Uniform bounds for diffeomorphisms of the torus and a conjecture of P. Boyland

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Abstract

We consider $C^{1+\epsilon}$ diffeomorphisms of the torus, denoted f, homotopic to the identity and whose rotation sets have interior. We give some uniform bounds on the displacement of points in the plane under iterates of a lift of f, relative to vectors in the boundary of the rotation set and we use these estimates in order to prove that if such a diffeomorphism f preserves area, then the rotation vector of the area measure is an interior point of the rotation set. This settles a strong version of a conjecture proposed by P. Boyland. We also present some new results on the realization of extremal points of the rotation set by compact f-invariant subsets of the torus.

Key words: sub-linear displacement, topological Horseshoes, Atkinson's lemma

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1 Introduction and main results

The main motivation for this paper is to study how rigid is the displacement of points in the plane under the action of a lift of a homeomorphism of the two dimensional torus homotopic to the identity (more precise explanations will be given below). The similar problem for an orientation preserving homeomorphism of the circle was already studied by H. Poincaré. He proved that given an orientation preserving circle homeomorphism $f : S^1 \to S^1$ and a lift of f to the real line, denoted $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, there exists a number $\omega \in \mathbb{R}$, called the rotation number of $\widetilde{f}$, such that

$$\left|\widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.\omega\right| < 2, \text{ for all } \widetilde{x} \in \mathbb{R} \text{ and any integer } n > 0.$$

The situation for homeomorphisms of the torus is more complicated. In general there is no such ω as above and some points may not even have a rotation vector, the generalization of rotation number to this new setting. In order to make things precise and to present our main results and some motivation, a few definitions are necessary:

Basic notation and some definitions:

1. Let $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus and let $p : \mathbb{R}^2 \longrightarrow \mathrm{T}^2$ be the associated covering map. Coordinates are denoted as $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$ and $(x, y) \in \mathrm{T}^2$.

2. Let $Diff_0^{1+\epsilon}(\mathrm{T}^2)$ be the set of $C^{1+\epsilon}$ (for some $\epsilon > 0$) diffeomorphisms of the torus homotopic to the identity and let $Diff_0^{1+\epsilon}(\mathbb{R}^2)$ be the set of lifts of elements from $Diff_0^{1+\epsilon}(\mathrm{T}^2)$ to the plane. Maps from $Diff_0^{1+\epsilon}(\mathrm{T}^2)$ are denoted f and their lifts to the plane are denoted $\widetilde{f}$. By $Diff_0^0(\mathrm{T}^2)$ we mean the set of homeomorphisms of T^2 homotopic to the identity and $Diff_0^0(\mathbb{R}^2)$ is the set of lifts of elements from $Diff_0^0(\mathrm{T}^2)$ to the plane. In this C^0-setting, maps of the torus are also denoted f and their lifts to the plane are denoted $\widetilde{f}$.

3. Let $p_{1,2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the standard projections; $p_1(\tilde{x}, \tilde{y}) = \tilde{x}$ and $p_2(\tilde{x}, \tilde{y}) = \tilde{y}$.

4. Given $f \in Diff_0^0(\mathrm{T}^2)$ and a lift $\widetilde{f} \in Diff_0^0(\mathrm{I\!R}^2)$, the so called rotation set of $\widetilde{f}$, $\rho(\widetilde{f})$, can be defined as follows (see [11]):

$$\rho(\widetilde{f}) = \bigcap_{i \geq 1} \overline{\bigcup_{n \geq i} \left\{ \frac{\widetilde{f}^n(\widetilde{z}) - \widetilde{z}}{n} : \widetilde{z} \in \mathrm{I\!R}^2 \right\}} \tag{1}$$

This set is a compact convex subset of $\mathrm{I\!R}^2$ (see [11]), and it was proved in [5] and [11] that all points in its interior are realized by compact f-invariant subsets of T^2, which can be chosen as periodic orbits in the rational case. By saying that some vector $\rho \in \rho(\widetilde{f})$ is realized by a compact f-invariant set, we mean that there exists a compact f-invariant subset $K \subset \mathrm{T}^2$ such that for all $z \in K$ and any $\widetilde{z} \in p^{-1}(z)$

$$\lim_{n \to \infty} \frac{\widetilde{f}^n(\widetilde{z}) - \widetilde{z}}{n} = \rho. \tag{2}$$

Moreover, the above limit, whenever it exists, is called the rotation vector of the point z, denoted $\rho(z)$.

As the rotation set is a compact convex subset of the plane, there are three possibilities for its shape:

1. it is a point;

2. it is a linear segment;

3. it has interior;

An important problem in this set up is to decide which subsets can be realized as rotation sets of homeomorphisms of the torus homotopic to the identity. For instance, with a simple rotation, all points can be realized. Some linear segments can be realized, for others it is not know. And what about the case when the rotation set has interior. Which sets can be realized? Rational polygons [9] can, but what else? We do not consider this problem, but we refer to [6] and [10].

In the first possibility above, Fábio Tal and Andrés Koropecki [8] presented an example of an area preserving C^∞ diffeomorphism of the torus homotopic to the identity, denoted f, which has a lift $\widetilde{f}$ to the plane such that $\rho(\widetilde{f}) = \{0\}$ and some points in the plane have unbounded orbits in every direction. In particular, there exists a point $\widetilde{x}_0 \in \mathbb{R}^2$ such that

$$\left|\widetilde{f}^n(\widetilde{x}_0) - \widetilde{x}_0 - n.0\right| \text{ is unbounded with } n > 0.$$

This type of behavior is usually called sub-linear displacement because, although there are unbounded $\widetilde{f}$-orbits in the plane, this behavior is not captured by the rotation set.

Related to the second possibility for the shape of the rotation set, Pablo Davalos [4] analyzed the following situation: Assume $f : \mathrm{T}^2 \to \mathrm{T}^2$ is a homeomorphism of the torus homotopic to the identity and $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of f such that some linear segment $\overline{AB}$ is contained in the boundary of $\rho(\widetilde{f})$ for some A, B rational vectors. He considered two situations:

- $\rho(\widetilde{f}) = \overline{AB}$;
- $\rho(\widetilde{f})$ has interior;

In the first case, let $\overrightarrow{v}^\perp$ be a unit vector orthogonal to $\overline{AB}$ with any of the two possible orientations and in the second, let $\overrightarrow{v}^\perp$ be the unit vector orthogonal to $\overline{AB}$ such that $-\overrightarrow{v}^\perp$points towards $\rho(\widetilde{f})$. Then Davalos proved the following:

Theorem [Davalos] : There exists a number $M > 0$ such that

$$\left\langle \widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.A, \overrightarrow{v}^\perp \right\rangle \leq M, \text{ for all } \widetilde{x} \in \mathbb{R}^2 \text{ and any integer } n > 0.$$

Our main result is similar to the above one, but it deals with all the possible situations when $\rho(\widetilde{f})$ has interior. As our methods rely on some results from [1], we need a stronger hypothesis, namely we assume $f \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$.

In order to state our main results, let us introduce a little more notation: Given a compact convex subset $K \subset \mathbb{R}^2$, for every $\alpha \in \partial K$, there exists a straight line r containing α such that $K \subset r \cup \{$one connected component of $r^c\}$. This line is called a supporting line at α. For instance, in case α is a vertex, there are infinitely many supporting lines at α.

Theorem 1 *: Let $f \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$ be such that $\rho(\widetilde{f})$ has interior. Then, there exists a number $M_f > 0$ such that for any $\omega \in \partial\rho(\widetilde{f})$ and any supporting line r at ω, if $\overrightarrow{v}^{\perp}$ is the unitary vector orthogonal to r, pointing towards the connected component of r^c which does not intersect $\rho(\widetilde{f})$, then*

$$\left\langle \widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.\omega, \overrightarrow{v}^{\perp} \right\rangle \leq M_f, \text{ for all } \widetilde{x} \in \mathbb{R}^2 \text{ and any integer } n > 0.$$

Remarks:

- Our proof will show that M_f can be precisely computed from f and moreover, the same number works for any map in $Diff_0^{1+\epsilon}(\mathrm{T}^2)$ sufficiently C^1-close to f;
- This theorem may be used as a tool to numerically estimate rotation sets. For instance if one is considering a family of maps $f_t \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$ an interesting problem connected to our result is to study how and when $\rho(\widetilde{f_t})$ changes as t varies;

As a corollary of the above result, we prove a stronger version of Boyland's conjecture in the torus case:

Theorem 2 *: Let $f \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$ be a Lebesgue measure preserving diffeomorphism such that $\rho(\widetilde{f})$ has interior. Then the rotation vector of the Lebesgue measure is an interior point of $\rho(\widetilde{f})$.*

