Representation type via quiver Grassmannians

[largely in collaboration with Thorsten Weist]

I Motivation

2000s: Fomin & Zelevinsky introduce cluster algebras

$$\mathcal{A}_Q \subset \mathbb{Z}[x_q^{\pm 1} \mid q \in Q_0]$$

[Ex: $Q = 1 \rightrightarrows 2$

with $Q_0 = \{1, 2\}$]

(where Q is a quiver), which are generated over $\mathbb{Z}$ by recursively defined cluster variables.

2005: Caldero-Chapoton & Keller identify the cluster variables with $\{x_q\}_{q\in Q_0}$ and the elements

$$X_M = \sum_{\underline{e}=(e_q)_{q\in Q_0}} \chi_{\underline{e}}(M) \frac{\prod_{v:s\to t} x_s^{m_t - e_t} \cdot x_t^{e_s}}{\prod_{q\in Q_0} x_q^{m_q}} \in \mathbb{Z}[x_q^{\pm 1} \mid q \in Q_0]$$

where
- M is a (complex) representation of Q with dimension vector $\underline{m} = (m_q)_{q\in Q_0}$;
- $\chi_{\underline{e}}(M)$ is the Euler characteristic of quiver Grassmannian

$$Gr_{\underline{e}}(M) = \{ N \subset M \text{ subrepresentation with } \underline{\dim}\, N = \underline{e} \}.$$

Central problems at that time where:

$\rightarrow$ Find formulas for $\chi_{\underline{e}}(\mu)$!

($\rightsquigarrow$ derive formulas for products

$$x_\mu \cdot x_{\mu'} = \sum_{\nu} c^{\nu}_{\mu,\mu'} \, x_\nu$$
)

$\rightarrow$ For which Q is $\chi_{\underline{e}}(\mu) \geq 0$ for all μ & $\underline{e}$?

($\rightsquigarrow$ non-negativity: $c^{\nu}_{\mu,\mu'} \geq 0$)

II Results

For simplicity:

Q acyclic & connected

(i.e. no oriented cycles)

Caldero–Reineke 2007:

M exceptional

(i.e. $\mathrm{Ext}^1(M,M)=0$)

Then $Gr_{\underline{e}}(M)$ is smooth

and $\chi_{\underline{e}}(M) \geq 0$.

A_n : $1 - 2 - 3 - \cdots - n$

$\tilde{A}_n$: $1 - 2 - 3 - \cdots - n$

Haupt 2010:

Q of (extended) Dynkin type A

Then

$$\chi_{\underline{e}}(M) = \#\left\{ \begin{array}{l} \text{fixed pts. of} \\ \text{a torus action} \end{array} \right\} \geq 0$$

for all M and $\underline{e}$.

Example (Zelevinsky 2010):

$Q = K(4)$ $\quad 1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; 2$

4-Kronecker quiver

M "generic" representation with $\underline{\dim}\, M = (3,4)$

$\underline{e} = (1,3)$

Then $Gr_{\underline{e}}(M)$ is a smooth curve of degree 4 in $\mathbb{P}^2$, and its Euler characteristic is $\chi_{\underline{e}}(M) = -4$.

Reineke '12 / Hille '15 / Ringel '17:

Every projective variety is a quiver Grassmannian.

Reineke: for

$$Q = 1 \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; 2 \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; 3$$

Hille: $X \subset \mathbb{P}^n \rightsquigarrow Q = K(n)$

Ringel: any wild Q

L – Weist 2019/21:

D_n: $\begin{smallmatrix}1\\2\end{smallmatrix} > 3 - 4 - \dots - n$

$\tilde{D}_n$: $\begin{smallmatrix}1\\2\end{smallmatrix} > 3 - 4 - \dots n-1 < \begin{smallmatrix}n\\n+1\end{smallmatrix}$

