde{f}$ be its natural lift. Then there exists a constant $C_f \geq 0$ such that, for all $g \in \mathrm{Deck}(\pi)$ and any fundamental domain $\widetilde{Q} \subset \mathbb{D}$ of S, there exist arbitrarily large natural numbers $N > 0$, a point $\widetilde{r} = \widetilde{r}(N) \in \mathbb{D}$ and a compact set $K = K(N) \subset V_{C_f}(\widetilde{Q})$ (the open C_f-neighborhood of $\widetilde{Q}$ in the metric $d_{\mathbb{D}}$ of $\mathbb{D}$) such that*

$$\widetilde{f}^N(\widetilde{r}) = g(\widetilde{r}) \quad \textit{and} \quad \widetilde{f}^N(K) = g(K).$$

Remark. Note that if, for some $\widetilde{p} \in \mathbb{D}$, $\widetilde{f}^n(\widetilde{p}) = g(\widetilde{p})$ for some $g \in \mathrm{Deck}(\pi)$, then, for every $h \in \mathrm{Deck}(\pi)$, $\widetilde{f}^n(h(\widetilde{p})) = hgh^{-1}(h(\widetilde{p}))$.

So, if $f : S \to S$ is a homeomorphism isotopic to the identity that has a fully essential system of curves, then $\bigcup_{n>0} \widetilde{f}^n(V_{C_f}(\widetilde{Q}))$ accumulates in the whole boundary of $\mathbb{D}$, and, given any compact set $M \subset \mathbb{D}$, if

$$D_M \stackrel{\text{def.}}{=} \{g(V_{C_f}(\widetilde{Q})) : g \text{ is some deck transformation for which } g(V_{C_f}(\widetilde{Q})) \cap M \neq \emptyset\},$$

then there exists $N_M > 0$ such that, for all $n \geq N_M$, $\widetilde{f}^n(V_{C_f}(\widetilde{Q}))$ intersects all expanded fundamental domains contained in D_M. By expanded fundamental domains, we mean translates of $V_{C_f}(\widetilde{Q})$ by deck transformations.

In the case of a torus, if (0, 0) belongs to the interior of the rotation set, an analogous property holds (with $C_f = 0$). Therefore, in the situation when the fundamental group is not Abelian (surfaces of genus larger than one), our hypotheses, the fully essential system of curves $\mathscr{C}$ (see Definition 1.3), are an analog for (0, 0) being in the interior of the rotation set when the surface is the torus.

Proof of Theorem 1. Let $f : S \to S$ be a homeomorphism satisfying the theorem hypotheses. If we remember §2.8 and Lemma 18, for any fixed $g \in \mathrm{Deck}(\pi)$ and any fundamental domain $\widetilde{Q} \subset \mathbb{D}$ of S, there exist arbitrarily small rectangles $R \subset S$ such that a connected component $\widetilde{R}$ of $\pi^{-1}(R)$ is contained in interior$(\widetilde{Q})$ (we may have to perturb $\widetilde{Q}$ a little bit) and, for some $N > 0$, $\phi^N(R) \cap R \supseteq R_0 \cup R_1$. Remember that $\widetilde{\phi}^N(\widetilde{R}) \cap \widetilde{R} \supseteq \widetilde{R}_0$ and $\widetilde{\phi}^N(\widetilde{R}) \cap g(\widetilde{R}) \supseteq \widetilde{R}_1$, where $\widetilde{R}_0$ and $\widetilde{R}_1$ are connected components of $\pi^{-1}(R_0)$ and $\pi^{-1}(R_1)$. Associated with this horseshoe, if we consider the ϕ^N-fixed point $q \in R_1$, then, for $\widetilde{q} = \pi^{-1}(q) \cap \widetilde{R}$,

$$\widetilde{\phi}^N(\widetilde{q}) = g(\widetilde{q}) \Rightarrow \text{for all } j > 0, \quad \widetilde{\phi}^{jN}(\widetilde{q}) = g^j(\widetilde{q}).$$

Let $s : W \to S$ be the semi-conjugacy given by Theorem 17 and let $\widetilde{s} : \pi^{-1}(W) \to \mathbb{D}$ be its lift which relates the natural lifts $\widetilde{f}$ and $\widetilde{\phi}$. Fix $\widetilde{z} \in \widetilde{s}^{-1}(\widetilde{q})$. Since $\widetilde{s} \circ \widetilde{f}(\widetilde{z}) = \widetilde{\phi} \circ \widetilde{s}(\widetilde{z})$,

$$\widetilde{s}(\widetilde{f}^{jN}(\widetilde{z})) = \widetilde{\phi}^{jN}(\widetilde{s}(\widetilde{z})) = \widetilde{\phi}^{jN}(\widetilde{q}) = g^j(\widetilde{q}).$$

As we explained in §2.7, the fact that s is isotopic to the inclusion implies the existence of $C_f > 0$ such that $d_{\mathbb{D}}(\widetilde{s}(\widetilde{w}), \widetilde{w}) < C_f$, for all $\widetilde{w} \in \pi^{-1}(W)$. In particular,

$$d_{\mathbb{D}}(\widetilde{f}^{jN}(\widetilde{z}), \widetilde{s}(\widetilde{f}^{jN}(\widetilde{z}))) = d_{\mathbb{D}}(\widetilde{f}^{jN}(\widetilde{z}), g^j(\widetilde{q})) < C_f \quad \text{for all } j > 0.$$

As $g^{-1} \in \mathrm{Deck}(\pi)$ is an isometry of $d_{\mathbb{D}}$,

$$d_{\mathbb{D}}(\widetilde{f}^{jN}(\widetilde{z}), g^j(\widetilde{q})) = d_{\mathbb{D}}(g^{-j}(\widetilde{f}^{jN}(\widetilde{z})), \widetilde{q}) < C_f.$$

This means that, for any $\widetilde{z} \in \widetilde{s}^{-1}(\widetilde{q})$ and for all $j > 0$, $(g^{-1}\widetilde{f}^N)^j(\widetilde{z}) \in B_{C_f}(\widetilde{q})$. So the positive orbit of $\widetilde{z}$ with respect to $g^{-1}\widetilde{f}^N$ is bounded. Thus, defining $\widetilde{K}_g$ as the ω-limit set of the point $\widetilde{z}$ under $g^{-1}\widetilde{f}^N$, $\widetilde{K}_g$ is a compact $g^{-1}\widetilde{f}^N$-invariant set contained in $V_{C_f}(\widetilde{Q})$, and hence $\widetilde{f}^N(\widetilde{K}_g) = g(\widetilde{K}_g)$. By Brouwer's lemma on translation arcs [**5**], $g^{-1}\widetilde{f}^N$ has a fixed point, that is, there exists $\widetilde{r} \in \mathbb{D}$ with $g^{-1}\widetilde{f}^N(\widetilde{r}) = \widetilde{r}$, and so

$$\widetilde{f}^N(\widetilde{r}) = g(\widetilde{r}). \qquad \square$$

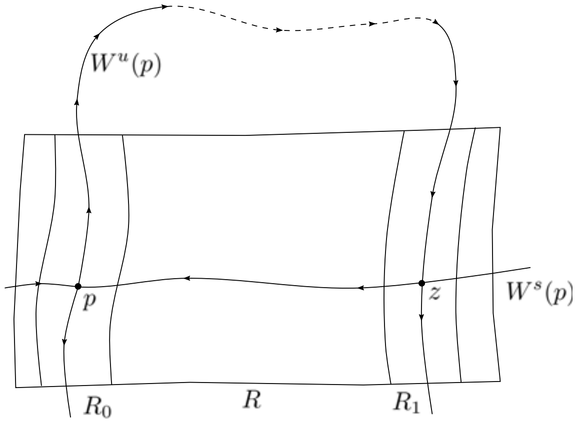

FIGURE 14. Horseshoe associated to $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$.

4. *Proof of Theorem 2*

Let $f : S \to S$ be a $C^{1+\epsilon}$ diffeomorphism isotopic to the identity with a fully essential system of curves $\mathscr{C}$. As in Theorem 1, let $\phi : S \to S$ be the pseudo-Anosov map relative to P that is isotopic to f^{m_0} relative to P (for some $m_0 > 0$, which, as before, to simplify the notation, we assume to be one). The finite set P is the set of periodic points associated with the geodesics in $\mathscr{C}$. By Lemma 13, for any given $g \in \mathrm{Deck}(\pi)$ and any fundamental domain $\widetilde{Q}$ of S, if $\widetilde{\phi} : \mathbb{D} \to \mathbb{D}$ is the natural lift of ϕ, there exists a hyperbolic $\widetilde{\phi}$-periodic point $\widetilde{p} \in \widetilde{Q} \subset \mathbb{D}$ such that

$$W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p})).$$

Again, as we did in previous results, without loss of generality, and considering an iterate of $\widetilde{\phi}$, if necessary, we assume that $\widetilde{p}$ is fixed under $\widetilde{\phi}$ and that each branch at $\widetilde{p}$ is also $\widetilde{\phi}$-invariant.

Using Lemma 18, if the transversal intersection $W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p}))$ at some $\widetilde{z} \in \mathbb{D}$ is projected to the surface S, it corresponds to a transversal homoclinic point $z = \pi(\widetilde{z}) \in W^u(p) \cap W^s(p)$. Associated with this intersection, a horseshoe in S can be obtained, i.e., on the surface there exists a small rectangle R, containing the arc β in $W^s(p)$ from p to z (as always R is very close to β) and a positive integer $N > 0$ such that $\phi^N(R) \cap R \supseteq R_0 \cup R_1$, where R_0 is a rectangle inside R containing p and R_1 is another rectangle inside R containing z (see Figure 14).