Remember that the rotation vector of the Lebesgue measure is defined as:

$$\rho(Leb) \overset{def.}{=} \int_{\mathrm{T}^2} \phi(x) dLeb, \tag{3}$$

where $\phi : \mathrm{T}^2 \to \mathbb{R}^2$ is the displacement function given by $\phi(x) = \widetilde{f}(\widetilde{x}) - \widetilde{x}$, for any $\widetilde{x} \in p^{-1}(x)$. In general, if we denote by

$$M_{inv}(f) = \{\text{subset of all } f\text{-invariant Borel probability measures in } \mathrm{T}^2\},$$

then for any $\mu \in M_{inv}(f)$, we define the rotation vector of μ, $\rho(\mu)$, as

$$\rho(\mu) \overset{def.}{=} \int_{\mathrm{T}^2} \phi(x) d\mu.$$

These definitions are clearly motivated by Birkhoff's ergodic theorem, since for every $x \in \mathrm{T}^2$ and any integer $n > 0$,

$$\frac{1}{n}\sum_{i=1}^{n-1} \phi \circ f^i(x) = \frac{\widetilde{f}^n(\widetilde{x}) - \widetilde{x}}{n}, \text{ for any } \widetilde{x} \in p^{-1}(x).$$

So, given $\mu \in M_{inv}(f)$, for μ *a.e.* $x \in \mathrm{T}^2$, Birkhoff's ergodic theorem implies that the following limit exists

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} \phi \circ f^i(x) = \lim_{n\to\infty} \frac{\widetilde{f}^n(\widetilde{x}) - \widetilde{x}}{n} = \rho(x) \text{ (the rotation vector of } x)$$

and

$$\int_{\mathrm{T}^2} \rho(x)d\mu = \int_{\mathrm{T}^2} \phi(x)d\mu = \rho(\mu).$$

One last remark about theorem 2 is the following: the original problem posed by P. Boyland was to prove that if $interior(\rho(\widetilde{f})) \neq \emptyset$ and $\rho(Leb) = (0,0)$, then $(0,0) \in interior(\rho(\widetilde{f}))$, but in the homeomorphism setting. A proof of this result in this C^0-setting was obtained by Fábio Tal [12].

The next result is another easy corollary of theorem 1 and lemma 6. Before stating it, we have to define a few more concepts. Let $K \subset \mathbb{R}^2$ be a compact and convex subset. We say that some point $z \in K$ is an extremal point if, whenever z is the convex combination of two other points $z_1, z_2 \in K$, then either $z = z_1$ or $z = z_2$. Clearly, extremal points are always in the boundary of K. We say that $z \in K$ is a vertex if z is an extremal point and there are at least two, which implies infinitely many, supporting lines at z.

Corollary 3 *: Let $f \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$ be such that $\rho(\widetilde{f})$ has interior. Suppose for some $\mu \in M_{inv}(f)$, $\rho(\mu) \in \partial\rho(\widetilde{f})$ is a vertex. Then, $supp(\mu)$ is a compact f-invariant set which realizes the rotation vector $\rho(\mu)$. Moreover, there exists $M_\mu > 0$ such that for every $x \in supp(\mu)$, for any $\widetilde{x} \in p^{-1}(x)$ and any integer $n > 0$, $\left\|\widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.\rho(\mu)\right\| < M_\mu$, that is, there is no sub-linear displacement in $supp(\mu)$. In case $\rho(\mu)$ is an extremal point, but not a vertex, if we assume that the intersection of the (unique) supporting line at $\rho(\mu)$ with $\rho(\widetilde{f})$ is just $\rho(\mu)$, then $supp(\mu)$ also realizes the rotation vector $\rho(\mu)$. But in this case there may be sub-linear displacement in $supp(\mu)$.*

Remarks:

- in the general case when $\rho(\mu)$ is an extremal point, we do not know if the above corollary holds;
- It was proved by Franks [7] that rational extremal points of the rotation set are realized by periodic orbits, but for general extremal points, the problem was open.

In an ongoing work with Andre de Carvalho we are generalizing some results from [1] to other surfaces. After that, using the methods from this paper, we plan to prove a version of theorem 2 to surfaces of $genus \geq 2$.

For homeomorphisms of the torus homotopic to Dehn twists, results analog to theorem 1 and 2 were proved in [2].

This paper is organized as follows. In the second section we present a result from [1] important for us and an idea of the proof of theorem 1 in an easy case. In the third section we prove some auxiliary lemmas and after that, we prove our main theorems.

2 An important result and some ideas on the proofs

In [1], we considered diffeomorphisms $f \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$ which preserve area. But the preservation of area is not necessary to prove the following result, whose proof is contained in the proof of theorem 6 of [1].

Theorem 4 *: Suppose f belongs to $Diff_0^{1+\epsilon}(\mathrm{T}^2)$ and $(0,0) \in int(\rho(\widetilde{f}))$. Then, f has a hyperbolic periodic saddle point $Q \in \mathrm{T}^2$ such that any $\widetilde{Q} \in p^{-1}(Q)$ is $\widetilde{f}$-periodic and for any pair of integers (a,b), $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q}+(a,b))$.*

Remarks:

1. Clearly, the rotation vector of Q is $(0,0)$.

2. By saying that $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q}+(a,b))$ we mean that they have a topologically transverse intersection, which of course is not necessarily C^1 transversal. See figure 1 for a picture which clarifies this. For a precise explanation, see definition 9 (right before the statement of lemma 1) of [1].

3. In the proof of theorem 4, we obtain a C^1-transverse intersection at least when $(a,b)=(0,0)$.

The converse of this result is also true, namely if some map $f \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$ has a hyperbolic periodic point $\widetilde{Q}$ such that for three non collinear integer vectors $(a_1,b_1),(a_2,b_2),(a_3,b_3)$ we have $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q}+(a_i,b_i))$, for $i=1,2,3$ and $(0,0)$ belongs to the convex hull of $\{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$, then $(0,0) \in int(\rho(\widetilde{f}))$. This follows from the following: The fact that $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q}+(a_i,b_i))$ implies that we can produce a topological horseshoe at $Q \in \mathrm{T}^2$ such that for some sequence in the symbolic dynamics (one corresponding to points visiting only one particular rectangle in the horseshoe), there is a periodic orbit for f whose rotation vector is $\left(\frac{a_i}{N_i},\frac{b_i}{N_i}\right)$, for some integer $N_i>0$. And so,

$$(0,0) \in interior\ of\ the\ Conv.Hull\{\left(\frac{a_1}{N_1},\frac{b_1}{N_1}\right),\left(\frac{a_2}{N_2},\frac{b_2}{N_2}\right),\left(\frac{a_3}{N_3},\frac{b_3}{N_3}\right)\},$$

which is contained in the interior of $\rho(\widetilde{f})$ because of its convexity.

The argument used to prove theorem 1 can be summarized as follows in the specific situation when $(0,0) \in int(\rho(\widetilde{f}))$, $\omega = (0,1) \in \partial\rho(\widetilde{f})$ and there is a horizontal supporting line denoted r at $(0,1)$. This is clearly not a general setting: Both the point $\omega \in \partial\rho(\widetilde{f})$ and the direction of the supporting line may be irrational, but it is illustrative of the general strategy.

Let $\widetilde{Q} \in \mathbb{R}^2$ be a hyperbolic periodic point for $\widetilde{f}$ as in theorem 4 which by remark 3 after it, has a C^1-transverse homoclinic intersection. Without loss of generality, we can assume that $\widetilde{Q}$ is fixed, otherwise we consider the map $\widetilde{g}=\widetilde{f}^{n_{\widetilde{Q}}}$, where $n_{\widetilde{Q}}$ is the period of $\widetilde{Q}$ (maybe twice the period if the eigenvalues at Q are negative). The rotation set changes as $\rho(\widetilde{g})=n_{\widetilde{Q}}.\rho(\widetilde{f})$. So $(0,0) \in int(\rho(\widetilde{g}))$, $(0,n_{\widetilde{Q}}) \in \partial\rho(\widetilde{g})$ and there is a horizontal supporting line denoted r' at $(0, n_{\widetilde{Q}})$. In the beginning of the proof of theorem 1 we show that the statement of the theorem holds for $\widetilde{f}$, if and only if, it holds for $\widetilde{g}$, which is actually something very easy to prove. So, let us assume that $n_{\widetilde{Q}} = 1$.

The existence of such a point $\widetilde{Q}$ as above implies that there are arbitrarily small topological rectangles $D_{\widetilde{Q}} \subset \mathbb{R}^2$ such that:

$$\begin{array}{c}\widetilde{Q} \text{ is a vertex of } D_{\widetilde{Q}} \text{ and the sides of } D_{\widetilde{Q}}, \text{denoted } \alpha_{\widetilde{Q}}, \beta_{\widetilde{Q}}, \gamma_{\widetilde{Q}} \text{ and } \delta_{\widetilde{Q}} \\ \text{are contained in } W^s(\widetilde{Q}), W^u(\widetilde{Q}), W^s(\widetilde{Q}) \text{ and } W^u(\widetilde{Q}) \text{ respectively,}\end{array} \tag{4}$$

see figure 2. As $D_{\widetilde{Q}}$ is arbitrarily small, we can assume that

$$D_{\widetilde{Q}} \cap (D_{\widetilde{Q}} + (a,b)) = \emptyset, \text{ for all integer pairs } (a,b) \neq (0,0),$$

which means that $p(D_{\widetilde{Q}}) \subset \mathrm{T}^2$ is also a topological rectangle.