Q of (extended) Dynkin type D

M indecomposable, $\underline{e}$ arbitrary

Then $Gr_{\underline{e}}(M)$ has a decomposition into affine spaces (= DIAS), i.e. there is a bijective morphism

$$\coprod_{i=1,\dots,s} \mathbb{A}^{n_i}_{\mathbb{C}} \longrightarrow Gr_{\underline{e}}(M) \quad (\text{for some } s \geq 0).$$

Cor:

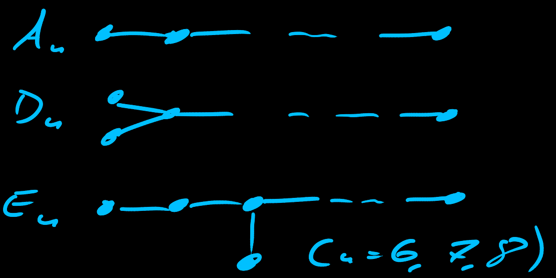
$$\chi_{\underline{e}}(M) = s \geq 0.$$

Cerulli Irelli – Esposito – Franzen – Reineke 2021:

DIAS for (extended) Dynkin type E.

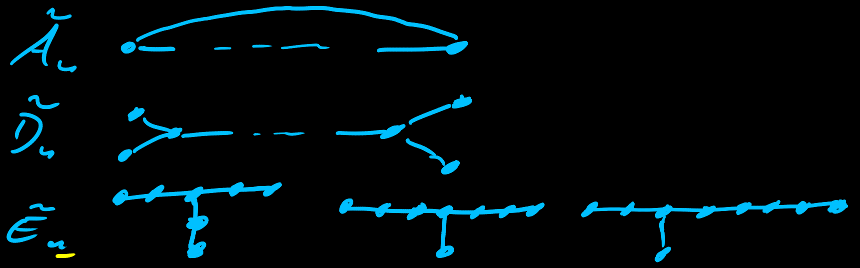
L–Weist 2017:

(1) Q is representation finite (= Dynkin type ADE)



$\Longleftrightarrow$ $Gr_{\underline{e}}(M)$ has a DIAS and is smooth for all $\underline{e}$ and indecomposable M.

(2) Q is tame (= extended Dynkin type ADE)



$\Longleftrightarrow$ $Gr_{\underline{e}}(M)$ has a DIAS for all $\underline{e}$ and indecomposable M,

but $Gr_{\underline{e}}(M)$ is singular for some $\underline{e}$ and indecomposable M.

(3) Q is wild

$\Longleftrightarrow$ $\{\chi_{\underline{e}}(M) \mid \text{all } \underline{e} \text{ \& indecomposable } M\} = \mathbb{Z}$

$\Longleftrightarrow$ $\{Gr_{\underline{e}}(M) \mid \text{all } \underline{e} \text{ \& indecomposable } M\} = \{\text{all projective varieties}\}$

III The proof

note: it's enough to prove "Q rep. finite/tame/wild $\Rightarrow$ properties"

$\boxed{(3)\ Q \text{ wild}}$

A

L-Weist 2017:

Q wild with $\# Q_0 \geq 3$, $m \geq 1$.

Then there are exceptional M and N with disjoint support and $\operatorname{ext}(M,N) \geq m$.

$$1 \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} 2$$

(m' arrows)

Ringel 2017:

Same for $K(m')$ $(m' \geq 3)$.

B Schofield induction:

$$\operatorname{Rep}^{exc}(K(m)) \longrightarrow \operatorname{Rep}^{exc}(Q)$$

$$X \longmapsto \tilde{X} \text{ such that } \exists$$

$\underline{\dim}\, X = (d_1, d_2)$ $\qquad 0 \to M^{d_1} \to \tilde{X} \to N^{d_2} \to 0$

get: $Gr_{\underline{e}}(X) = Gr_{e_1 \cdot \underline{\dim}\, M + e_2 \cdot \underline{\dim}\, N}(\tilde{X})$

C Hille 2015:

Every projective $X \subset \mathbb{P}^{m-1}$ is a quiver Grassmannian for $K(m)$.