As $\widetilde{z} \in W^u(\widetilde{p}) \cap W^s(g(\widetilde{p}))$ can be chosen as close as we want to $g(\widetilde{p})$, the rectangle $R \subset S$ can be chosen small enough so that all the singular points of the stable and unstable foliations of ϕ do not belong to R. Moreover, considering the compact set $\Omega = \bigcap_{k\in\mathbb{Z}} \phi^{kN}(R_0 \cup R_1)$, we know that if R is sufficiently close to β, then for every bi-infinite sequence in $\{0, 1\}^{\mathbb{Z}}$, denoted by $(a_n)_{n\in\mathbb{Z}}$, there is a single point $z_* \in \Omega$ which realizes it: that is, $\phi^{kN}(z_*)$ belongs to R_{a_k} for all integers k.

LEMMA 22. *There exists a point $\widetilde{q}_2 \subset \widetilde{R}$, the connected component of $\pi^{-1}(R)$ that contains $\widetilde{p}$, such that $\widetilde{\phi}^{3N}(\widetilde{q}_2) = g(\widetilde{q}_2)$ and $W^u(\widetilde{q}_2) \pitchfork W^s(\widetilde{q}_2)$.*

Proof. Let $q_1 = p$, q_2 and q_3 be the ϕ^N-periodic points in Ω satisfying

$$\begin{aligned} \text{sequence}(q_1) &= \cdots 000000000000 \cdots\ , \\ \text{sequence}(q_2) &= \cdots 001001001001 \cdots\ , \\ \text{sequence}(q_3) &= \cdots 011011011011 \cdots\ . \end{aligned}$$

Since there are no singular points of the stable and unstable foliations inside R, the points q_2, q_3 are regular points of the stable and unstable foliations and are ϕ^{3N}-periodic. Moreover, there is a local product structure inside R : given $i, j \in \{1, 2, 3\}$, $i \neq j$, the intersection $W^u_{\mathrm{loc}}(q_i) \cap W^s_{\mathrm{loc}}(q_j)$ is transversal and consists of exactly one point. By $W^{s,u}_{\mathrm{loc}}(q_i)$, we mean the connected components of $W^{s,u}(q_i) \cap R$ containing q_i.

Let $\widetilde{p} = \widetilde{q}_1 \in \widetilde{R}$ and fix $\widetilde{q}_2 = \pi^{-1}(q_2) \cap \widetilde{R}$ and $\widetilde{q}_3 = \pi^{-1}(q_3) \cap \widetilde{R}$. By construction,

$$\begin{aligned} \widetilde{\phi}^{3N}(\widetilde{q}_1) &= \widetilde{q}_1, \\ \widetilde{\phi}^{3N}(\widetilde{q}_2) &= g(\widetilde{q}_2), \\ \widetilde{\phi}^{3N}(\widetilde{q}_3) &= g^2(\widetilde{q}_3). \end{aligned}$$

If we set $\widetilde{\psi} = g^{-1}\widetilde{\phi}^{3N}$, then

$$\begin{aligned} \widetilde{\psi}(\widetilde{q}_1) &= g^{-1}(\widetilde{q}_1), \\ \widetilde{\psi}(\widetilde{q}_2) &= \widetilde{q}_2, \\ \widetilde{\psi}(\widetilde{q}_3) &= g(\widetilde{q}_3). \end{aligned}$$

In particular, $\widetilde{q}_2$ is a hyperbolic fixed saddle point for $\widetilde{\psi}$. As $W^u_{\mathrm{loc}}(q_2) \pitchfork W^s_{\mathrm{loc}}(q_1)$, we get that $W^u(\widetilde{q}_2) \pitchfork W^s(\widetilde{q}_1)$ (note that, for all $m > 0$, $W^{s,u}(g^{-m}(\widetilde{q}_1))$ is the lift of $W^{s,u}(q_1)$ to $g^{-m}(\widetilde{R})$). Since $\widetilde{\psi}(\widetilde{q}_1) = g^{-1}(\widetilde{q}_1)$, using that $W^u(\widetilde{q}_2)$ is invariant under $\widetilde{\psi}$ and the fact that g^{-1} commutes with $\widetilde{\psi}$, we conclude that, for all $m > 0$,

$$W^u(\widetilde{q}_2) \pitchfork W^s(g^{-m}(\widetilde{q}_1)). \tag{13}$$

Note that, as $W^u_{\mathrm{loc}}(q_1)$ intersects $W^s_{\mathrm{loc}}(q_2)$ transversely, there exists $m' > 0$ such that

$$W^u(\widetilde{q}_2) \pitchfork W^s(g^{-m'}(\widetilde{q}_2)). \tag{14}$$

The same argument considering the point q_3 instead of q_1, gives an integer $m'' > 0$ such that

$$W^u(\widetilde{q}_2) \pitchfork W^s(g^{m''}(\widetilde{q}_2)). \tag{15}$$

So, by the λ-lemma,

$$W^u(\widetilde{q}_2) \pitchfork W^s(g^{m'm''}(\widetilde{q}_2)) \quad \text{and} \quad W^u(\widetilde{q}_2) \pitchfork W^s(g^{-m'm''}(\widetilde{q}_2)),$$

which finally imply that

$$W^u(\widetilde{q}_2) \pitchfork W^s(\widetilde{q}_2). \qquad \square$$

Associated with the transversal intersection $W^u(\widetilde{q}_2) \pitchfork W^s(\widetilde{q}_2)$, there is a compact $\widetilde{\psi}^{N'}$-invariant set Ω_g, for some $N' > 0$, such that $h(\widetilde{\psi}^{N'}|_{\Omega_g}) > 0$. Defining $\Omega^*_g = \bigcup_{i=0}^{N'-1} \widetilde{\psi}^i(\Omega_g)$, it is a $\widetilde{\psi}$-invariant compact set with $h(\widetilde{\psi}|_{\Omega^*_g}) > 0$. We are looking for a similar statement for the map $g^{-1}\widetilde{f}^{3N}$.

LEMMA 23. *There exists a set $K_g \subset S$ which is analogous to f of the set Ω_g^* with respect to ϕ.*

Proof. Theorem 14 implies that there exists a compact f-invariant set W and a continuous surjective map $s : W \to S$, homotopic to the inclusion, such that $s \circ f|_W = \phi \circ s$. Instead of W, we will consider a compact f-invariant subset $W'' \subseteq W$ constructed in the following way: since ϕ is pseudo-Anosov relative to a finite set, there exists a point z_0 in S such that $\mathrm{Orb}_\phi^+(z_0) = \{\phi^n(z_0) | n \geq 0\}$ is dense in S. Choose some point $w_0 \in s^{-1}(z_0)$ and let $W' = \overline{\mathrm{Orb}_f^+(w_0)}$. Clearly, $f(W') \subseteq W'$, and defining $W'' = \bigcap_{n \geq 0} f^n(W')$, we get that $f(W'') = W''$, it is compact and $s(W'') = S$. In particular, $\mathrm{Orb}_f^+(w_0)$ is dense in W''.

Lifting $s : W'' \to S$ to $\widetilde{s} : \pi^{-1}(W'') \to \mathbb{D}$, we obtain a compact $g^{-1}\widetilde{f}^{3N}$-invariant set

$$K_g \subseteq \bigcup_{n \geq 0} (g^{-1}\widetilde{f}^{3N})^n(\widetilde{s}^{-1}(\Omega_g^*)) \subset \pi^{-1}(W'') \subset \mathbb{D} \quad \text{with } h(g^{-1}\widetilde{f}^{3N}|_{K_g}) > 0. \qquad \square$$

As we explained in §2.9, the fact that $h(g^{-1}\widetilde{f}^{3N}|_{K_g}) > 0$ implies the existence of a non-atomic, hyperbolic, ergodic $g^{-1}\widetilde{f}^{3N}$-invariant Borel probability measure μ_g with positive entropy, whose support is contained in K_g.

As $\mu_g(R^\pm \cap \mathrm{supp}(\mu_g)) = 1$, for $0 < \delta < 1$, choosing any point $\widetilde{r} \in \Lambda_\delta \cap \mathrm{supp}(\mu_g) \cap R^\pm$, we get that $\mu_g(V_{\widetilde{r}} \cap \Lambda_\delta \cap R^\pm) > 0$. So Lemmas 19 and 20 and Theorem 21 assure that, given $\eta > 0$, there exists an inaccessible point $\widetilde{z}_g \in \mathrm{supp}(\mu_g) \subset K_g$ (see definition (2.9)) such that arbitrarily small rectangles enclosing $\widetilde{z}_g$ can be obtained, where the sides of these rectangles are contained in the invariant manifolds of two hyperbolic $g^{-1}\widetilde{f}^{3N}$-periodic saddle points, $\widetilde{r}_g'$, $\widetilde{r}_g''$, whose orbits are contained in the η-neighborhood of $\mathrm{supp}(\mu_g)$. Moreover, $W^u(\widetilde{r}_g') \pitchfork W^s(\widetilde{r}_g'')$ and $W^u(\widetilde{r}_g'') \pitchfork W^s(\widetilde{r}_g')$ in a C^1-transverse way. So $W^u(\widetilde{r}_g') \pitchfork W^s(\widetilde{r}_g')$ and $W^u(\widetilde{r}_g'') \pitchfork W^s(\widetilde{r}_g'')$, also in a C^1-transverse way. An important observation that will be used later is that each of these rectangles contains infinitely many points belonging to $\mathrm{supp}(\mu_g) \subset \pi^{-1}(W'')$ because μ_g is non-atomic. For these two periodic points, there exists k', $k'' > 0$ such that

$$\widetilde{f}^{3Nk'}(\widetilde{r}_g') = g^{k'}(\widetilde{r}_g') \quad \text{and} \quad \widetilde{f}^{3Nk''}(\widetilde{r}_g'') = g^{k''}(\widetilde{r}_g'').$$

Going back to the surface S, we define $z_g = \pi(\widetilde{z}_g)$, $r_g' = \pi(\widetilde{r}_g')$ and $r_g'' = \pi(\widetilde{r}_g'')$. Then r_g' and r_g'' are hyperbolic f-periodic saddles for which $W^u(r_g') \pitchfork W^s(r_g'')$ and $W^u(r_g'') \pitchfork W^s(r_g')$ in a C^1-transverse way and so $\overline{W^s(r_g')} = \overline{W^s(r_g'')}$ and $\overline{W^u(r_g')} = \overline{W^u(r_g'')}$. Associated with these points there are small rectangles in S whose sides are contained in their invariant manifolds and enclose the point z_g; they are the projection under π of the rectangles in $\mathbb{D}$.