Moreover, there exists an integer

$$N'' > 0 \text{ such that for all } n \geq N'' \text{ we have:} \tag{5}$$

1. $\widetilde{f}^n(\beta_{\widetilde{Q}})$ and $\widetilde{f}^n(\delta_{\widetilde{Q}})$ have topologically transverse intersections with $\alpha_{\widetilde{Q}} + (0,1), \gamma_{\widetilde{Q}} + (0,1)$ and with $\alpha_{\widetilde{Q}} + (1,0), \gamma_{\widetilde{Q}} + (1,0)$;

2. $\widetilde{f}^n(\gamma_{\widetilde{Q}}) \subset \alpha_{\widetilde{Q}}$;

Now we construct a closed path connected set $\theta \subset \mathbb{R}^2$ such that:

1. $\theta = \theta + (1,0)$;

2. θ contains $D_{\widetilde{Q}} + i(1,0)$, for all integers i;

3. θ contains two compact simple arcs η_1 and η_2 of the following form: The arc η_1 starts at $\widetilde{Q}$, goes through $\widetilde{f}^{N''}(\beta_{\widetilde{Q}})$ until it crosses $\alpha_{\widetilde{Q}} + (1,0)$ and $\gamma_{\widetilde{Q}} + (1,0)$. The arc η_2 starts at $\widetilde{f}^{N''}(\delta_{\widetilde{Q}} \cap \alpha_{\widetilde{Q}})$, goes through $\widetilde{f}^{N''}(\delta_{\widetilde{Q}})$ until it crosses $\alpha_{\widetilde{Q}} + (1,0)$ and $\gamma_{\widetilde{Q}} + (1,0)$, see figure 3;

4. clearly, θ contains $\eta_{1(or\ 2)} + i(1,0)$, for all integers i;

5. θ is bounded in the $(0,1)$ direction, that is, θ is contained between two straight lines, both parallel to $(1,0)$, and the distance between them is denoted $d_{(1,0)}$;

Now, assume that the uniform bound in the statement of theorem 1 does not hold. This means that for every $M > 0$, there exists $\widetilde{x}_M \in \mathbb{R}^2$ and an integer $n_M > 0$, $n_M \stackrel{M \to \infty}{\to} \infty$, such that

$$p_2 \circ \widetilde{f}^{n_M}(\widetilde{x}_M) - p_2(\widetilde{x}_M) - n_M > M.$$

If we choose a sufficiently large $M > 0$ and the point $\widetilde{x}_M$ below θ satisfying $dist.(\widetilde{x}_M, \theta) \leq 2 + 2.d_{(1,0)}$, then we get that

$$\widetilde{f}^{n_M}(\theta) \text{ intersects } \theta + (0, n_M + \lfloor M - 4 - 4.d_{(1,0)} \rfloor).$$

More precisely, for some integer a,

$$\widetilde{f}^{n_M + N''}(D_{\widetilde{Q}}) \cap \left(D_{\widetilde{Q}} + (a, n_M + \lfloor M - 5 - 4.d_{(1,0)} \rfloor) \right)$$

contains a connected topological rectangle $\widetilde{R}^*$ as in figure 4. So, there is a topological horseshoe in $D_Q = p(D_{\widetilde{Q}}) \subset \mathrm{T}^2$ and in particular, this topological horseshoe has a point which is fixed under iterates of $f^{n_M + N''}$ and this point belongs to $p(\widetilde{R}^*) \subset D_Q$. So, it has a rotation vector whose second coordinate is equal to

$$\frac{n_M + \lfloor M - 5 - 4.d_{(1,0)} \rfloor}{n_M + N''},$$

which is larger than one, if $M > 0$ is sufficiently large. So we produced a point whose rotation vector belongs to the connected component of r^c which does not intersect the rotation set. This contradiction proves the theorem.

3 Proofs

In the first subsection, we prove some auxiliary results.

3.1 Auxiliary results

In the next lemma we are going to produce, for every possible direction $\overrightarrow{v}$, an unbounded closed connected set $\theta_{\overrightarrow{v}} \subset \mathbb{R}^2$ which separates the plane into two special unbounded connected components (maybe there are other components in the complement of $\theta_{\overrightarrow{v}}$), by concatenating integer translates of appropriate pieces of the stable and unstable manifolds of the hyperbolic $\widetilde{f}$-periodic point $\widetilde{Q}$ given in theorem 4 (in the applications, the direction $\overrightarrow{v}$ is that of the supporting line at the rotation vector ω in the boundary of $\rho(\widetilde{f})$ we are considering). The subset $\theta_{\overrightarrow{v}}$ is a general version of the set θ considered in the previous section.

As we already explained, for any $\widetilde{Q} \in p^{-1}(Q)$, where Q is given in theorem 4, there are arbitrarily small topological rectangles $D_{\widetilde{Q}} \subset \mathbb{R}^2$ whose sides are contained in $W^s(\widetilde{Q})$ and $W^u(\widetilde{Q})$, see (4). In order to construct the sets $\theta_{\overrightarrow{v}}$, let us first consider the following basic pieces, denoted $\Gamma_{(1,0)}$ and $\Gamma_{(0,1)}$ (suppose some $\widetilde{Q} \in p^{-1}(Q)$ is fixed):

1. $\Gamma_{(1,0)}$ is given by the union of $D_{\widetilde{Q}}$ with $D_{\widetilde{Q}}+(1,0)$ and the region bounded by them and two simple arcs η_1^H, η_2^H defined as follows: the arc η_1^H starts at $\widetilde{Q}$, goes through $\widetilde{f}^{N''}(\beta_{\widetilde{Q}})$ until it crosses $\alpha_{\widetilde{Q}} + (1,0)$ and $\gamma_{\widetilde{Q}} + (1,0)$. The arc η_2^H starts at $\widetilde{f}^{N''}(\delta_{\widetilde{Q}} \cap \alpha_{\widetilde{Q}})$, goes through $\widetilde{f}^{N''}(\delta_{\widetilde{Q}})$ until it crosses $\alpha_{\widetilde{Q}} + (1,0)$ and $\gamma_{\widetilde{Q}} + (1,0)$. We say that the beginning of $\Gamma_{(1,0)}$ is at $D_{\widetilde{Q}}$ and the end is at $D_{\widetilde{Q}} + (1,0)$.

2. $\Gamma_{(0,1)}$ is given by the union of $D_{\widetilde{Q}}$ with $D_{\widetilde{Q}}+(0,1)$ and the region bounded by them and two simple arcs η_1^V, η_2^V analogously defined: the arc η_1^V starts at $\widetilde{Q}$, goes through $\widetilde{f}^{N''}(\beta_{\widetilde{Q}})$ until it crosses $\alpha_{\widetilde{Q}} + (0,1)$ and $\gamma_{\widetilde{Q}} + (0,1)$. The arc η_2^V starts at $\widetilde{f}^{N''}(\delta_{\widetilde{Q}} \cap \alpha_{\widetilde{Q}})$, goes through $\widetilde{f}^{N''}(\delta_{\widetilde{Q}})$ until it crosses $\alpha_{\widetilde{Q}}+(0,1)$ and $\gamma_{\widetilde{Q}}+(0,1)$, see figure 5. As above, we say that the beginning of $\Gamma_{(0,1)}$ is at $D_{\widetilde{Q}}$ and the end is at $D_{\widetilde{Q}} + (0,1)$.

Remember that the definition of N'' appears in expression (5) and below it. Also note that all crosses mentioned above are topologically transverse intersections in the sense of theorem 4.

Lemma 5 *: Given a vector $\overrightarrow{v} \in \mathbb{R}^2$, we can construct a path connected closed set $\theta_{\overrightarrow{v}} \subset \mathbb{R}^2$ such that $\theta_{\overrightarrow{v}}$ is obtained by the union of integer translates of $\Gamma_{(1,0)}$ and $\Gamma_{(0,1)}$ in a way that:*

1. $\theta_{\overrightarrow{v}}$ intersects every straight line parallel to $\overrightarrow{v}^{\perp}$, a vector orthogonal to $\overrightarrow{v}$;

2. $\theta_{\overrightarrow{v}}$ is bounded in the direction of $\overrightarrow{v}^{\perp}$, that is, $\theta_{\overrightarrow{v}}$ is contained between two straight lines l_- and l_+, both parallel to $\overrightarrow{v}$, and the distance between these lines is less then $3+2.\max\{diameter(\Gamma_{(1,0)}), diameter(\Gamma_{(0,1)})\}$. So, in particular $(\theta_{\overrightarrow{v}})^c$ has at least two unbounded connected components, one containing l_- and the other containing l_+;

Proof:

To prove this lemma, we fix some $\widetilde{Q} \in \mathbb{R}^2$ as in theorem 4 and consider a straight line r passing through $\widetilde{Q}$ parallel to $\overrightarrow{v}$. Without loss of generality, we can assume that $\widetilde{Q} = (0,0)$ and $\overrightarrow{v} = (a,b)$ (so let $\overrightarrow{v}^{\perp} = (-b,a)$), with $a \geq 0, b \in \mathbb{R}$ and $a^2+b^2=1$. If $a=0$, then

$$\theta_{\overrightarrow{v}} = \bigcup_{i \in integers} \left(\Gamma_{(0,1)} + (0,i)\right)$$
and if $b=0$,
$$\theta_{\overrightarrow{v}} = \bigcup_{i \in integers} \left(\Gamma_{(1,0)} + (i,0)\right),$$

so first, let us consider the case $a,b>0$. We denote the Euclidean distance between two points in the plane by $d_{Euc}(\bullet,\bullet)$.