(1) Q of Dynkin type ADE

$\boxed{A}$ All indecomposable M are exceptional

$\Rightarrow \mathrm{Gr}_{\underline{e}}(M)$ smooth.

[Caldero–Reineke]

$\boxed{B}$ Type A: 

M indecomposable $\Rightarrow M$ thin

(i.e. $\dim M_q \leq 1 \ \forall q \in Q_0$)

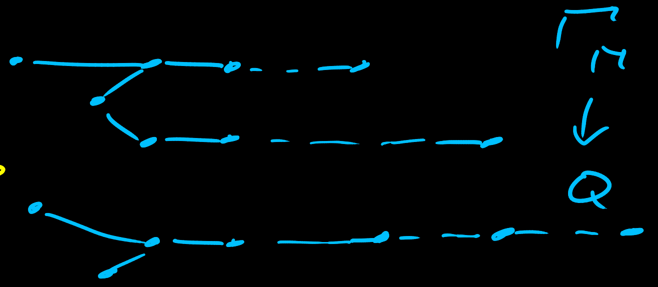
$\Rightarrow \mathrm{Gr}_{\underline{e}}(M) = *$ or $\emptyset$,

which has a DIAS.

Type D: 

M indecomposable $\Rightarrow M$ thin or has

coefficient quiver



$\Rightarrow \dim M_q \leq 2 \ \forall q \in Q_0$

$\Rightarrow \mathrm{Gr}_{\underline{e}}(M) = \emptyset, *$

or $\prod_{\text{finite}} \mathbb{P}^1$; all have DIAS.

Type E: (CIEFR 21)

Go through all indecomposable representations.

(2) Q tame

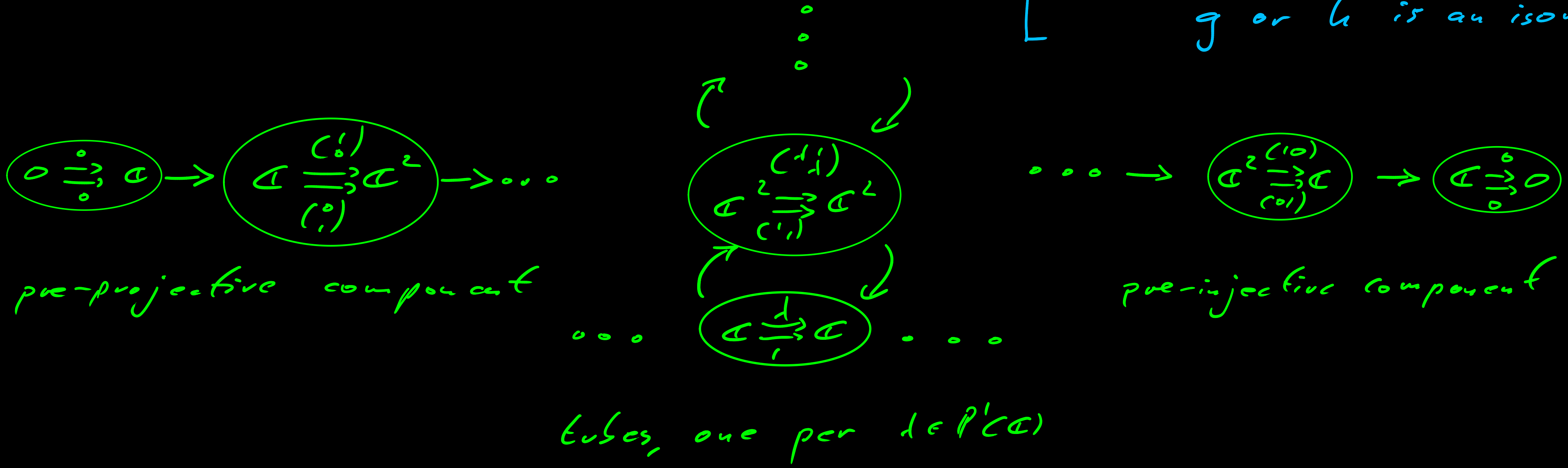
The Auslander-Reiten quiver of Q

Vertices = {isomorphism classes of indecomposable representations}

Arrows = $\{[M] \longrightarrow [N]$ if there is an irreducible $M \xrightarrow{f} N\}$

↳ [i.e. • f is not an isom. & $\neq 0$
• if $f = g \circ h$, then g or h is an isom.]