LEMMA 24. *There exists a contractible periodic hyperbolic saddle point $\widetilde{r}_0 \in \mathbb{D}$ and deck transformations g_1'' and g_2'' such that $g_1''g_2'' \neq g_2''g_1''$ and $W^u(\widetilde{r}_0) \pitchfork W^s(g_1''(\widetilde{r}_0))$ and $W^u(\widetilde{r}_0) \pitchfork W^s(g_2''(\widetilde{r}_0))$.*

Proof. First, choose $g_1, g_2 \in \mathrm{Deck}(\pi)$ such that they correspond to different geodesics in S. In this way, $g_1g_2 \neq g_2g_1$, and their powers are never conjugated: i.e., for all $h \in \mathrm{Deck}(\pi)$ and n, m integers, $hg_1^nh^{-1} \neq g_2^m$.

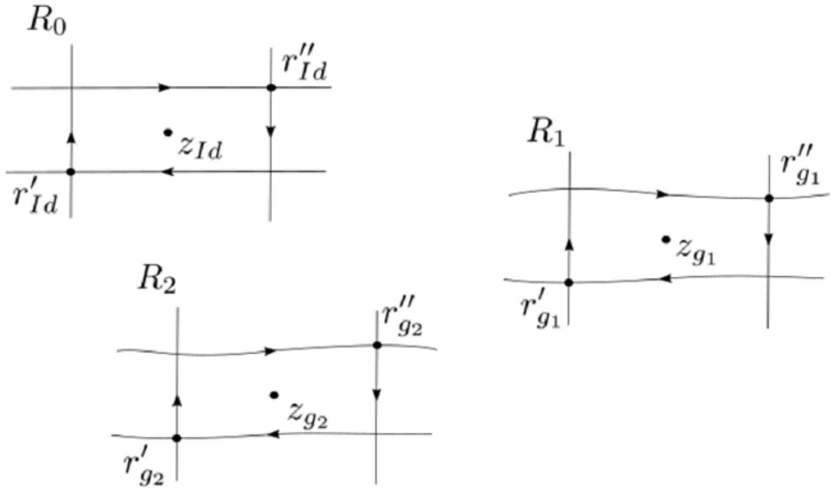

FIGURE 15. The rectangles R_0, R_1 and R_2.

For the maps Id, g_1, g_2, we consider the compact sets K_{Id}, K_{g_1} and K_{g_2} contained in $\mathbb{D}$ and inaccessible points $\widetilde{z}_{\mathrm{Id}} \in K_{\mathrm{Id}}$, $\widetilde{z}_{g_1} \in K_{g_1}$ and $\widetilde{z}_{g_2} \in K_{g_2}$. From what we just did, there are hyperbolic f-periodic saddles r'_{Id}, r''_{Id}, r'_{g_1}, r''_{g_1}, r'_{g_2} and r''_{g_2} in S, with R_0 being a small rectangle in S whose sides are contained in the invariant manifolds of r'_{Id} and r''_{Id} and enclose the point $\pi(\widetilde{z}_{\mathrm{Id}}) = z_{\mathrm{Id}} \in W''$. Similarly, for $i \in \{1, 2\}$, R_i is a small rectangle in S whose sides are contained in the invariant manifolds of r'_{g_i} and r''_{g_i} and enclose the point $\pi(\widetilde{z}_{g_i}) = z_{g_i} \in W''$, see Figure 15.

Let $n^{\mathrm{comm}} > 0$ be a natural number that is a common period of all the points r'_{Id}, r''_{Id}, r'_{g_1}, r''_{g_1}, r'_{g_2} and r''_{g_2}, which also leaves invariant all stable and unstable branches of these points. Clearly, the orbits of all the previous points can be assumed to be disjoint.

As we said, R_0 is the small rectangle enclosing the point z_{Id}. For all $0 \le i \le n^{\mathrm{comm}} - 1$, $f^i(R_0)$ is a rectangle in S. If we denote the arcs in the boundary of R_0 as $\alpha'_0 \in W^s(r'_{\mathrm{Id}})$, $\omega'_0 \in W^u(r'_{\mathrm{Id}})$, $\alpha''_0 \in W^s(r''_{\mathrm{Id}})$ and $\omega''_0 \in W^u(r''_{\mathrm{Id}})$, then, for large $m > 0$ and for all $0 \le i \le n^{\mathrm{comm}} - 1$,

$$\partial(f^{n^{\mathrm{comm}}m}(f^i(R_0))) \subset f^{n^{\mathrm{comm}}m}(f^i(\alpha'_0)) \cup W^u(r'_{\mathrm{Id}}) \cup f^{n^{\mathrm{comm}}m}(f^i(\alpha''_0)) \cup W^u(r''_{\mathrm{Id}}),$$

and the sets $f^{n^{\mathrm{comm}}m}(f^i(\alpha'_0))$, $f^{n^{\mathrm{comm}}m}(f^i(\alpha''_0))$ are as close as we want to the points $f^i(r'_{\mathrm{Id}})$ and $f^i(r''_{\mathrm{Id}})$, respectively. Using an analogous notation with the rectangles R_1 and R_2, we can find a natural number $m_0 > 0$ such that, for $0 \le i, j \le n^{\mathrm{comm}} - 1$, $k, t \in \{0, 1, 2\}$, $k \ne t$ and $m > m_0$,

$$\begin{aligned}
f^{n^{\mathrm{comm}}m}(f^i(\alpha'_k)) \cap f^j(\omega'_t) &= \emptyset,\\
f^{n^{\mathrm{comm}}m}(f^i(\alpha'_k)) \cap f^j(\omega''_t) &= \emptyset,\\
f^{n^{\mathrm{comm}}m}(f^i(\alpha''_k)) \cap f^j(\omega'_t) &= \emptyset,\\
f^{n^{\mathrm{comm}}m}(f^i(\alpha''_k)) \cap f^j(\omega''_t) &= \emptyset.
\end{aligned} \tag{16}$$

As $f^i(z_{\mathrm{Id}})$ and $f^i(z_{g_1})$ are in the interior of $f^i(R_0)$ and $f^i(R_1)$, respectively, they are both accumulated by points in W'', and as there exists a point whose orbit is dense in W'', we get that, for all $0 \le i \le n^{\text{comm}} - 1$, there exist integers $l_0(i)$, $l_1(i) > m_0 n^{\text{comm}}$ such that

$$f^{l_1(i)}(f^i(R_1)) \cap R_0 \neq \emptyset \quad \text{and} \quad f^{l_0(i)}(f^i(R_0)) \cap R_1 \neq \emptyset.$$

So, for any $0 \le i \le n^{\text{comm}} - 1$, there exist integers $m_0(i), m_1(i) \ge m_0$ and other integers $0 \le j_0(i)$, $j_1(i) \le n^{\text{comm}} - 1$ such that

$$f^{n^{\text{comm}} m_1(i)}(f^i(R_1)) \cap f^{j_0(i)}(R_0) \neq \emptyset \quad \text{and} \quad f^{n^{\text{comm}} m_0(i)}(f^i(R_0)) \cap f^{j_1(i)} R_1 \neq \emptyset.$$

Since the boundary of rectangle R_0 is contained in the invariant manifolds of r'_{Id} and r''_{Id}, and

$$W^u(r'_{\text{Id}}) \pitchfork W^s(r''_{\text{Id}}),$$
$$W^u(r''_{\text{Id}}) \pitchfork W^s(r'_{\text{Id}}),$$

with the same being true for R_1, r'_{g_1} and r''_{g_1}, we conclude by expression (16) that, for all $0 \le i \le n^{\text{comm}} - 1$, $W^u(f^i(r'_{\text{Id}})) \pitchfork W^s(f^{j_1(i)}(r'_{g_1}))$ and $W^u(f^i(r'_{g_1})) \pitchfork W^s(f^{j_0(i)}(r'_{\text{Id}}))$. Then a combinatorial argument implies that there exist $0 \le i, j \le n^{\text{comm}} - 1$ such that $W^u(f^i(r'_{g_1})) \pitchfork W^s(f^j(r'_{\text{Id}}))$ and $W^u(f^j(r'_{\text{Id}})) \pitchfork W^s(f^i(r'_{g_1}))$ (see [**1**]).

Doing the same for R_2, r'_{g_2} and r''_{g_2}, and using the fact that topologically transverse intersections are mapped into themselves under f, we can find $0 \le k \le n^{\text{comm}} - 1$ such that, for the same j as above, $W^u(f^k(r'_{g_2})) \pitchfork W^s(f^j(r'_{\text{Id}}))$ and $W^u(f^j(r'_{\text{Id}})) \pitchfork W^s(f^k(r'_{g_2}))$.