We start building the piece of $\theta_{\overrightarrow{v}}$ which follows the semi-line contained in r given by $\{y=(b/a).x : x \geq 0\}$. Our strategy is the following. We compute the numbers

$$\begin{gathered} \left|\left\langle \overrightarrow{v}^{\perp}, (1,0)\right\rangle\right| \overset{def.}{=} a_0 = d_{Euc}(\widetilde{Q}+(1,0), r) = |-b| \\ \text{and} \\ \left|\left\langle \overrightarrow{v}^{\perp}, (0,1)\right\rangle\right| \overset{def.}{=} b_0 = d_{Euc}(\widetilde{Q}+(0,1), r) = a. \end{gathered} \tag{6}$$

If $a_0 \leq b_0$, then we start with $\Gamma_{(1,0)}$. In this case $n_0 \overset{def.}{=} (1,0)$. If $a_0 > b_0$, then we start with $\Gamma_{(0,1)}$. In this case $n_0 \overset{def.}{=} (0,1)$.

So we have our first approximation, which is $\theta_{\overrightarrow{v}}^{0+} \overset{def.}{=} \Gamma_{n_0}$, where $n_0 \in \{(0,1),(1,0)\}$ is chosen as explained above. Now, in order to decide which of the subsets, $\Gamma_{(1,0)}+n_0$ or $\Gamma_{(0,1)}+n_0$ we add, we make the following computations analogous to the ones in (6):

$$\begin{gathered} \left|\left\langle \overrightarrow{v}^{\perp}, n_0+(1,0)\right\rangle\right| \overset{def.}{=} a_1 = d_{Euc}(\widetilde{Q}+n_0+(1,0), r) \\ \text{and} \\ \left|\left\langle \overrightarrow{v}^{\perp}, n_0+(0,1)\right\rangle\right| \overset{def.}{=} b_1 = d_{Euc}(\widetilde{Q}+n_0+(0,1), r). \end{gathered}$$

If $a_1 \leq b_1$, then we add $\Gamma_{(1,0)}+n_0$. If $a_1 > b_1$, then we add $\Gamma_{(0,1)}+n_0$. Now we have $\theta^{1+}_{\overrightarrow{v}} \overset{def.}{=} \Gamma_{n_0} \cup (\Gamma_{n_1}+n_0)$, where as before $n_1 \in \{(0,1),(1,0)\}$. Continuing, in order to decide which of the subsets $\Gamma_{(1,0)}+n_0+n_1$ or $\Gamma_{(0,1)}+n_0+n_1$ we add, we compute:

$$\left|\left\langle \overrightarrow{v}^{\perp}, n_0+n_1+(1,0)\right\rangle\right| \overset{def.}{=} a_2 = d_{Euc}(\widetilde{Q}+n_0+n_1+(1,0),r)$$
and
$$\left|\left\langle \overrightarrow{v}^{\perp}, n_0+n_1+(0,1)\right\rangle\right| \overset{def.}{=} b_2 = d_{Euc}(\widetilde{Q}+n_0+n_1+(0,1),r).$$

If $a_2 \leq b_2$, then we add $\Gamma_{(1,0)}+n_0+n_1$. If $a_2 > b_2$, then we add $\Gamma_{(0,1)}+n_0+n_1$. Now we have $\theta^{2+}_{\overrightarrow{v}} \overset{def.}{=} \Gamma_{n_0} \cup (\Gamma_{n_1}+n_0) \cup (\Gamma_{n_2}+n_0+n_1)$, again for some $n_2 \in \{(0,1),(1,0)\}$. After l steps we arrive at

$$\theta^{l+}_{\overrightarrow{v}} \overset{def.}{=} \Gamma_{n_0} \cup (\Gamma_{n_1}+n_0) \cup ... \cup (\Gamma_{n_l}+n_0+n_1+...+n_{l-1}).$$

By construction, the points $\widetilde{Q}, \widetilde{Q}+n_0, \widetilde{Q}+n_0+n_1, ..., \widetilde{Q}+n_0+n_1+...+n_l$ all belong to $\theta^{l+}_{\overrightarrow{v}}$. Now let us prove that for all integers $l \geq 0$, $d_{Euc}(\widetilde{Q}+n_0+n_1+...+n_l,r) \leq 1$. Clearly, $d_{Euc}(\widetilde{Q},r)=0$ and $d_{Euc}(\widetilde{Q}+n_0,r) \leq \min\{|a|,|b|\} \leq 1$. So, suppose by induction that for some integer $i' \geq 0$, $d_{Euc}(\widetilde{Q}+n_0+n_1+...+n_i,r) \leq 1$, for all $0 \leq i \leq i'$. This means that if we define $\Delta_{i'} \overset{def.}{=} \left\langle \overrightarrow{v}^{\perp}, n_0+n_1+...+n_{i'}\right\rangle$, then $|\Delta_{i'}| \leq 1$.

If $\Delta_{i'} > 0$, then

$$-1 \leq \left\langle \overrightarrow{v}^{\perp}, n_0+n_1+...+n_{i'}+(1,0)\right\rangle = \Delta_{i'}-b < \Delta_{i'} \leq 1.$$

If $\Delta_{i'} < 0$, then

$$-1 \leq \Delta_{i'} < \left\langle \overrightarrow{v}^{\perp}, n_0+n_1+...+n_{i'}+(0,1)\right\rangle = \Delta_{i'}+a \leq 1.$$

These estimates clearly imply that $d_{Euc}(\widetilde{Q}+n_0+n_1+...+n_{i'+1},r) \leq 1$ because $n_{i'+1} \in \{(0,1),(1,0)\}$ is chosen in a way to minimize the distance. So, our claim is proved.

If $\Delta_{i'}=0$, this means that $\widetilde{Q}+n_0+n_1+...+n_{i'}$ belongs to r, which means that $\overrightarrow{v}$ is a rational direction and so

$$\theta_{\overrightarrow{v}} = \bigcup_{i \in integers} \left(\begin{array}{c} \Gamma_{n_0} \cup (\Gamma_{n_1}+n_0) \cup ... \cup (\Gamma_{n_{i'}}+n_0+n_1+...+n_{i'-1})+ \\ +i.(n_0+n_1+...+n_{i'-1}+n_{i'}) \end{array} \right)$$

In case $\Delta_i \neq 0$ for all integers $i > 0$, we define $\theta^{+}_{\overrightarrow{v}} \overset{def.}{=} \bigcup_{i \geq 0} \theta^{i+}_{\overrightarrow{v}}$. In order to get the whole $\theta_{\overrightarrow{v}}$, we have to construct the other side of it. For this, let

$$\theta^{i-}_{\overrightarrow{v}} \overset{def.}{=} (\Gamma_{n_0} - n_0) \cup (\Gamma_{n_1} - n_1 - n_0) \cup ... \cup (\Gamma_{n_i} - n_i - n_{i-1} ... - n_1 - n_0)$$

and analogously $\theta^{-}_{\overrightarrow{v}} \overset{def.}{=} \bigcup_{i \geq 0} \theta^{i-}_{\overrightarrow{v}}$. As we did above, for any integer $i \geq 0$ points of the form $\widetilde{Q} - n_0 - n_1 ... - n_i$ all belong to $\theta^{-}_{\overrightarrow{v}}$ and

$$d_{Euc}(\widetilde{Q} - n_0 - n_1 ... - n_i, r) = \left| \left\langle \overrightarrow{v}^{\perp}, -n_0 - n_1 ... - n_i \right\rangle \right| =$$

$$= \left| \left\langle \overrightarrow{v}^{\perp}, n_0 + n_1 + ... + n_i \right\rangle \right| = |\Delta_i| \leq 1.$$

So, finally we make $\theta_{\overrightarrow{v}} \overset{def.}{=} \theta^{-}_{\overrightarrow{v}} \cup \theta^{+}_{\overrightarrow{v}}$. It is a closed, connected subset of the plane and from the properties obtained above, the projection of $\theta_{\overrightarrow{v}}$ in the direction of $\overrightarrow{v}^{\perp}$ has diameter smaller than $3+2.\max\{diameter(\Gamma_{(0,1)}), diameter(\Gamma_{(1,0)})\}$, so it is contained between two straight lines parallel to $\overrightarrow{v}$, whose distance is less than $3 + 2.\max\{diameter(\Gamma_{(0,1)}), diameter(\Gamma_{(1,0)})\}$. The fact that $\theta_{\overrightarrow{v}}$ intersects every straight line parallel to $\overrightarrow{v}^{\perp}$ is easy. If $a > 0$ and $b < 0$, the proof is analogous. $\square$

The next lemma uses theorem 1 and easily implies theorem 2:

Lemma 6 *: Suppose $f \in Diff_0^{1+\epsilon}(\mathrm{T}^2)$ has a rotation set $\rho(\widetilde{f})$ with interior. Let $\mu \in M_{inv}(f)$ be such that the rotation vector of μ, $\rho(\mu) \in \partial\rho(\widetilde{f})$. Let r be a supporting line at $\rho(\mu)$ and $\overrightarrow{v}^{\perp}$ be the unitary vector orthogonal to r, pointing towards the connected component of r^c which does not intersect $\rho(\widetilde{f})$. Then, if $x' \in supp(\mu)$, for any $\widetilde{x}' \in p^{-1}(x')$ and any integer $n > 0$,*

$$\left| \left\langle \widetilde{f}^n(\widetilde{x}') - \widetilde{x}' - n.\rho(\mu), \overrightarrow{v}^{\perp} \right\rangle \right| \leq 2 + M_f, \tag{7}$$

where M_f comes from theorem 1.

