Boyland's conjecture for rotationless homeomorphisms of the annulus with two fixed points

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Abstract

Let f be a homeomorphism of the closed annulus A that preserves the orientation, the boundary components and the Lebesgue measure. Suppose that f has a lift $\tilde{f}$ to the infinite strip $\tilde{A}$ which has zero Lebesgue measure rotation number and $\tilde{f}$ has only two fixed points in each fundamental domain, both in the interior of $\tilde{A}$. In this case, zero is an interior point of the rotation set of $\tilde{f}$.

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1 Introduction and statement of the main result

In this note we consider homeomorphisms f of the closed annulus $A = S^1 \times [0,1]$, which satisfy certain special conditions, namely:

1. f preserves the orientation and the boundary components of A;
2. f preserves the Lebesgue measure;
3. there exists a special lift $\widetilde{f}$ of f to the universal cover of the annulus $\widetilde{A} = \mathbb{R} \times [0,1]$, satisfying the following:

   If $p_1 : \widetilde{A} \to \mathbb{R}$ is the projection on the first coordinate and $p : \widetilde{A} \to A$ is the covering mapping, we can define the displacement function $\phi : A \to \mathbb{R}$ as

$$\phi(x,y) = p_1 \circ \widetilde{f}(\widetilde{x},\widetilde{y}) - \widetilde{x}, \tag{1}$$

   for any $(\widetilde{x},\widetilde{y}) \in p^{-1}(x,y)$. Then the rotation number of the Lebesgue measure λ satisfies

$$\rho(\lambda) \stackrel{def.}{=} \int_A \phi d\lambda = 0.$$

Following the usual definition (see [3]), we refer to such mappings as rotationless homeomorphisms. Every time we say that f is a rotationless homeomorphism, a special lift $\widetilde{f}$ is fixed, and is used to define ϕ and the rotation set $\rho(\widetilde{f}) \stackrel{def.}{=} \{\omega \in \mathbb{R} : \omega = \int_A \phi d\mu$ for some Borel probability f-invariant measure $\mu\}$.

Boyland's Conjecture says that if the rotation set of a rotationless homeomorphism of the annulus is not reduced to a point, then zero is an interior point of it.

Our theorem is the following:

Theorem 1 *: If $\widetilde{f}$ has exactly two fixed points in each fundamental domain of $\widetilde{A}$, both not in $\partial\widetilde{A}$, then its rotation set is an interval and 0 is an interior point of it.*

Remark:

The above theorem is not complete solution to Boyland's conjecture because of our assumption on the fixed points of $\widetilde{f}$. Although this may not look like a very restrictive assumption, a glance at our proof shows that it can not be easily adapted to the general setting.

2 An auxiliary lemma

Lemma 1 *: If f is a rotationless homeomorphism, then it has at least one fixed point of zero rotation number in the interior of the annulus.*

Proof:

Consider some $\widetilde{z} \in [0,1]\times]0,1[$ which is not fixed under $\widetilde{f}$. Then there exists a ball $B \subset [0,1]\times]0,1[$ centered at $\widetilde{z}$ such that $B \cap \widetilde{f}(B) = \emptyset$. If for some integer $n > 1$, $B \cap \widetilde{f}^n(B) \neq \emptyset$, then Brouwer's theory implies that $\widetilde{f}$ has a fixed point in $interior(\widetilde{A})$. So assume that B is wandering. In this case, as $\widetilde{f}$ preserves area, for each integer $k > 0$, if N is the first integer larger than $10k/\lambda(B) > 0$, then there exists $n_k \in \{1, 2, ..., N\}$ such that

$$\lambda(\widetilde{f}^{n_k}(B)) \cap (]-\infty, -k-1] \times [0,1] \cup [k+1, +\infty[\times[0,1]) > \frac{1}{2}\lambda(B).$$

Taking a subsequence if necessary, we can suppose without loss of generality, that for each integer $k > 0$, $\lambda(\widetilde{f}^{n_k}(B)) \cap [k+1, +\infty[\times[0,1] > \frac{1}{2}\lambda(B)$. But this means that there exists a subset $B_k \subset B$, $\lambda(B_k) > \lambda(B)/2$, such that for any $\widetilde{z} \in B_k$,

$$\frac{p_1(\widetilde{f}^{n_k}(\widetilde{z})) - p_1(\widetilde{z})}{n_k} \geq \frac{k}{11k/\lambda(B)} = \frac{\lambda(B)}{11} > 0.$$