Set $r_0 = f^j(r'_{\text{Id}})$, $r_1 = f^i(r'_{g_1})$ and $r_2 = f^k(r'_{g_2})$. Then these are hyperbolic f-periodic saddle points and, for $i \in \{1, 2\}$,

$$W^u(r_0) \pitchfork W^s(r_i) \quad \text{and} \quad W^u(r_i) \pitchfork W^s(r_0).$$

Fix any $\widetilde{r}_0$ in $\pi^{-1}(r_0)$. By our construction, since there is a point $\widetilde{r}'_0 \in \pi^{-1}(r_0)$ whose orbit is forever close to K_{Id}, we get that $\widetilde{f}^{n^{\text{comm}}}(\widetilde{r}_0) = \widetilde{r}_0$.

Recall that n^{comm} is a common period for r_0, r_1 and r_2. The fact that $W^u(r_0) \pitchfork W^s(r_1)$ implies that there exists a point $\widetilde{r}_1 \in \pi^{-1}(r_1)$ for which $W^u(\widetilde{r}_0) \pitchfork W^s(\widetilde{r}_1)$. Moreover, arguing as above, there exists an integer $n_1 > 0$ such that $\widetilde{f}^{n^{\text{comm}}}(\widetilde{r}'_1) = g_1^{n_1}(\widetilde{r}'_1)$ for some $\widetilde{r}'_1 \in \pi^{-1}(r_1)$, close to K_{g_1}. As $\widetilde{r}_1, \widetilde{r}'_1 \in \pi^{-1}(\widetilde{r}_1)$, there exists $h_1 \in \text{Deck}(\pi)$ with $\widetilde{r}_1 = h_1(\widetilde{r}'_1)$. Hence, $\widetilde{f}^{n^{\text{comm}}}(\widetilde{r}_1) = h_1 g_1^{n_1} h_1^{-1}(\widetilde{r}_1)$.

Set $g'_1 = h_1 g_1^{n_1} h_1^{-1}$. As before, since $W^u(\widetilde{r}_0) \pitchfork W^s(\widetilde{r}_1)$ and $\widetilde{f}^{n^{\text{comm}}}(\widetilde{r}_1) = g'_1(\widetilde{r}_1)$, for all $m \ge 0$

$$W^u(\widetilde{r}_0) \pitchfork W^s((g'_1)^m(\widetilde{r}_1)).$$

As $W^u(r_1)$ intersects $W^s(r_0)$ in a topologically transverse way, there is a compact connected piece of a branch of $W^u(r_1)$, denoted by λ_1, such that one of its endpoints is r_1 and it has a topologically transversal intersection with $W^s(r_0)$. If $\widetilde{\lambda}_1$ is the lift of λ_1 starting at the point $\widetilde{r}_1$, then there exists $h'_1 \in \text{Deck}(\pi)$ such that

$$\widetilde{\lambda}_1 \pitchfork W^s(h'_1(\widetilde{r}_0)).$$

This implies that, if $m_1 > 0$ is sufficiently large, then a piece of $W^u(\widetilde{r}_0)$ is sufficiently close in the Hausdorff topology to $(g'_1)^{m_1}(\widetilde{\lambda}_1)$, something that forces $W^u(\widetilde{r}_0)$ to have a topological transverse intersection with $W^s((g'_1)^{m_1} h'_1(\widetilde{r}_0))$. Summarizing, for all $m_1 > 0$ sufficiently large,

$$W^u(\widetilde{r}_0) \pitchfork W^s((g'_1)^{m_1} h'_1(\widetilde{r}_0)).$$

Arguing in an analogous way with respect to the point r_2, we find $h_2, h'_2 \in \mathrm{Deck}(\pi)$ and an integer $n_2 > 0$ such that, if $g'_2 = h_2 g_2^{n_2} h_2^{-1}$, then, for all $m_2 > 0$ sufficiently large,

$$W^u(\widetilde{r}_0) \pitchfork W^s((g'_2)^{m_2} h'_2(\widetilde{r}_0)).$$

In order to conclude, let us show that $m_1, m_2 > 0$ can be chosen in a way that $(g'_1)^{m_1} h'_1$ and $(g'_2)^{m_2} h'_2$ do not commute. We started with deck transformations g_1 and g_2 for which $g_1 g_2 \neq g_2 g_1$ and g_1^n is not conjugated to g_2^m, for all integers n, m. As we already explained, the above conditions follow from the fact that g_1 and g_2 correspond, in S, to different geodesics.

In particular, this implies that the deck transformations g'_1 and g'_2 do not commute and the fixed points of g'_1 and g'_2 at the boundary at infinity $\partial\mathbb{D}$ are all different, i.e., $\mathrm{Fix}(g'_1) \cap \mathrm{Fix}(g'_2) = \emptyset$.

Fix two large integers $m_1, m_2 > 0$ and let us analyze $(g'_1)^{m_1} h'_1$ and $(g'_2)^{m_2} h'_2$. If they do not commute, there is nothing to do.

So assume that $(g'_1)^{m_1} h'_1$ and $(g'_2)^{m_2} h'_2$ commute. Since they commute, $\mathrm{Fix}((g'_1)^{m_1} h'_1) = \mathrm{Fix}((g'_2)^{m_2} h'_2)$. Observe that either g'_1 does not commute with $(g'_1)^{m_1} h'_1$ or g'_2 does not commute with $(g'_2)^{m_2} h'_2$.

In fact, if they both commute, then

$$\mathrm{Fix}(g'_1) = \mathrm{Fix}((g'_1)^{m_1} h'_1) = \mathrm{Fix}((g'_2)^{m_2} h'_2) = \mathrm{Fix}(g'_2),$$

and this contradicts the fact that g'_1 and g'_2 do not commute. So, without loss of generality, assume that g'_1 and $(g'_1)^{m_1} h'_1$ do not commute. Hence $\mathrm{Fix}(g'_1) \cap \mathrm{Fix}((g'_1)^{m_1} h'_1) = \emptyset$.

We claim that $(g'_1)^{m_1+1} h'_1 = g'_1 (g'_1)^{m_1} h'_1$ and $(g'_2)^{m_2} h'_2$ do not commute. Otherwise,

$$\mathrm{Fix}((g'_1)^{m_1+1} h'_1) = \mathrm{Fix}((g'_2)^{m_2} h'_2) = \mathrm{Fix}((g'_1)^{m_1} h'_1).$$

So, for all $\widetilde{q} \in \mathrm{Fix}((g'_1)^{m_1} h'_1)$,

$$\widetilde{q} = g'_1((g'_1)^{m_1} h'_1(\widetilde{q})) = g'_1(\widetilde{q}),$$

which means that $\mathrm{Fix}((g'_1)^{m_1} h'_1) = \mathrm{Fix}(g'_1)$, which contradicts our previous assumption that g'_1 and $(g'_1)^{m_1} h_1$ do not commute. Hence $(g'_1)^{m_1+1} h'_1$ and $(g'_2)^{m_2} h'_2$ do not commute.

So, we can always find arbitrarily large integers $m_1, m_2 > 0$ such that if $g''_1 = (g'_1)^{m_1} h'_1$ and $g''_2 = (g'_2)^{m_2} h'_2$, then $g''_1 g''_2 \neq g''_2 g''_1$ and

$$W^u(\widetilde{r}_0) \pitchfork W^s(g''_1(\widetilde{r}_0)) \quad \text{and} \quad W^u(\widetilde{r}_0) \pitchfork W^s(g''_2(\widetilde{r}_0)). \qquad \square$$

Now, as in the proof of Lemma 13, we construct the path connected sets θ and θ' using the point $\widetilde{r}_0$ and the deck transformations g''_1 and g''_2. Since f has a fully essential system of curves $\mathscr{C}$ and the periodic points P associated to $\mathscr{C}$, the exact same proof of Lemma 13 without any modifications shows that, for every $g \in \mathrm{Deck}(\pi)$,

$$W^u(\widetilde{r}_0) \pitchfork W^s(g(\widetilde{r}_0)).$$

As $\widetilde{r}_0 \in \pi^{-1}(r_0)$ was arbitrary, after redefining $p = r_0$, the proof is complete. $\square$

5. *Proof of Theorem 3*

Let $\widetilde{p} \in \mathbb{D}$ be a hyperbolic periodic saddle point for $\widetilde{f}$ given by Theorem 2 (as before, assume, without loss of generality, that $\widetilde{p}$ is fixed and all four branches at $\widetilde{p}$ are $\widetilde{f}$-invariant; otherwise, consider some iterate of $\widetilde{f}$). For all $g \in \mathrm{Deck}(\pi)$,

$$W^u(\widetilde{p}) \pitchfork W^s(g(\widetilde{p})).$$

In fact, a stronger statement holds: the proof of Theorem 2 gives an unstable branch $\widetilde{\lambda}_u$ of $W^u(\widetilde{p})$ and a stable branch $\widetilde{\beta}_s$ of $W^s(\widetilde{p})$ such that, for all $g \in \mathrm{Deck}(\pi)$,

$$\widetilde{\lambda}_u \pitchfork g(\widetilde{\beta}_s). \tag{17}$$

Fix some $0 < \epsilon < 1/10$ small enough so that, for any $z \in S$, if $\widetilde{z}_1, \widetilde{z}_2 \in \pi^{-1}(z), \widetilde{z}_1 \neq \widetilde{z}_2$, then $B_{2\epsilon}(\widetilde{z}_1) \cap B_{2\epsilon}(\widetilde{z}_2) = \emptyset$. Let $\widetilde{\lambda}$ be a compact subarc of $\widetilde{\lambda}_u$, small enough so that one of its endpoints is $\widetilde{p}$ and $\widetilde{\lambda} \subset B_\epsilon(\widetilde{q})$. In a similar way, let $\widetilde{\beta}$ be a compact subarc of $\widetilde{\beta}_s$, so that $\widetilde{p}$ is one of its endpoints and $\widetilde{\beta} \subset B_\epsilon(\widetilde{p})$. The arc $\widetilde{\beta}$ satisfies another property: its endpoint which is not $\widetilde{p}$ belongs to $W^u(\widetilde{p})$ and, actually, this endpoint is a C^1-transversal homoclinic point. It is possible to choose $\widetilde{\beta}$ in this way because the proof of Theorem 2 implies the existence of a C^1-transversal intersection between $W^s(\mathrm{Id}(\widetilde{p}))$ and $W^u(\widetilde{p})$. When, instead of Id, we consider any other deck transformation, only topologically transverse intersections are assured, but for the Id, C^1-transversality was obtained.