Proof:

Let us denote $r^c = \Omega_1 \cup \Omega_2$, in a way that $\rho(\widetilde{f}) \subset r \cup \Omega_1$.

Fact 3.1: Every ergodic measure ξ that appears in the ergodic decomposition of μ has rotation vector contained in r.

Proof:

This follows from $\rho(\widetilde{f}) \cap \Omega_2 = \emptyset$ and $\rho(\mu) \in r$. By contradiction, assume that for some ξ in the ergodic decomposition of μ, $\rho(\xi)$ does not belong to r. Then $\rho(\xi) \in \Omega_1$. Here we are using the non-obvious fact that

$$\rho(\widetilde{f}) = \{\omega \in \mathbb{R}^2 : \exists \eta \in M_{inv}(f) \text{ such that } \rho(\eta) = \int_{\mathrm{T}^2} \phi(x) d\eta = \omega\},$$

see [11]. Therefore, as ξ is in the ergodic decomposition of μ, the fact that $\rho(\xi) \in \Omega_1$ would imply the existence of another ergodic measure ξ' also in the ergodic decomposition of μ such that $\rho(\xi') \in \Omega_2$ (because $\rho(\mu) \in r$). This contradiction proves the fact. $\square$

To prove lemma 6, we again argue by contradiction. So let us suppose that there exists $x' \in supp(\mu)$ and some integer $n_0 > 0$, such that for any $\widetilde{x}' \in p^{-1}(x')$,

$$\left\langle \widetilde{f}^{n_0}(\widetilde{x}') - \widetilde{x}' - n_0.\rho(\mu), \overrightarrow{v}^{\perp} \right\rangle < -2 - M_f. \tag{8}$$

Theorem 1 implies that if the present lemma does not hold, then the above is the only possibility.

Expression (8) and a simple continuity argument clearly imply that there exists $\epsilon' > 0$ such that for all $x \in B_{\epsilon'}(x')$ (the ball of radius ϵ' centered at x') and any $\widetilde{x} \in p^{-1}(x)$,

$$\left\langle \widetilde{f}^{n_0}(\widetilde{x}) - \widetilde{x} - n_0.\rho(\mu), \overrightarrow{v}^{\perp} \right\rangle < -2 - M_f. \tag{9}$$

Now let $\nu \in M_{inv}(f)$ be an ergodic measure in the ergodic decomposition of μ such that $x' \in supp(\nu)$. As $\rho(\nu) \in r$ (see fact 3.1), $\rho(\nu) = \rho(\mu) + \lambda.\overrightarrow{v}$, where $\overrightarrow{v}$ is parallel to r and λ is some adequate real number. So, $\left\langle \rho(\nu), \overrightarrow{v}^{\perp} \right\rangle = \left\langle \rho(\mu), \overrightarrow{v}^{\perp} \right\rangle$.

We also define the relative to μ displacement function in the direction of $\overrightarrow{v}^{\perp}$ as $\phi_{\mu,\overrightarrow{v}^{\perp}} : \mathrm{T}^2 \to \mathbb{R}$ given by $\phi_{\mu,\overrightarrow{v}^{\perp}}(x) = \left\langle \widetilde{f}(\widetilde{x}) - \widetilde{x} - \rho(\mu), \overrightarrow{v}^{\perp} \right\rangle$, for any $\widetilde{x} \in p^{-1}(x)$. Then the following consequences hold:

1. $\int_{\mathrm{T}^2} \phi_{\mu,\overrightarrow{v}^{\perp}}(x) d\nu = 0$;

2. for any $\widetilde{x} \in \mathbb{R}^2$ and any integer $n > 0$, if $x = p(\widetilde{x})$, then

$$\left\langle \widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.\rho(\mu), \overrightarrow{v}^{\perp} \right\rangle = \sum_{i=0}^{n-1} \phi_{\mu,\overrightarrow{v}^{\perp}}(f^i(x));$$

So from Atkinson's lemma (see [3]) we get that for every $0 < \epsilon < \epsilon'$, there exists $x^* \in B_\epsilon(x')$, such that for some integer $n_1 > n_0$ and any $\widetilde{x}^* \in p^{-1}(x^*)$,

$$\left| \left\langle \widetilde{f}^{n_1}(\widetilde{x}^*) - \widetilde{x}^* - n_1.\rho(\mu), \overrightarrow{v}^{\perp} \right\rangle \right| < 1.$$

Thus, from expressions (9) and the above one, we finally obtain that

$$\left\langle \widetilde{f}^{n_1 - n_0}(\widetilde{f}^{n_0}(\widetilde{x}^*)) - \widetilde{f}^{n_0}(\widetilde{x}^*) - (n_1 - n_0).\rho(\mu), \overrightarrow{v}^{\perp} \right\rangle > 1 + M_f,$$

a contradiction with theorem 1. So expression (8) does not hold and the lemma is proved. □

3.2 Proof of theorem 1

First, let us consider a map $\widetilde{g}(\bullet) \overset{def.}{=} \widetilde{f}^q(\bullet) - (p, s)$ for some rational vector $\left(\frac{p}{q}, \frac{s}{q}\right) \in int(\rho(\widetilde{f}))$, not necessarily in irreducible form, in a way that g has a FIXED hyperbolic saddle point $Q \in \mathrm{T}^2$ with positive eigenvalues, as in theorem 4. For example, $\left(\frac{p}{q}, \frac{s}{q}\right)$ could be equal to $\left(\frac{1}{3}, \frac{2}{3}\right)$, but $q = 30, p = 10$ and $s = 20$. It is easy to see that $\rho(\widetilde{g}) = q.\rho(\widetilde{f}) - (p, s)$. So if we fix some $\omega \in \partial\rho(\widetilde{f})$ and a supporting line r at ω, parallel to some unitary vector $\overrightarrow{v}$, the corresponding rotation vector and supporting line for $\widetilde{g}$ are: $q.\omega - (p, s) \in \partial\rho(\widetilde{g})$ and a straight line r' passing through $q.\omega - (p, s)$, also parallel to $\overrightarrow{v}$.

Let us show that if the theorem holds for g, then it also holds for f. For this, assume there exists a number $M_g > 0$ such that for any $\tau \in \partial\rho(\widetilde{g})$ and any supporting line r at τ, if $\overrightarrow{v}^{\perp}$ is the unitary vector orthogonal to r, pointing towards the connected component of r^c which does not intersect $\rho(\widetilde{g})$, then

$$\left\langle \widetilde{g}^n(\widetilde{x}) - \widetilde{x} - n.\tau, \overrightarrow{v}^{\perp} \right\rangle \leq M_g, \text{ for all } \widetilde{x} \in \mathbb{R}^2 \text{ and any integer } n > 0. \quad (10)$$

From the relation between $\rho(\widetilde{g})$ and $\rho(\widetilde{f})$,

$$\omega \in \partial\rho(\widetilde{f}) \;\Leftrightarrow\; q.\omega - (p,s) \in \partial\rho(\widetilde{g}).$$

Expression (10) implies that

$$\left\langle \widetilde{f}^{n.q}(\widetilde{x}) - \widetilde{x} - n.q.\frac{\tau + (p,s)}{q}, \overrightarrow{v}^{\perp} \right\rangle \leq M_g, \text{ for all } \widetilde{x} \in \mathbb{R}^2 \text{ and any integer } n > 0.$$

Which gives,

$$\left\langle \widetilde{f}^{n}(\widetilde{x}) - \widetilde{x} - n.\tfrac{\tau+(p,s)}{q}, \overrightarrow{v}^{\perp} \right\rangle \leq M_g + q.\left(\sup_{\widetilde{z}\in\mathbb{R}^2} \left\| \widetilde{f}(\widetilde{z}) - \widetilde{z} \right\| + \sup_{\iota\in\rho(\widetilde{f})} \|\iota\| \right),$$

$$\text{for all } \widetilde{x} \in \mathbb{R}^2 \text{ and any integer } n > 0.$$

As $\sup_{\iota\in\rho(\widetilde{f})} \|\iota\| \leq \sup_{\widetilde{z}\in\mathbb{R}^2} \left\| \widetilde{f}(\widetilde{z}) - \widetilde{z} \right\|$ and the map $\tau \to \frac{\tau+(p,s)}{q}$ is a bijection from $\partial\rho(\widetilde{g})$ to $\partial\rho(\widetilde{f})$, if we choose $M_f = M_g + 2q.\left(\sup_{\widetilde{z}\in\mathbb{R}^2} \left\| \widetilde{f}(\widetilde{z}) - \widetilde{z} \right\| \right)$, then we are done.