As $\rho(\lambda) = 0$, the facts that

- the above expression is true for every integer $k > 0$;
- $\lambda(B_k) > \lambda(B)/2$;

imply that 0 is an interior point of $\rho(\widetilde{f})$. Finally using the main result of [6], which says that if $\widetilde{f}$ has positively and negatively returning disks, then there are fixed points in the interior of the annulus, we obtain a fixed point for $\widetilde{f}$ in $interior(\widetilde{A})$. $\square$

3 Proof of the main theorem

First, note that there is a version of the Conley-Zehnder theorem for the annulus which implies that every rotationless homeomorphism g has at least two fixed points with zero rotation number, that is, the special lift $\widetilde{g}$ has at least two fixed points in each fundamental domain. See, for instance, corollary 3.3 of [4] to obtain at least one fixed point, and the main theorem of [7] to get the second fixed point. We are assuming that f has exactly two fixed points with zero rotation number, say z_1 and z_2, and they both belong to the $interior(A)$.

If the rotations in $S^1 \times \{0\}$ and $S^1 \times \{1\}$ have opposite sign, then the theorem is proved because in this case $\rho(\widetilde{f})$ would have negative and positive elements. So, as we are assuming that there are no fixed points in the boundary of $\widetilde{A}$, without loss of generality we can suppose that the rotation numbers in the boundary components of the annulus are both positive.

As in [1] and [2], let B^- be the union of all unbounded connected components of

$$B = \bigcap_{n \leq 0} \widetilde{f}^n(]-\infty, 0] \times [0,1]),$$

which by theorem 1 of [1] is not empty and intersects $\{0\} \times [0,1]$. Clearly, $B^- \cap \partial\widetilde{A} = \emptyset$ and $\widetilde{f}(B^-) \subset B^-$.

Lemma 3.1 of [2] says that if $\omega(B^-) \stackrel{def}{=} \bigcap_{n=0}^{\infty} \overline{\left(\bigcup_{i=n}^{\infty} \widetilde{f}^i(B^-)\right)} = \emptyset$, then $\rho(\widetilde{f})$ contains negative values and we are done. Thus, we can suppose that $\omega(B^-) \neq \emptyset$.

Since the rotation number of $\widetilde{f}$ restricted to $S^1 \times \{0\}$ and $S^1 \times \{1\}$ is strictly positive, there exists $\sigma > 0$ such that $p_1(\widetilde{f}(\widetilde{x}, i)) > \widetilde{x} + 2\sigma$ for all $\widetilde{x} \in \mathbb{R}$ and $i = 0, 1$. Let $\epsilon > 0$ be sufficiently small such that for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times \{[0,\epsilon] \cup [1-\epsilon, 1]\}$, $p_1 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) > \widetilde{x} + \sigma$.

Let us first consider the case when $S^1 \times \{[0, \epsilon/2] \cup [1-\epsilon/2, 1]\}$ intersects $\overline{p(\omega(B^-))}$. Then there is a real a such that

$$\omega(B^-) \cap \{a\} \times [0,\epsilon] \neq \emptyset \text{ or } \omega(B^-) \cap \{a\} \times [1-\epsilon, 1] \neq \emptyset. \tag{2}$$

Without loss of generality, we can suppose that the first intersection in expression (2) is non-empty. The fact that $\omega(B^-) \subset B^-$ is closed implies that there must be a $\delta \leq \epsilon$ such that $(a, \delta) \in \omega(B^-)$, and such that for all $0 \leq \tilde{y} < \delta, (a, \tilde{y}) \notin \omega(B^-)$ (remember that $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ do not intersect B^-). In other words, (a, δ) is the "lowest" point of $\omega(B^-)$ in $\{a\} \times [0, \epsilon]$. We denote by v the interval $\{a\} \times [0, \delta[$.