Now choose $h_1, h_2, \ldots, h_{2g} \in \mathrm{Deck}(\pi)$, where $g > 0$ is the genus of S, such that the geodesics in S associated to $\{h_1, h_2, \ldots, h_{2g}\}$ generate the first homotopy group, $\pi_1(S)$. Expression (17) implies the existence of a compact arc $\widetilde{\Lambda}$ such that $\widetilde{\lambda}_u \supset \widetilde{\Lambda} \supset \widetilde{\lambda}$ and

$$\begin{gathered}\widetilde{\Lambda} \text{ contains both endpoints of } \widetilde{\beta},\ \widetilde{\Lambda} \pitchfork h_i(\widetilde{\beta}), \text{ for all } 1 \leq i \leq 2g, \text{ the} \\ \text{endpoint of } \widetilde{\Lambda} \text{ which is not } \widetilde{p} \text{ is contained in the interior of } \widetilde{\beta} \text{ and} \\ \text{it is a } C^1\text{-transversal homoclinic point.}\end{gathered} \tag{18}$$

Clearly, the above choice implies that every connected component of the complement of $\pi(\widetilde{\Lambda} \cup \widetilde{\beta})$ is an open disk in S.

Let $R \subset B_{2\epsilon}(p)$ be a closed rectangle which has p as a vertex and $\beta = \pi(\widetilde{\beta})$ as one side; R is very thin, close to β in the Hausdorff topology. ∂R is given by the union of four arcs: α, α', β and β'. The arcs α and α' are contained in $W^u(p) : \alpha \subset \pi(\widetilde{\lambda})$ and α' contains the endpoint of β which is not p.

From the choice of $\widetilde{\Lambda}$,

$$\pi(\widetilde{\Lambda}) \supset \alpha'. \tag{19}$$

Clearly, β and β' are contained in $W^s(p)$ and β' is C^1-close to β. As was explained when defining $\widetilde{\beta}$, the existence of such a rectangle R follows from Theorem 2, which says that $W^u(\widetilde{p})$ has C^1-transverse intersections with $W^s(\widetilde{p})$.

At this point, we need to determine the size of α and α' and a number $N > 0$, as follows: we know (from (18)) that $\widetilde{\Lambda} \pitchfork h_i(\widetilde{\beta})$, for all $1 \leq i \leq 2g$. Choose β' sufficiently close to β (so α and α' are very small) in such a way that if $\widetilde{R}$ is the connected component of $\pi^{-1}(R)$ that contains $\widetilde{\beta}$ (the sides of $\widetilde{R}$ are denoted by $\widetilde{\alpha} \subset \widetilde{\lambda}$, $\widetilde{\alpha}'$, $\widetilde{\beta}$ and $\widetilde{\beta}'$; $\widetilde{\alpha}, \widetilde{\alpha}' \subset W^u(\widetilde{p})$ and $\widetilde{\beta}, \widetilde{\beta}' \subset W^s(\widetilde{p})$), then

$$\widetilde{\Lambda} \pitchfork h_i(\widetilde{\beta}') \quad \text{for all } 1 \leq i \leq 2g.$$

Now, fix some $N > 0$ such that

$$\begin{array}{c}\widetilde{f}^N(\widetilde{\beta}') \subset \widetilde{\beta},\ \widetilde{f}^N(\widetilde{\alpha}) \supset \widetilde{\Lambda} \supset \widetilde{\alpha}' \text{ and } \widetilde{f}^N(\widetilde{\alpha}') \supset \widetilde{\Lambda}',\ \text{ an arc} \\ \text{sufficiently } C^1\text{-close to } \widetilde{\Lambda}, \text{ whose endpoints} \\ \text{are also in } \widetilde{\beta}, \text{ in a way that } \widetilde{\Lambda}' \pitchfork h_i(\widetilde{\beta}) \text{ and} \\ \widetilde{\Lambda}' \pitchfork h_i(\widetilde{\beta}'), \text{ for all } 1 \leq i \leq 2g. \text{ Moreover, the arcs in } \widetilde{\beta} \\ \text{connecting the appropriate endpoints (the ones} \\ \text{which are closer) of } \widetilde{\Lambda} \text{ and } \widetilde{\Lambda}' \text{ are disjoint from } \widetilde{\Lambda} \cup \widetilde{\Lambda}'.\end{array} \tag{20}$$

Changing the subject a little, remember that §1.2 implies that

$$\rho_m(f) = \mathrm{Conv}(\rho_{\mathrm{erg}}(f)) = \mathrm{Conv}(\rho_{mz}(f)).$$

We also know that every extremal point of the convex hull of $\rho_{mz}(f)$ is the rotation vector of some recurrent point.

Let w be a extremal point of $\mathrm{Conv}(\rho_{mz}(f))$, and let $q_w \in S$ be a recurrent point with

$$w = \lim_{n\to\pm\infty} \frac{\Psi_f^n(q_w)}{n}. \tag{21}$$

From the existence of a fully essential system of curves $\mathscr{C}$, it is easy to see that $\underline{0} = (0, \ldots, 0)$ belongs to the interior of the $\mathrm{Conv}(\rho_{\mathrm{erg}}(f))$. So, $w \neq \underline{0}$.

Since q_w is a recurrent point, fix a fundamental domain $\widetilde{Q} \subset \mathbb{D}$. If we pick $\widetilde{q}_w \in \pi^{-1}(q_w) \cap \widetilde{Q}$, then there exists a sequence $n_k \to \infty$ such that, for some $g_k \in \mathrm{Deck}(\pi)$,

$$\widetilde{f}^{n_k}(\widetilde{q}_w) \in g_k(\widetilde{Q}) \quad \text{and} \quad d_{\mathbb{D}}(\widetilde{f}^{n_k}(\widetilde{q}_w), g_k(\widetilde{q}_w)) < \frac{1}{k} \quad \text{for all } k > 0.$$

For all $k > 0$, let β_k be a path in S joining $f^{n_k}(q_w)$ to q_w with $l(\beta_k) < 1/k$. As $\|[I_{q_w}^{n_k} * \beta_k] - \Psi_f^{n_k}(q_w)\| \leq 2C_{\mathcal{A}} + 1$ and $w = \lim_{k\to\infty} \Psi_f^{n_k}(q_w)/n_k$, we get that

$$w = \lim_{k\to\infty} \frac{[I_{q_w}^{n_k} * \beta_k]}{n_k}.$$

Let $\widetilde{I}_{\widetilde{q}_w}^{n_k}$ be the lift of $I_{q_w}^{n_k}$ with base point $\widetilde{q}_w$. Then $\widetilde{I}_{\widetilde{q}_w}^{n_k}$ is a path in $\mathbb{D}$ joining $\widetilde{q}_w$ to $\widetilde{f}^{n_k}(\widetilde{q}_w)$ and so the loop $I_{q_w}^{n_k} * \beta_k$ lifts to a path $\widetilde{I}_{\widetilde{q}_w}^{n_k} * \widetilde{\beta}_k$ joining $\widetilde{q}_w$ to $g_k(\widetilde{q}_w)$.

For any $g \in \mathrm{Deck}(\pi)$, a path $\widetilde{\gamma}_g$ joining any point $\widetilde{q} \in \mathbb{D}$ to $g(\widetilde{q})$ projects into a loop $\gamma_g = \pi(\widetilde{\gamma}_g)$ whose free homotopy class (and, in particular, its homology class) is determined only by g. We denote by $[g] = [\gamma_g]$ this homology class. Hence, we can write

$$w = \lim_{k\to\infty} \frac{[g_k]}{n_k}. \tag{22}$$

LEMMA 25. *There exist deck transformations $\{m_1, m_2, \ldots, m_J\}$ for some $J > 0$ such that, for all sufficiently large $k > 0$ and for a fixed fundamental domain $\widetilde{Q} \subset \mathbb{D}$, with $n_k > 2 \cdot N > 0$ (see (22) and (20)), if $\widetilde{f}^{n_k}(\widetilde{Q})$ intersects $g_k(\widetilde{Q})$ for some deck transformation g_k (see expressions (21) and (22)), then there exist $i_0, i_1 \in \{1, \ldots, J\}$, which depend on k, such that $\widetilde{f}^{N+n_k}(\widetilde{R}) \cap (m_{i_0}^{-1} g_k m_{i_1}(\widetilde{R})) \supset \widetilde{R}_1$, where $N > 0$ is given in (20) and $\widetilde{R}_1$ is a 'vertical rectangle' in $m_{i_0}^{-1} g_k m_{i_1}(\widetilde{R})$: two of its sides are contained, one in $m_{i_0}^{-1} g_k m_{i_1}(\widetilde{\beta})$ and the other in $m_{i_0}^{-1} g_k m_{i_1}(\widetilde{\beta}')$ and the two other sides are contained in the interior of $m_{i_0}^{-1} g_k h_{i_1}(\widetilde{R})$, each one connecting a point from one of the previous sides to the other. Clearly, $\widetilde{f}^{N+N_{\widetilde{Q}}}(\widetilde{R}) \cap \widetilde{R} \supset \widetilde{R}_0$, a rectangle similar to $\widetilde{R}_1$, but contained in $\widetilde{R}$.*

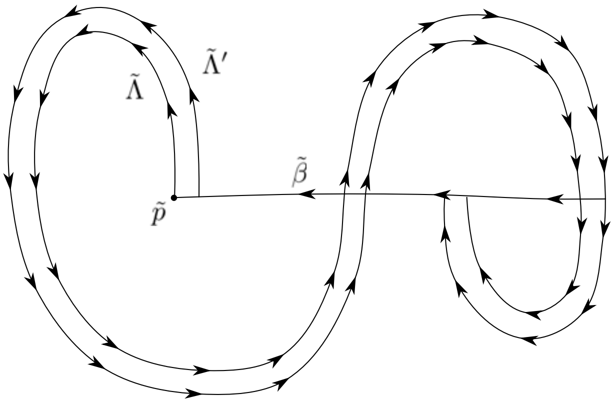

FIGURE 16. How to obtain the sets $\widetilde{M}_{\widetilde{\Lambda}}$, $\widetilde{M}_{\widetilde{\Lambda}'}$ and $\widetilde{M}_{\min}$.