So it remains to show that the present theorem holds for g. Let us fix some $\tau \in \partial\rho(\widetilde{g})$ and any supporting line r at τ, parallel to some vector $\overrightarrow{v}$. Also, let $\overrightarrow{v}^{\perp}$ be the unitary vector orthogonal to r, pointing towards the connected component of r^c which does not intersect $\rho(\widetilde{g})$. From lemma 5, fixed some $\widetilde{Q} \in p^{-1}(Q)$, there exists a subset $\theta_{\overrightarrow{v}} \subset \mathbb{R}^2$ as in the statement of that lemma, containing $\widetilde{Q}$. This means that there are straight lines, l_- and l_+, both parallel to $\overrightarrow{v}$, at a distance less then $d_\theta \overset{def.}{=} 3 + 2.\max\{diameter(\Gamma_{(0,1)}), diameter(\Gamma_{(1,0)})\}$ such that $\theta_{\overrightarrow{v}}$ is contained between l_- and l_+ and $\overrightarrow{v}^{\perp}$ points from l_- to l_+. Let $U_{\overrightarrow{v}}^-$ and $U_{\overrightarrow{v}}^+$ be the unbounded connected components of $(\theta_{\overrightarrow{v}})^c$, such that $\overrightarrow{v}^{\perp}$ points towards $U_{\overrightarrow{v}}^+$, or equivalently $l_- \subset U_{\overrightarrow{v}}^-$ and $l_+ \subset U_{\overrightarrow{v}}^+$. Note that $\theta_{\overrightarrow{v}}$ also intersects all straight lines parallel to $\overrightarrow{v}^{\perp}$.

If (a,b) is an integer vector such that $\left|\left\langle (a,b), \overrightarrow{v}^{\perp} \right\rangle\right| > d_\theta$, then $\theta_{\overrightarrow{v}} \cap (\theta_{\overrightarrow{v}} + (a,b)) = \emptyset$. More precisely, if $\left\langle (a,b), \overrightarrow{v}^{\perp} \right\rangle > d_\theta$, then $(\theta_{\overrightarrow{v}} + (a,b)) \subset U_{\overrightarrow{v}}^+$ and if $\left\langle (a,b), \overrightarrow{v}^{\perp} \right\rangle < -d_\theta$, then $(\theta_{\overrightarrow{v}} + (a,b)) \subset U_{\overrightarrow{v}}^-$.

Now let us suppose by contradiction that there exists $\widetilde{x}^* \in \mathbb{R}^2$ and an integer $n^* > N'' > 0$ such that

$$\left\langle \widetilde{g}^{n^*}(\widetilde{x}^*) - \widetilde{x}^* - n^*.\tau, \overrightarrow{v}^{\perp} \right\rangle > 100 + 20.d_\theta + N''.\left\langle \tau, \overrightarrow{v}^{\perp} \right\rangle. \tag{11}$$

Remember that N'' was defined in expression (5) and $\langle \tau, \overrightarrow{v}^{\perp} \rangle > 0$ because $(0,0) \in int(\rho(\widetilde{g}))$. Without loss of generality, we can assume that $\widetilde{x}^*$ belongs to the connected component of $(l_-)^c$ contained in $U^-_{\overrightarrow{v}}$ and $d_{Euc}(\widetilde{x}^*, l_-) < 1$.

Now let us choose some integer vector (a_+, b_+) such that

$$50+3.d_\theta+(n^*+N''). \langle \tau, \overrightarrow{v}^{\perp} \rangle < \langle (a_+, b_+), \overrightarrow{v}^{\perp} \rangle < 70+10.d_\theta+(n^*+N''). \langle \tau, \overrightarrow{v}^{\perp} \rangle .$$

From what we explained above, $(\theta_{\overrightarrow{v}} + (a_+, b_+)) \subset U^+_{\overrightarrow{v}}$ and $\widetilde{g}^{n^*}(\widetilde{x}^*)$ belongs to the connected component of $(l_+ + (a_+, b_+))^c$ contained in $U^+_{\overrightarrow{v}} + (a_+, b_+)$. So, $\widetilde{g}^{n^*}(\theta_{\overrightarrow{v}})$ intersects $\theta_{\overrightarrow{v}} + (a_+, b_+)$. More precisely, there exist integer vectors $(a_i, b_i), (a_f, b_f)$ such that $\widetilde{Q} + (a_i, b_i) \in \theta_{\overrightarrow{v}}$, $\widetilde{Q} + (a_f, b_f) \in \theta_{\overrightarrow{v}} + (a_+, b_+)$ and at least one of the following possibilities hold:

- $\widetilde{g}^{n^*}(\eta_1^H + (a_i, b_i))$ and $\widetilde{g}^{n^*}(\eta_2^H + (a_i, b_i))$ intersect both $\alpha_{\widetilde{Q}} + (a_f, b_f)$ and $\gamma_{\widetilde{Q}} + (a_f, b_f)$;
- $\widetilde{g}^{n^*}(\eta_1^V + (a_i, b_i))$ and $\widetilde{g}^{n^*}(\eta_2^V + (a_i, b_i))$ intersect both $\alpha_{\widetilde{Q}} + (a_f, b_f)$ and $\gamma_{\widetilde{Q}} + (a_f, b_f)$;

In any of the above cases, from the definition of N'' (see expression (5)), we get that $\widetilde{g}^{n^*+N''}(D_{\widetilde{Q}} + (a_i, b_i)) \cap (D_{\widetilde{Q}} + (a_f, b_f))$ contains a topological rectangle $\widetilde{R}_{fast}$ with one side contained in $\alpha_{\widetilde{Q}} + (a_f, b_f)$, another one contained in $\gamma_{\widetilde{Q}} + (a_f, b_f)$ and the two other sides contained in the interior of $D_{\widetilde{Q}} + (a_f, b_f)$, see figure 6. This implies that there is a compact g-invariant set contained in $D_Q = p(D_{\widetilde{Q}}) \subset \mathrm{T}^2$ whose dynamics is semi-conjugate to that of a horseshoe. In particular, there exists a fixed point for $g^{n^*+N''}$ in $p(\widetilde{R}_{fast})$ (such a point must exist, but it may not be unique) and its rotation vector with respect to $\widetilde{g}$ is

$$\rho_{fast} = \left(\frac{a_f - a_i, b_f - b_i}{n^* + N''} \right).$$

As $\widetilde{Q}, \widetilde{Q} + (a_i, b_i) \in \theta_{\overrightarrow{v}}$ and $\widetilde{Q} + (a_f, b_f) \in \theta_{\overrightarrow{v}} + (a_+, b_+)$ we get that

$$\left\langle \left(\frac{a_f - a_i, b_f - b_i}{n^* + N''} \right) - \tau, \overrightarrow{v}^{\perp} \right\rangle = \left\langle \left(\frac{\widetilde{Q} + (a_f, b_f) - (\widetilde{Q} + (a_i, b_i))}{n^* + N''} \right) - \tau, \overrightarrow{v}^{\perp} \right\rangle >$$

$$> \frac{50 + 3.d_\theta + (n^* + N''). \langle \tau, \overrightarrow{v}^{\perp} \rangle - 2.d_\theta}{n^* + N''} - \langle \tau, \overrightarrow{v}^{\perp} \rangle = \frac{50 + d_\theta}{n^* + N''} > 0.$$

And this is a contradiction, because from $\left\langle \rho_{fast} - \tau, \overrightarrow{v}^{\perp} \right\rangle > 0$ we get that $\rho_{fast} \notin \rho(\widetilde{g})$. So expression (11) does not hold and thus, for all $\widetilde{x} \in \mathbb{R}^2$ and any integer $n > 0$,

$$\left\langle \widetilde{g}^n(\widetilde{x}) - \widetilde{x} - n.\tau, \overrightarrow{v}^{\perp} \right\rangle \leq \max\{\left(100 + 20.d_\theta + N''.\left\langle \tau, \overrightarrow{v}^{\perp} \right\rangle\right), \ M_{g,N''}\},$$

where the number

$$M_{g,N''} \overset{def.}{=} 2N''.\left(\sup_{\widetilde{x} \in \mathbb{R}^2} \|\widetilde{g}(\widetilde{x}) - \widetilde{x}\|\right)$$

appears because for any $\widetilde{x} \in \mathbb{R}^2$ and any $0 \leq n \leq N''$,

$$\left\langle \widetilde{g}^n(\widetilde{x}) - \widetilde{x} - n.\tau, \overrightarrow{v}^{\perp} \right\rangle \leq \|\widetilde{g}^n(\widetilde{x}) - \widetilde{x}\| + n.\|\tau\| \leq$$

$$\leq n.\left(\sup_{\widetilde{x} \in \mathbb{R}^2} \|\widetilde{g}(\widetilde{x}) - \widetilde{x}\|\right) + n.\sup_{\iota \in \rho(\widetilde{g})} \|\iota\| \leq 2n.\left(\sup_{\widetilde{x} \in \mathbb{R}^2} \|\widetilde{g}(\widetilde{x}) - \widetilde{x}\|\right) \leq M_{g,N''}.$$