Let Ω be the connected component of $(\omega(B^-) \cup v)^c$ that contains $]-\infty, a[\times\{0\}$. Now we remember two results from [2] (proposition 3.2 and corollary 2.7):

Lemma 2 *: The following holds:* $\Omega \subset \tilde{f}(\Omega)$ *and* $\lambda(\tilde{f}(\Omega) \setminus \Omega) > 0$.

Lemma 3 *: Let* $(f, \tilde{f})$ *be a rotationless homeomorphism and let* Ω *be an open subset of the strip,* $\Omega \subset (-\infty, 0] \times [0,1]$, *such that* $\Omega \subset \tilde{f}(\Omega)$ *and* $\lambda(\tilde{f}(\Omega) \setminus \Omega) > 0$. *Then* 0 *is an interior point of the rotation set of* $\tilde{f}$.

So, in this case, our main theorem follows.

Let us deal with the remaining case, when $S^1 \times \{[0, \epsilon/2] \cup [1-\epsilon/2, 1]\} \cap \overline{p(\omega(B^-))} = \emptyset$, but $\omega(B^-)$ is not empty. Let A_* be the connected component of $\left(\overline{p(\omega(B^-))}\right)^c$ which contains $S^1 \times \{0\}$. The next result is lemma 3.3 of [2]:

Lemma 4 *: The set* A_* *is a* f*-invariant open sub-annulus.*

The boundary of A_* has two connected components, one is $S^1 \times \{0\}$ and the other is denoted by K. Clearly, $K \subset S^1 \times [\epsilon/2, 1-\epsilon/2]$ and as A_* is a f-invariant annulus, we can compute the rotation number of the Lebesgue measure restricted to it, $\int_{A_*} \phi d\lambda$. If it is negative, then the proof is over. If it is positive, then $\int_{A_*^c} \phi d\lambda < 0$ and the proof is also over. So, in the remainder of our proof we suppose it is zero.

An important thing to consider is the rotation set of $\widetilde{f}$ restricted to K, defined as follows:

$$\rho(\widetilde{f}\mid_K) = \{\omega \in \mathbb{R} : \omega = \textstyle\int_A \phi d\mu \text{ for some Borel probability } f\text{-invariant measure } \mu \text{ supported on } K\}$$

It is clearly a non-empty closed interval or a single point. If $\rho(\widetilde{f}\mid_K)$ does not contain 0, then either it is contained in $]-\infty,0[$ and our theorem is proved because we assumed that the boundary components of the annulus have positive rotation number or $\rho(\widetilde{f}\mid_K)$ is contained in $]0,\infty[$. In this case, clearly $B^- \cap (S^1 \times \{0\} \cup K) = \emptyset$ and if we define

$$B_*^- = B^- \cap A_*,$$

we get that $B_*^- \neq \emptyset$. This follows from the proof of theorem 1 of [1] (remember that $B^- \cap (S^1 \times \{0\} \cup K) = \emptyset$) and the fact that the rotation number of the Lebesgue measure restricted to A_* is zero. As $\widetilde{f}(B_*^-) \subset B_*^-$, we get that

$$\omega(B_*^-) \stackrel{def}{=} \bigcap_{n=0}^{\infty} \overline{\left(\bigcup_{i=n}^{\infty} \widetilde{f}^i(B_*^-) \right)} = \omega(B^-) \cap A_* = \emptyset,$$

so our theorem follows from lemma 3.1 of [2]. Thus we are left to consider the case when $0 \in \rho(\widetilde{f}\mid_K)$. In this case, proposition 5.4 of [5] says that a fixed point of f with zero rotation number belongs to K. Using lemma 1, one concludes that f has a fixed point with zero rotation number in A_*. So, it follows that $\{z_1, z_2\} \subset closure(A_*)$.

Finally, let A_{**} be the connected component of $\left(\overline{p(\omega(B^-))}\right)^c$ which contains $S^1 \times \{1\}$. As above, A_{**} is a f-invariant open sub-annulus. As it does not have fixed points with zero rotation number, lemma 1 implies that $\int_{A_{**}} \phi d\lambda$ can not be zero, proving our main theorem because $\left(\int_{A_{**}} \phi d\lambda\right) . \left(\int_{A_{**}^c} \phi d\lambda\right) < 0$.

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