Proof. Define

$$\widetilde{M}_{\widetilde{\Lambda}} = \text{filled}(\widetilde{\beta} \cup \widetilde{\Lambda}), \quad \widetilde{M}_{\widetilde{\Lambda}'} = \text{filled}(\widetilde{\beta} \cup \widetilde{\Lambda}') \quad \text{and} \quad \widetilde{M}_{\min} = \widetilde{M}_{\widetilde{\Lambda}} \cap \widetilde{M}_{\widetilde{\Lambda}'}, \tag{23}$$

where, for any compact connected subset $\widetilde{K}$ of $\mathbb{D}$,

$$\text{filled}(\widetilde{K}) = \widetilde{K} \cup \{\text{all bounded connected components of } \widetilde{K}^c\}.$$

It is well known that $\text{fill}(\widetilde{K})^c$ is open, connected and unbounded.

From the choice of $\widetilde{\Lambda}$, $\widetilde{\Lambda}'$ and $\widetilde{\beta}$, the sets $\widetilde{M}_{\widetilde{\Lambda}}$, $\widetilde{M}_{\widetilde{\Lambda}'} and \widetilde{M}_{\min}$ are connected and the complement of any of the three sets $\pi(\widetilde{M}_{\widetilde{\Lambda}})$, $\pi(\widetilde{M}_{\widetilde{\Lambda}'})$, $\pi(\widetilde{M}_{\min})$ is a union of open disks, see Figure 16. So, given a fundamental domain $\widetilde{Q} \subset \mathbb{D}$ of S, there exist deck transformations $\{m_1, m_2, \ldots, m_J\}$, for some $J > 0$, such that

$$\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})$$

is a (bounded) connected closed set and its complement has a bounded connected component denoted $\widetilde{\theta}$ which contains $\widetilde{Q}$. Moreover,

$$d_{\mathbb{D}}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min}),\ \widetilde{Q}\right) > 1.$$

In particular, this implies that $\widetilde{Q} \subset \text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min}))$.

The reason why the above is true is as follows: $\pi^{-1}(\pi(\widetilde{M}_{\min}))$ is a closed connected equivariant subset of $\mathbb{D}$ and its complement has only open topological disks as connected components, all with diameters uniformly bounded from above. Let $\widetilde{\Gamma}$ be a simple closed curve which surrounds $\widetilde{Q}$ and such that

$$d_{\mathbb{D}}(\widetilde{\Gamma}, \widetilde{Q}) > 1.$$

As $\widetilde{\Gamma}$ is compact and $\widetilde{\Lambda} \pitchfork h_i(\widetilde{\beta})$, $\widetilde{\Lambda}' \pitchfork h_i(\widetilde{\beta})$, for all $1 \le i \le 2g$, there exists deck transformations $\{m_1, m_2, \ldots, m_J\}$, for some $J > 0$, such that $\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})$ is connected and its complement has a bounded connected component (the one we previously denoted by $\widetilde{\theta}$) which contains $\widetilde{\Gamma}$. Moreover, if ν is a simple arc which avoids unstable manifolds of periodic saddle points and ν connects a point in the unbounded connected component of $(\text{Neighborhood}_{1/5}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})))^c$ to a point in $\widetilde{Q}$, then, for some $i \in \{1, 2, \ldots, J\}$, ν must cross $m_i(\widetilde{R})$ from $m_i(\widetilde{\beta})$ to $m_i(\widetilde{\beta}')$ or *vice-versa*. This happens because, as $\text{diameter}(R) < 2 \cdot \epsilon < 1/5$,

$$\text{Neighborhood}_{1/5}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})\right) \supset \bigcup_{i=1}^{J} m_i(\widetilde{R}).$$

Assume that $k > 0$ is sufficiently large, so that $n_k > 2 \cdot N$ and

$$\text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'})\right) \cap g_k\left(\text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'} \cup \widetilde{R})\right)\right) = \emptyset.$$

This implies the following remark.

Claim 5. $\widetilde{f}^{n_k}(\text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'})))$ does not intersect $g_k(\bigcup_{i=1}^{J} m_i(\widetilde{\Lambda} \cup \widetilde{\Lambda}'))$.

Proof of the claim. Otherwise, if some point

$$\widetilde{z} \in g_k\left(\bigcup_{i=1}^{J} m_i(\widetilde{\Lambda} \cup \widetilde{\Lambda}')\right) \cap \widetilde{f}^{n_k}\left(\text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'})\right)\right),$$

then $\widetilde{f}^{-n_k}(\widetilde{z}) \in g_k(\bigcup_{i=1}^{J} m_i(\widetilde{\alpha})) \cap \text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'}))$, which is contained in

$$g_k\left(\text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'} \cup \widetilde{R})\right)\right) \cap \text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'})\right) = \emptyset.$$

But this is a contradiction. □

The previous claim, although simple, will be very important.

As $\widetilde{q}_w \in \widetilde{Q} \subset \text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min}))$ and $\widetilde{f}^{n_k}(\widetilde{q}_w) \in g_k(\widetilde{Q})$, we can argue as follows: consider the connected components of

$$\text{interior}\left\{\text{filled}\left[\widetilde{f}^{n_k}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})\right) \cup \left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})\right)\right] \cap \left(\text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})\right)\right)^c\right\}.$$

From the existence of $\widetilde{q}_w$, as above, there is one such connected component, denoted by $\widetilde{C}_k$, which intersects $g_k(\widetilde{Q})$. The boundary of $\widetilde{C}_k$ is a Jordan curve, made of two simple arcs which only intersect at their endpoints: one arc is contained in $\partial(\text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})))$ and its endpoints are in $\bigcup_{i=1}^{J} m_i(\widetilde{\beta})$ and the other arc is equal to $\widetilde{f}^{n_k}(\widetilde{\xi})$, where $\widetilde{\xi}$ is an arc either contained in $m_{i_0}(\widetilde{\Lambda})$ or $m_{i_0}(\widetilde{\Lambda}')$ (assume it is $m_{i_0}(\widetilde{\Lambda})$), for some $i_0 \in \{1, \ldots, J\}$.

As both endpoints of $\widetilde{f}^{n_k}(\widetilde{\xi})$ are contained in $\bigcup_{i=1}^{J} m_i(\widetilde{\beta}) \subset \text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min}))$, there exists some $i_1 \in \{1, \ldots, J'\}$ such that $\widetilde{f}^{n_k}(\widetilde{\xi})$ crosses $g_k.m_{i_1}(\widetilde{R})$ from outside $g_k(\text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'} \cup \widetilde{R})))$ to inside $g_k(\widetilde{Q})$: that is, it crosses $g_k.m_{i_1}(\widetilde{R})$ from $g_k.m_{i_1}(\widetilde{\beta})$ to $g_k.m_{i_1}(\widetilde{\beta}')$, or *vice-versa*, in order to intersect $g_k(\widetilde{Q})$.

From the definition of $\widetilde{M}_{\min}$ (see (23)), and our assumption that $\widetilde{\xi}$ is contained in $m_{i_0}(\widetilde{\Lambda})$, there exists an arc $\widetilde{\xi}' \subset m_{i_0}(\widetilde{\Lambda}')$, whose endpoints are also contained in $\bigcup_{i=1}^{J} m_i(\widetilde{\beta})$), such that

$$\widetilde{\xi} \subset \text{interior}\left(\text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min}) \cup \widetilde{\xi}'\right)\right). \tag{24}$$

This implies that

$$\text{Strip}_{[\widetilde{\xi},\widetilde{\xi}']} = \text{closure}\left(\text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min}) \cup \widetilde{\xi}'\right) \Big\backslash \text{filled}\left(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})\right)\right)$$

has two types of boundary points:

- an inner boundary, contained in $\partial(\text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\min})))$ and containing $\widetilde{\xi}$; and
- an outer boundary, equal to $\widetilde{\xi}'$.