So we can take

$$M_g \overset{def.}{=} \max\{\left(100 + 20.d_\theta + N''.\sup_{\widetilde{x} \in \mathbb{R}^2} \|\widetilde{g}(\widetilde{x}) - \widetilde{x}\|\right), M_{g,N''}\},$$

because $\sup_{\widetilde{x} \in \mathbb{R}^2} \|\widetilde{g}(\widetilde{x}) - \widetilde{x}\| \geq \sup_{\iota \in \rho(\widetilde{g})} \|\iota\| \geq \left\langle \tau, \overrightarrow{v}^{\perp} \right\rangle$ for every $\tau \in \partial\rho(\widetilde{g})$ and $\overrightarrow{v}^{\perp}$ an unitary vector orthogonal to the supporting line at τ oriented in an adequate way. □

3.3 Proof of theorem 2

By contradiction, suppose that the rotation vector of the Lebesgue measure, denoted ω, belongs to $\partial\rho(\widetilde{f})$. Let r be a supporting line at ω and let $\overrightarrow{v}^{\perp}$ be a unitary vector orthogonal to r, pointing towards the connected component of r^c that does not intersect $\rho(\widetilde{f})$. From lemma 6 we get that for all $\widetilde{x} \in \mathbb{R}^2$ and any integer $n > 0$ (remember that $supp(Lebesgue) = \mathrm{T}^2$),

$$\left| \left\langle \widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.\omega, \overrightarrow{v}^{\perp} \right\rangle \right| \leq 2 + M_f, \text{ where } M_f \text{ comes from theorem 1.} \quad (12)$$

Now pick some point $z \in \mathrm{T}^2$ which is periodic and has a rotation vector $\nu \in int(\rho(\widetilde{f}))$. As $\nu \notin r$, $\left\langle \nu, \overrightarrow{v}^{\perp} \right\rangle \neq \left\langle \omega, \overrightarrow{v}^{\perp} \right\rangle$. So, for any $\widetilde{z} \in p^{-1}(z)$,

$$\frac{\left\langle \widetilde{f}^n(\widetilde{z}) - \widetilde{z} - n.\omega, \overrightarrow{v}^{\perp} \right\rangle}{n} \overset{n \to \infty}{\to} \left\langle \nu - \omega, \overrightarrow{v}^{\perp} \right\rangle \neq 0,$$

a contradiction with expression (12). This proves the theorem. $\square$

3.4 Proof of corollary 3

First, assume $\rho(\mu) \in \partial\rho(\widetilde{f})$ is a vertex. Then there are 2 different supporting lines at $\rho(\mu)$ (in fact, there are infinitely many), denoted r_1 and r_2, and $\overrightarrow{v}_1^{\perp}$, $\overrightarrow{v}_2^{\perp}$ are unitary vectors orthogonal, respectively to r_1 and r_2, such that $-\overrightarrow{v}_1^{\perp}$ and $-\overrightarrow{v}_2^{\perp}$ point towards $\rho(\widetilde{f})$. From lemma 6, for every $x \in supp(\mu)$, for any $\widetilde{x} \in p^{-1}(x)$ and any integer $n > 0$,

$$\left| \left\langle \widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.\rho(\mu), \overrightarrow{v}_i^{\perp} \right\rangle \right| \leq 2 + M_f, \text{ for } i = 1, 2.$$

As $\overrightarrow{v}_1^{\perp}$ and $\overrightarrow{v}_2^{\perp}$ are not parallel, the above expression implies that there exists M_μ which depends only on M_f and $\overrightarrow{v}_1^{\perp}, \overrightarrow{v}_2^{\perp}$ such that for every $x \in supp(\mu)$, for any $\widetilde{x} \in p^{-1}(x)$ and any integer $n > 0$,

$$\left\| \widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n.\rho(\mu) \right\| < M_\mu.$$

This proves the first part of the corollary. Now suppose $\rho(\mu)$ is an extremal point of $\rho(\widetilde{f})$ and the intersection of the supporting line r at $\rho(\mu)$ with $\rho(\widetilde{f})$ is just $\rho(\mu)$. From lemma 6, we know that for every $x \in supp(\mu)$ and for any $\widetilde{x} \in p^{-1}(x)$,

$$\lim_{n\to\infty} \left\langle \frac{\widetilde{f}^n(\widetilde{x}) - \widetilde{x}}{n} - \rho(\mu), \overrightarrow{v}^{\perp} \right\rangle = 0, \tag{13}$$

where $\overrightarrow{v}^{\perp}$ is the unitary vector orthogonal to r oriented in a way that $-\overrightarrow{v}^{\perp}$ points towards $\rho(\widetilde{f})$. As the accumulation points of the sequence

$$\frac{\widetilde{f}^n(\widetilde{x}) - \widetilde{x}}{n}$$

belong both to r (this follows from expression (13)) and to $\rho(\widetilde{f})$, there is just one accumulation point and it is $\rho(\mu)$. So $\rho(x)$ exists and it is equal to $\rho(\mu)$. As x is any point in $supp(\mu)$, this proves the second part of the corollary. $\square$

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Figure captions.

Figure 1. Diagram showing a topologically transverse intersection between $W^u(\widetilde{Q})$ and $W^s(\widetilde{Q}+(a,b))$.

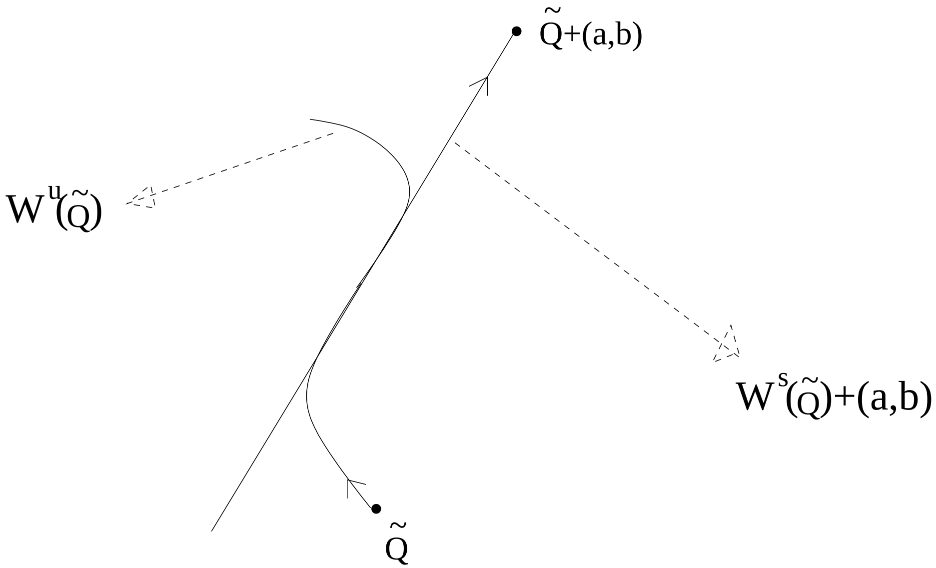
Figure 2. Diagram showing the topological rectangle $D_{\widetilde{Q}}$.

Figure 3. Diagram showing the set θ.

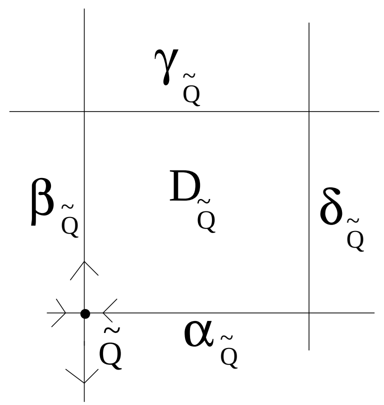
Figure 4. Diagram showing the topological rectangle $\widetilde{R}^*$ contained in $\widetilde{f}^{n_M+N''}(D_{\widetilde{Q}})\cap \left(D_{\widetilde{Q}}+(a, n_M+\lfloor M-5-4.d_{(1,0)}\rfloor)\right)$.

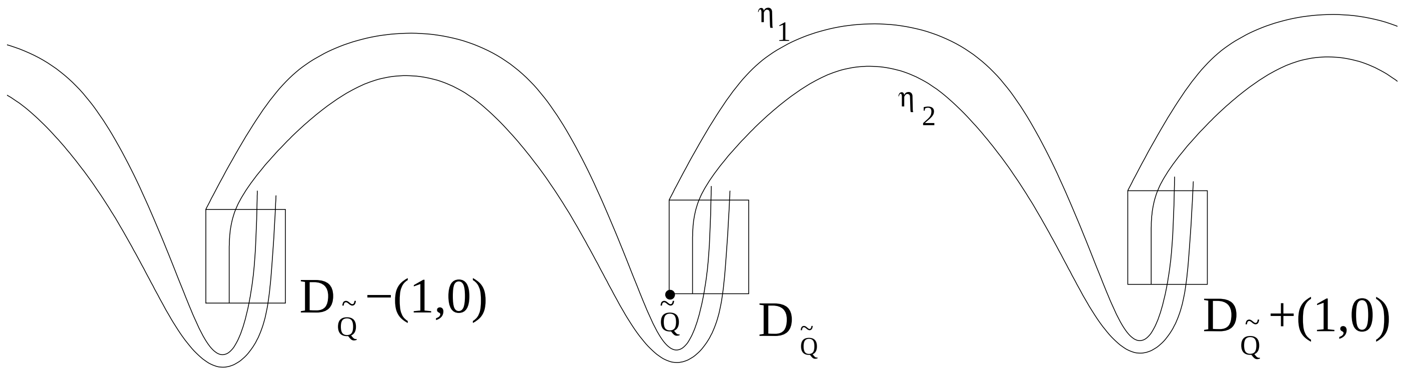
Figure 5. Diagram showing the sets: (a) $\Gamma_{(0,1)}$ and (b) $\Gamma_{(1,0)}$.