Example: $Q \quad 1 \rightrightarrows 2$ of type $\tilde{A}_2$



$\boxed{A}$ Singularities

→ M pre-injective or pre-projective

$\Rightarrow$ M exceptional

$\Rightarrow$ $Gr_{\underline{e}}(M)$ smooth for all $\underline{e}$

→ L-Weist 2017:

Every tube of the AR-quiver of Q contains an M such that $Gr_{\underline{e}}(M)$ is singular for some $\underline{e}$.

$\boxed{B}$ DIAS

Type $\tilde{A}$: "easy" (CIEFR21: type $\tilde{E}$)

Type $\tilde{D}$ & $\tilde{E}$: not so easy

(case consideration of different classes of M's)

Techiques for type $\tilde{D}$:

(i) Recursive solution of defining equations in linear terms.

(ii) Inductive argument for short exact sequences
$$0 \to N \to M \to P \to 0$$

(iii) Duality

Auslander – Reiten quiver for $\tilde{D}$

pre-projectives	tubes		pre-injectives
defect −2	homogeneous tubes	3 exceptional tubes	defect > 0
defect −1	quasi-simples		

(iii) Duality

L-Weist 15:

M^* dual representation of M

$\underline{e}^* = \underline{\dim} M - \underline{e}$

Then $Gr_{\underline{e}^*}(M^*) \simeq Gr_{\underline{e}}(M)$.

(ii) Induction over generalized Auslander-Reiten sequences

A diagram

$$(\ast)\quad \begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] & M \arrow[r, "p"] & P \arrow[r] & 0 \\ 0 \arrow[r] & i^{-1}(C) \arrow[r] \arrow[u, phantom, sloped, "\subseteq"] & C \arrow[r] \arrow[u, phantom, sloped, "\subseteq"] & p(C) \arrow[r] \arrow[u, phantom, sloped, "\subseteq"] & 0 \end{tikzcd}$$

yields a map

$$\Phi_{\underline{e}} : Gr_{\underline{e}}(M) \longrightarrow \coprod_{\underline{f}+\underline{g}=\underline{e}} Gr_{\underline{f}}(N) \times Gr_{\underline{g}}(P)$$

$$C \longmapsto (i^{-1}(C), p(C))$$

whose fibres are affine spaces or empty.

If $(\ast)$ is "nice enough" (a "generalized AR-sequence"), then $\Phi_{\underline{e}}$ is locally constant.

$\underset{(+\varepsilon)}{\Longrightarrow}$ DIAS for $Gr_{\underline{f}}(N)$ and $Gr_{\underline{g}}(P)$ imply DIAS for $Gr_{\underline{e}}(M)$.

(i) Explicit equations for Schubert cells

→ Choose ordered bases

$\mathcal{B}_q \subset M_q \quad (q \in Q_0)$;

We say that $\beta = (\beta_q)_{q \in Q_0}$ is of type $\underline{e}$ if $\beta_q \subset \mathcal{B}_q$ has cardinality e_q for all q.

→ Define C_β^M as intersection:

$$\begin{array}{ccc} Gr_{\underline{e}}(M) & \longrightarrow & \prod_{q \in Q_0} Gr(e_q, M_q) \\ \cup & \llcorner & \cup \\ C_\beta^M & \longrightarrow & \prod_{q \in Q_0} C(\beta_q) \end{array}$$

[quiver Schubert cell] → C_β^M

$\prod_{q \in Q_0} C(\beta_q)$ ← [usual Schubert cell]

Fact: $Gr_{\underline{e}}(M) = \coprod_{\beta \text{ of type } \underline{e}} C_\beta^M$

→ want: basis $\mathcal{B}$ such that $C_\beta^M = \mathbb{A}^{n_\beta}$ or $\emptyset$ for all β.