The inclusion in (24), together with the facts that $\widetilde{f}^{n_k}(\widetilde{\xi})$ is the part of the boundary of $\widetilde{C}_k$ which crosses $g_k.m_{i_1}(\widetilde{R})$ from outside $g_k(\text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'} \cup \widetilde{R})))$ to inside and that $\widetilde{f}^{n_k}(\text{Strip}_{[\widetilde{\xi},\widetilde{\xi}']}) \cap g_k(\bigcup_{i=1}^{J} m_i(\widetilde{\Lambda} \cup \widetilde{\Lambda}')) = \emptyset$ (true by Claim 5), imply that $\widetilde{f}^{n_k}(\widetilde{\xi}')$ also has to cross $g_k.m_{i_1}(\widetilde{R})$ from outside $g_k(\text{filled}(\bigcup_{i=1}^{J} m_i(\widetilde{M}_{\widetilde{\Lambda}} \cup \widetilde{M}_{\widetilde{\Lambda}'} \cup \widetilde{R})))$ to inside. This implies the existence of a 'rectangle', as in the statement of the lemma contained in

$$\widetilde{f}^{n_k}(\text{Strip}_{[\widetilde{\xi},\widetilde{\xi}']}) \cap g_k m_{i_1}(\widetilde{R}).$$

So $\widetilde{f}^{n_k+N}(m_{i_0}(\widetilde{R})) \cap g_k.m_{i_1}(\widetilde{R})$ contains such a 'rectangle' and thus

$$\widetilde{f}^{n_k+N}(\widetilde{R}) \cap m_{i_0}^{-1} g_k m_{i_1}(\widetilde{R}) \supset \widetilde{R}_1.$$

Clearly, $\widetilde{f}^{n_k+N}(\widetilde{R}) \cap (\widetilde{R}) \supset \widetilde{R}_0$, by our choice of $\widetilde{\Lambda}$ and $\widetilde{\Lambda}'$. □

So we can finally build a 'topological horseshoe': arguing exactly as when all crossings are C^1-transversal, it can be proved that for every bi-infinite sequence in $\{0, 1\}^{\mathbb{Z}}$, denoted by $(a_n)_{n\in\mathbb{Z}}$, there is a compact set which realizes it (not necessarily a point, as in the C^1-transverse case; see [**6**], and also [**2**], for a simpler application of the above construction).

If we denote by $M_k \subset R$ the compact set associated with the sequence $(1)_{\mathbb{Z}}$ and $\widetilde{M}_k = \pi^{-1}(M_k) \cap \widetilde{R}$, then, by our construction, $\widetilde{f}^{m(N+n_k)}(\widetilde{M}_k) = (m_{i_0}^{-1} g_k m_{i_1})^m(\widetilde{M}_k)$, for all $m > 0$. In particular, if $r \in M_k$ and $\widetilde{r} \in \pi^{-1}(r) \cap \widetilde{M}_k$, then $\widetilde{f}^{m(N+n_k)}(\widetilde{r}) \in (m_{i_0}^{-1} g_k m_{i_1})^m(\widetilde{M}_k)$, for all $m > 0$.

By our choice of R, for all $m > 0$, we can find β'_m, a path in R joining $f^{m(N+n_k)}(r)$ to r with $l(\beta'_m) < 2\epsilon$. Thus, if $\widetilde{I}_{\widetilde{r}}^{m(N+n_k)} * \widetilde{\beta}'_m$ is the lift of $I_r^{m(N+n_k)} * \beta'_m$ with base point $\widetilde{r}$, then $\widetilde{I}_{\widetilde{r}}^{m(N+n_k)} * \widetilde{\beta}'_m$ is a path in $\mathbb{D}$ joining $\widetilde{r}$ to $(m_{i_0}^{-1} g_k m_{i_1})^m(\widetilde{r})$. In particular,

$$\frac{[I_p^{m(N+n_k)} * \beta'_m]}{m(N+n_k)} = \frac{[(m_{i_0}^{-1} g_k m_{i_1})^m]}{m(N+n_k)} = \frac{m[m_{i_0}^{-1} g_k m_{i_1}]}{m(N+n_k)} = \frac{[m_{i_0}^{-1}] + [g_k] + [m_{i_1}]}{N+n_k}.$$

As $w = \lim_{k\to\infty}[g_k]/n_k$, $N > 0$ is fixed and there is just a finite number of possibilities for m_{i_0} and m_{i_1}, if $k > 0$ is large enough, then

$$\frac{[m_{i_0}^{-1}] + [g_k] + [m_{i_1}]}{N+n_k}$$

is as close as we want to w.

So given an error > 0, if $k > 0$ is sufficiently large, defining

$$g_w = m_{i_0}^{-1} g_k m_{i_1} \quad \text{and} \quad n_w = N + n_k, \tag{25}$$

we get that

$$\left\| \frac{[g_w]}{n_w} - w \right\| < \text{error}.$$

Using the above construction, we will show that $\rho_{mz}(f) = \text{Conv}(\rho_{mz}(f))$. For this we need Steinitz's theorem [**17**]. This theorem says that if a point is interior to the convex hull of a set X in $\mathbb{R}^n$, it is interior to the convex hull of some set of $2n$ or fewer points of X.

Since $\rho_{mz}(f)$ is a compact set, $\text{Conv}(\rho_{mz}(f)) = \text{Conv}(\text{Ext}(\rho_{mz}(f)))$, where $\text{Ext}(\rho_{mz}(f))$ is the set of all extremal points of $\text{Conv}(\rho_{mz}(f))$. Using Steinitz's theorem, any point in the interior of $\text{Conv}(\rho_{mz}(f))$ is a convex combination of at most $4g$ extremal points.

Rational case. Let v be a point in $\text{int}(\text{Conv}(\rho_{mz}(f))) \cap \mathbb{Q}^{2g}$. By the previous observation, there exists at most $4g$ extremal points (here, without loss of generality, we will assume that exactly $4g$ extremal points are used) $w_1, \ldots, w_{4g}$ such that

$$v = \sum_{i=1}^{4g} \lambda_i w_i,$$

where $\lambda_i \in]0, 1[$, for all $1 \le i \le 4g$ and $\lambda_1 + \cdots + \lambda_{4g} = 1$. By the previous construction, for some general w, choose deck transformations $g_{w_1}, \ldots, g_{w_{4g}}$ and natural numbers $n_{w_1}, \ldots, n_{w_{4g}}$ (as in expression (25)) such that

$$v \in \text{int}\left(\text{Conv}\left(\frac{[g_{w_1}]}{n_{w_1}}, \ldots, \frac{[g_{w_{4g}}]}{n_{w_{4g}}}\right)\right).$$

This is always possible since $[g_{w_i}]/n_{w_i}$ can be chosen as close as desired to w_i. As $[g_{w_i}]/n_{w_i} \in \mathbb{Q}^{2g}$ for all $1 \le i \le 4g$, and v is a rational point in the interior of the convex hull of these points, there exist $\lambda'_1, \ldots, \lambda'_{4g}$, with $\lambda'_i \in (0, 1) \cap \mathbb{Q}$, $\lambda'_1 + \cdots + \lambda'_{4g} = 1$, such that

$$v = \sum_{i=1}^{4g} \lambda'_i \frac{[g_{w_i}]}{n_{w_i}}.$$

Thus, multiplying both sides of the previous equation by an appropriate positive integer, we get positive integers $a_{\text{Total}}, a_1, \ldots, a_{4g}$ such that $a_{\text{Total}} = a_1 + \cdots + a_{4g}$ and

$$a_{\text{Total}} v = \sum_{i=1}^{4g} a_i \frac{[g_{w_i}]}{n_{w_i}},$$

For each $i \in \{1, 2, \ldots, 4g\}$, $\widetilde{f}^{n_{w_i}}(\widetilde{R})$ intersects $g_{w_i}(\widetilde{R})$ in a vertical rectangle, as in Lemma 25. Since $\widetilde{f}$ commutes with every deck transformation, $\widetilde{f}^{n_{w_j}}(g_{w_i}(\widetilde{R}))$ intersects $g_{w_i} g_{w_j}(\widetilde{R})$ in a similar rectangle, see Figure 17.

Let $N_{\text{product}} = n_{w_1} n_{w_2} \cdots n_{w_{4g}}$ and, for all $1 \le i \le 4g$, let $u_i = N_{\text{product}}/n_{w_i}$. By the previous definitions, $\widetilde{f}^{(a_i u_i) n_{w_i}}(\widetilde{R}) = \widetilde{f}^{a_i N_{\text{product}}}(\widetilde{R})$ satisfies that $\widetilde{f}^{(a_i u_i) n_{w_i}}(\widetilde{R}) \cap g_{w_i}^{a_i u_i}(\widetilde{R})$ contains a vertical rectangle, as in Lemma 25. So, considering all iterates of this type

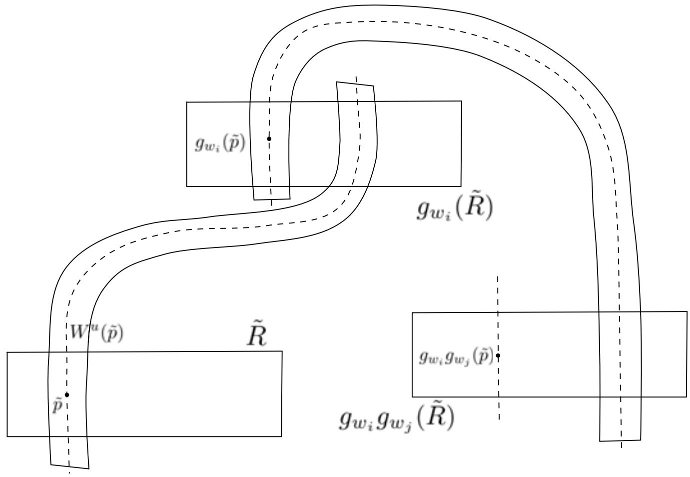

FIGURE 17. How to create intersections between iterates of $\widetilde{R}$ and its translates.

for $1 \leq i \leq 4g$ and composing them, we obtain that $\widetilde{f}^{a_{\text{Total}} N_{\text{product}}}(\widetilde{R}) \cap h_v(\widetilde{R})$ contains a vertical rectangle, as in Lemma 25, where $h_v = g_{w_1}^{a_1 u_1} \circ g_{w_2}^{a_2 u_2} \circ \cdots \circ g_{w_{4g}}^{a_{4g} u_{4g}}$.