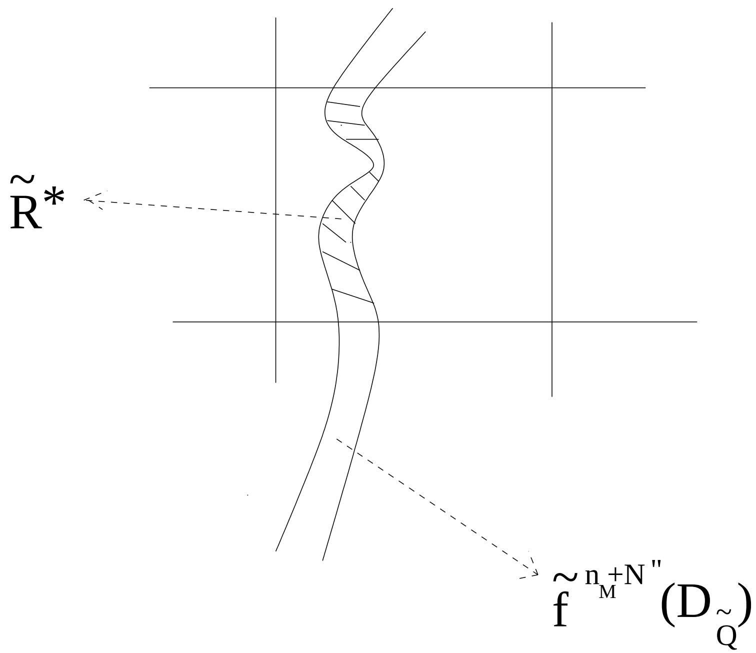
Figure 6. Diagram showing the the topological rectangle $\widetilde{R}_{fast}$.

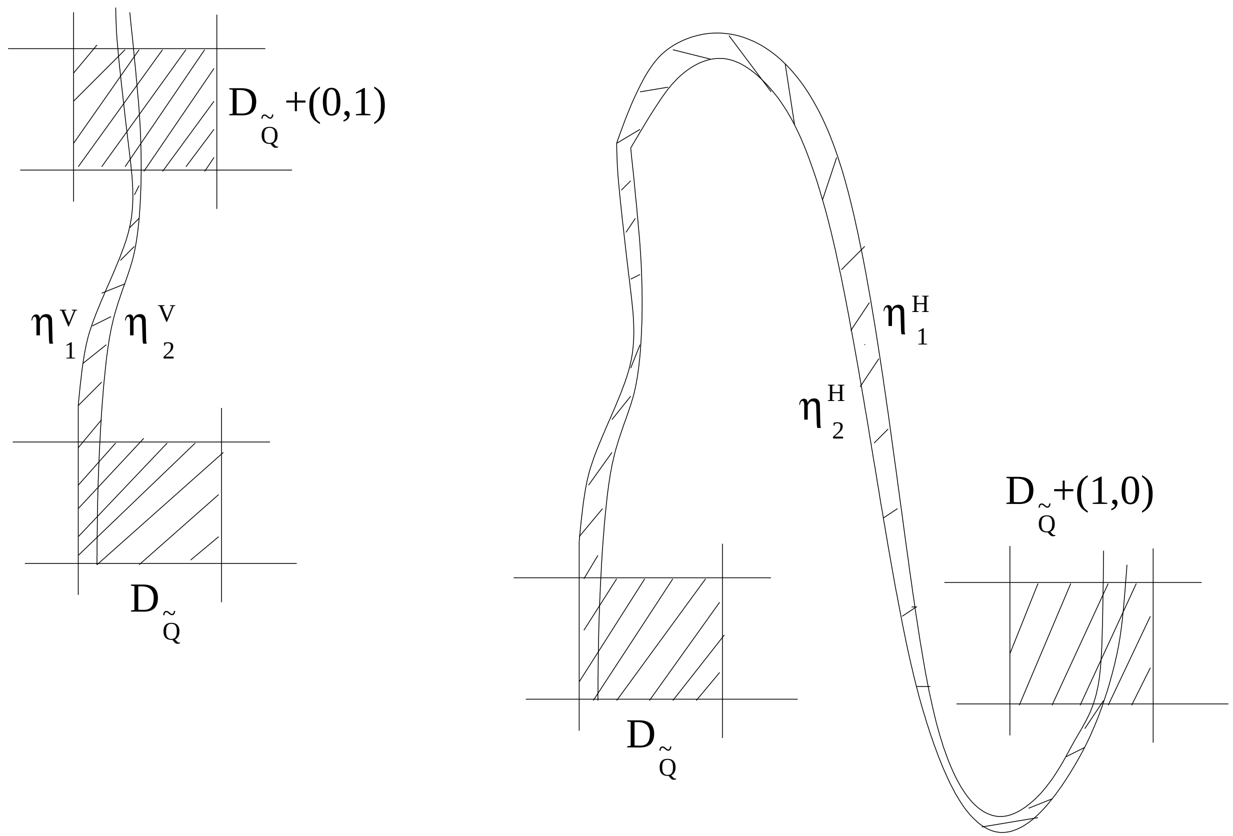


$\tilde{Q}+(a,b)$
$W^u(\tilde{Q})$
$W^s(\tilde{Q})+(a,b)$
$\tilde{Q}$

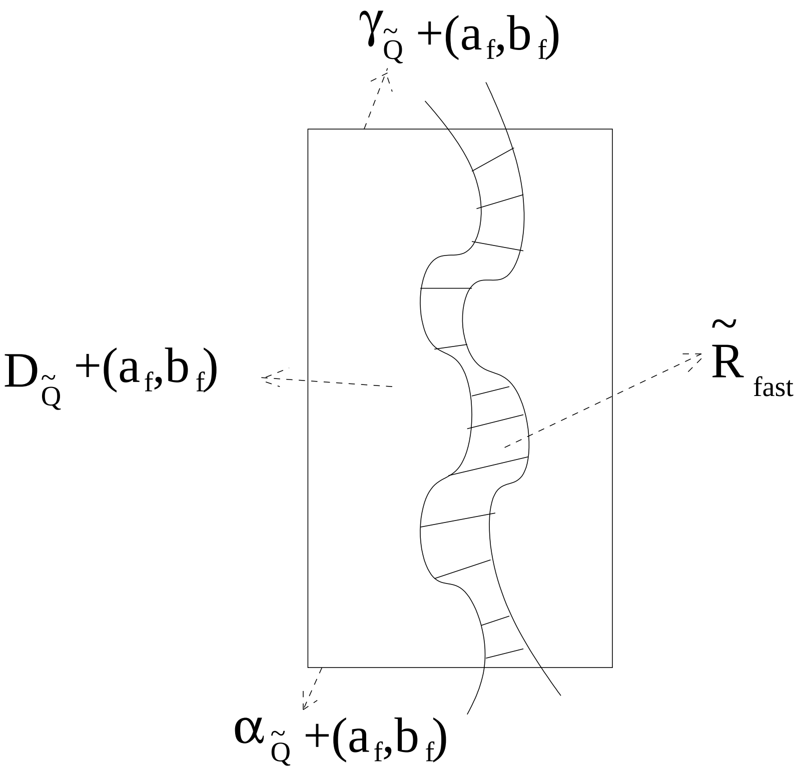

$\gamma_{\tilde{Q}}$
$\beta_{\tilde{Q}}$
$D_{\tilde{Q}}$
$\delta_{\tilde{Q}}$
$\tilde{Q}$
$\alpha_{\tilde{Q}}$


η1
η2
D_Q̃ −(1,0)
Q̃
D_Q̃
D_Q̃ +(1,0)


$\tilde{R}^*$
$\tilde{f}^{\,n_M+N''}(D_{\tilde{Q}})$


$D_{\tilde{Q}}+(0,1)$
η_1^V
η_2^V
$D_{\tilde{Q}}$
η_1^H
η_2^H
$D_{\tilde{Q}}+(1,0)$
$D_{\tilde{Q}}$

(a) (b)


$\gamma_{\tilde{Q}} + (a_f, b_f)$
$D_{\tilde{Q}} + (a_f, b_f)$
$\tilde{R}_{fast}$
$\alpha_{\tilde{Q}} + (a_f, b_f)$