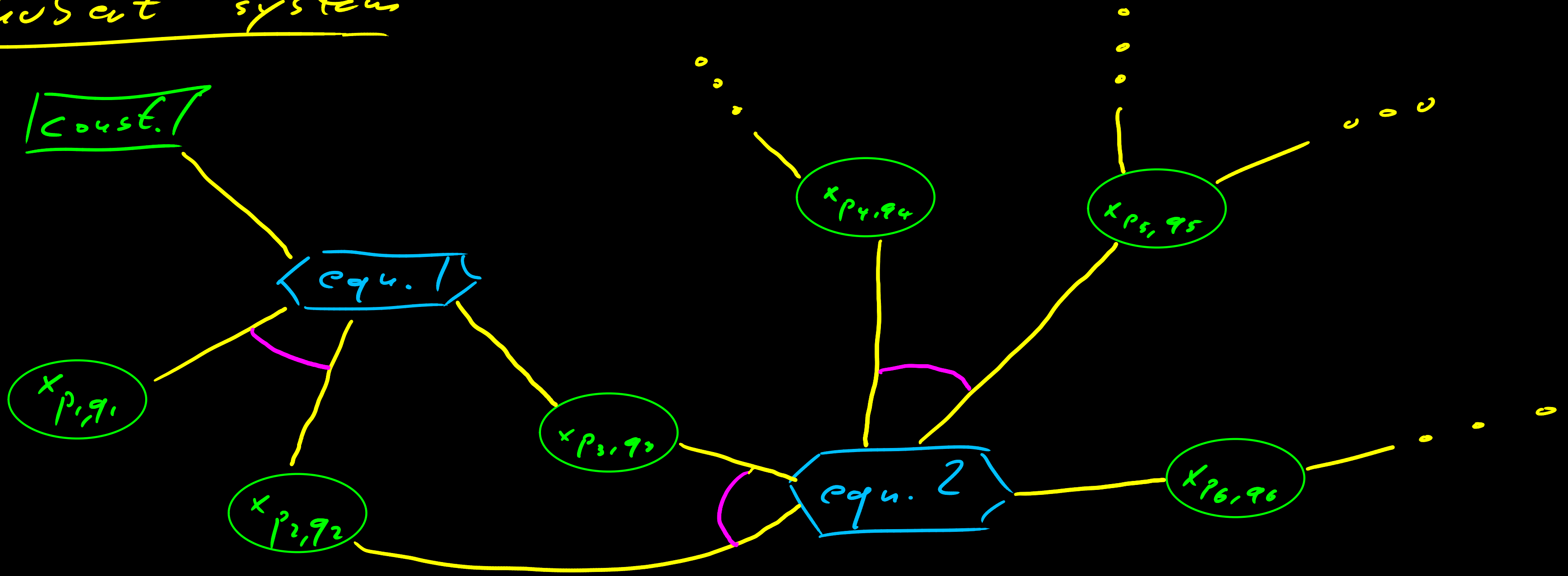
L 2012: Explicit defining equations for $C_\beta^M \xrightarrow{\text{closed}} \mathrm{Mat}_{\beta\times\beta}$ (degree ≤ 2).

$\rightarrow$ Schubert system

($c \hat{=}$ explicit nonzero constant)

equ. 1: $c\cdot x_{p_1,q_1}\cdot x_{p_2,q_2} + c\cdot x_{p_3,q_3} + c = 0$

equ. 2: $c\cdot x_{p_4,q_4}\cdot x_{p_5,q_5} + c\cdot x_{p_6,q_6} = 0$



$\rightsquigarrow$ Solve a combinatorial problem about certain graphs!