Clearly, as $\widetilde{f}^{a_{\text{Total}} N_{\text{product}}}(\widetilde{R}) \cap \widetilde{R}$ contains a vertical rectangle similar to $\widetilde{R}_0$, just thinner, we can consider the compact $f^{a_{\text{Total}} N_{\text{product}}}$-invariant subset $K_v \subset R$ of the topological horseshoe we just produced associated with the sequence $(1)_{\mathbb{Z}}$. If $\widetilde{K}_v = \widetilde{R} \cap \pi^{-1}(K_v)$, then

$$\widetilde{f}^{a_{\text{Total}} N_{\text{product}}}(\widetilde{K}_v) = h_v(\widetilde{K}_v).$$

So $h_v^{-1} \widetilde{f}^{a_{\text{Total}} N_{\text{product}}}(\widetilde{K}_v) = \widetilde{K}_v$, which implies, using Brouwer's lemma on translation arcs [**5**], that $h_v^{-1} \widetilde{f}^{a_{\text{Total}} N_{\text{product}}}$ has a fixed point $\widetilde{z}_v$. Since

$$\widetilde{f}^{a_{\text{Total}} N_{\text{product}}}(\widetilde{z}_v) = h_v(\widetilde{z}_v)$$

and

$$\frac{[h_v]}{a_{\text{Total}} N_{\text{product}}} = \frac{[g_{w_1}^{a_1 u_1} \circ g_{w_2}^{a_2 u_2} \circ \cdots \circ g_{w_{4g}}^{a_{4g} u_{4g}}]}{a_{\text{Total}} N_{\text{product}}} = \sum_{i=1}^{4g} \frac{a_i u_i [g_{w_i}]}{a_{\text{Total}} N_{\text{product}}}$$

$$= \frac{1}{a_{\text{Total}}} \sum_{i=1}^{4g} a_i \frac{[g_{w_i}]}{n_{w_i}} = v,$$

we conclude that the f-periodic point $z_v = \pi(\widetilde{z}_v)$ has a rotation vector $\rho(f, z_v) = v$. This shows that $v \in \rho_{mz}(f)$. Since $\rho_{mz}(f)$ is compact, $\rho_{mz}(f) = \text{Conv}(\rho_{mz}(f))$.

Irrational case. For any $v \in (\mathbb{Q}^{2g})^c \cap \text{int}(\rho_{mz}(f))$, exactly as in the rational case, one can find $4g$ rational points $w_1, \ldots, w_{4g}$ in $\rho_{mz}(f)$ for which

$$v \in \text{int}(\text{Conv}(\{w_1, \ldots, w_{4g}\}))$$

and such that, for positive integers $n_{w_1}, \ldots, n_{w_{4g}}$, $\widetilde{f}^{n_{w_i}}(\widetilde{R}) \cap g_{w_i}(\widetilde{R})$ contains a vertical rectangle, as in Lemma 25, for some $g_{w_i} \in \text{Deck}(\pi)$ such that $[g_{w_i}]/n_{w_i} = w_i$.

As above, let $N_{\text{product}} = n_{w_1} \cdots n_{w_{4g}}$ and $u_i = N_{\text{product}}/n_{w_i}$. Then, $\widetilde{f}^{N_{\text{product}}}(\widetilde{R}) \cap g_{w_i}^{u_i}(\widetilde{R})$ also contains a vertical rectangle $\widetilde{R}_i$ as in Lemma 25.

Clearly,

$$\frac{[g_{w_i}^{u_i}]}{N_{\text{product}}} = \frac{u_i \cdot [g_{w_i}]}{N_{\text{product}}} = w_i.$$

So, going back to the surface S,

$$f^{N_{\text{product}}}(R) \cap R \supseteq R_1 \cup \cdots \cup R_{4g} \quad \text{where } R_i = \pi(\widetilde{R}_i).$$

We claim that there exists an infinite sequence in $\{1, \ldots, 4g\}^{\mathbb{N}}$, denoted by

$$a_1 a_2 \cdots a_n \cdots$$

such that, for some constant $C^* > 0$,

$$\left\| \sum_{i=1}^{n} [g_{w_{a_i}}^{u_{a_i}}] - n N_{\text{product}} \cdot v \right\| < C^* \quad \text{for all } n > 0.$$

The existence of this kind of sequence is what is behind, in the case of a torus, the realization of irrational rotation vectors in the interior of the rotation set by compact invariant sets with bounded mean motion in the universal cover. This was done for relative pseudo-Anosov maps in [**25**, Lemma 3] and was extended to the original map using a shadowing result, similar to Theorem 17 (see [**25**] for details).

Now let $z \in R$ be any point which corresponds to the sequence $a_1 a_2 \cdots a_n \cdots$, namely, $f^{nN_{\text{product}}}(z) \in R_{a_n}$, for all $n \geq 1$. Clearly, for $\widetilde{z} \in \widetilde{R} \cap \pi^{-1}(z)$ and any $n \geq 1$,

$$\widetilde{f}^{n \cdot N_{\text{product}}}(\widetilde{z}) \in g_{w_{a_1}}^{u_{a_1}} \cdot g_{w_{a_2}}^{u_{a_2}} \cdots g_{w_{a_n}}^{u_{a_n}}(\widetilde{R}),$$

so not only the rotation vector of z is v, but

$$\begin{aligned} \|[\alpha_z^l] - l \cdot v\| &< C^* + N_{\text{product}} \cdot \|v\| + 2\epsilon \\ &\quad + \max\{d_{\mathbb{D}}(\widetilde{f}^i(\widetilde{z}), \widetilde{z}) : \widetilde{z} \in \mathbb{D} \text{ and } 0 \leq i \leq N_{\text{product}}\}. \end{aligned}$$

This implies that the ω-limit set of z, denoted by K_v, has the property we are looking for because, for any $z' \in K_v$, $\|[\alpha_{z'}^n] - n \cdot v\|$ is smaller than some constant which is independent of n and $z' \in K_v$. □

6. *Proof of Theorem 4*

Here we just make use of the machinery developed in the proof of Theorem 3.

Suppose, by contradiction, that, for every $M > 0$, there exists $\omega \in \partial\rho_{mz}(f)$, a supporting hyperplane $\omega \in H \subset \mathbb{R}^{2g}$, $z \in S$ and $n > 0$ such that

$$([\alpha_z^n] - n \cdot \omega) \cdot \overrightarrow{v_H} > M,$$

where $\overrightarrow{v_H}$ is the unitary normal vector to H pointing towards the connected component of H^c which does not intersect $\rho_{mz}(f)$.

We fix some fundamental domain of S, denoted by $\widetilde{Q} \subset \mathbb{D}$. Then there exists $\widetilde{z} = \pi^{-1}(z) \cap \widetilde{Q}$ such that, for some $g \in \text{Deck}(\pi)$,

$$\widetilde{f}^n(\widetilde{z}) \in g(\widetilde{Q}) \quad \text{and} \quad ([g] - n \cdot \omega) \cdot \overrightarrow{v_H} > M - C_{\widetilde{Q}},$$

where $C_{\widetilde{Q}} > 0$ is a constant which depends only on the shape of $\widetilde{Q}$. From the proof of the previous theorem, we know that there are deck transformations $\{m_1, m_2, \ldots, m_J\}$, for some $J > 0$, which do not depend on the choices of:

$$M > 0,\ \omega \in \partial\rho_{mz}(f), \quad \text{the supporting hyperplane } \omega \in H \subset \mathbb{R}^{2g},\ z \in S \text{ and } n > 0,$$

such that, for some i_0 and i_1 in $\{1, \ldots, J\}$, there exists a compact subset $\widetilde{K}_M$ for which

$$\widetilde{f}^{N+n}(\widetilde{K}_M) = m_{i_0}^{-1} g m_{i_1}(\widetilde{K}_M),$$

where $N > 0$ is given in expression (20). Thus for some point $\widetilde{z}_M \in \mathbb{D}$, $\widetilde{f}^{N+n}(\widetilde{z}_M) = m_{i_0}^{-1} g m_{i_1}(\widetilde{z}_M)$.

So, if $M > 0$ is large enough so that

$$([m_{i_0}^{-1} g m_{i_1}] - (n + N) \cdot \omega) \cdot \overrightarrow{v_H} > 0,$$

we get a contradiction. □

7. *Proof of Theorem 5*

This proof is very similar to the proof of Theorem 2 of [**2**]. In particular, the following lemma from that paper, which was proved for the torus, holds without any modifications under the hypotheses of the present paper.

LEMMA 26. (Adapted [**2**, Lemma 6]) *Suppose $f : S \to S$ is a $C^{1+\epsilon}$ diffeomorphism isotopic to the identity which has a fully essential system of curves $\mathscr{C}$. Let μ be a f-invariant Borel probability measure such that its rotation vector $\rho_m(f, \mu)$ belongs to $\partial\rho_{mz}(f)$. Let H be a supporting hyperplane at $\rho_m(f, \mu)$ and let $\overrightarrow{v_H}$ be the unitary vector orthogonal to H, pointing towards the connected component of H^c which does not intersect $\rho_{mz}(f)$. Then, if $x' \in \operatorname{supp}(\mu)$, for any integer $n > 0$,*

$$|([\alpha_{x'}^n] - n \cdot \rho_m(f, \mu)) \cdot \overrightarrow{v_H}| \leq 2 + M(f), \tag{26}$$

where $M(f) > 0$ comes from Theorem 4.

Now the proof continues exactly as the proof of Theorem 2 of [**2**].

Acknowledgements. The first author is grateful to Juliana Xavier for some conversations on this subject several years ago. After working on the torus case I asked her about ways to generalize the results to other surfaces and she explained several things to me. Both of us are grateful to André de Carvalho for several conversations and, as we already mentioned, to Alejandro Passeggi for pointing out that Theorem 3 was a consequence of Theorem 2. The first author is partially supported by CNPq, grant: 306348/2015-2 and the second author was supported by CNPq, grant: 159853/2013-3.

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