Relative strongly Gorenstein objects in abelian categories

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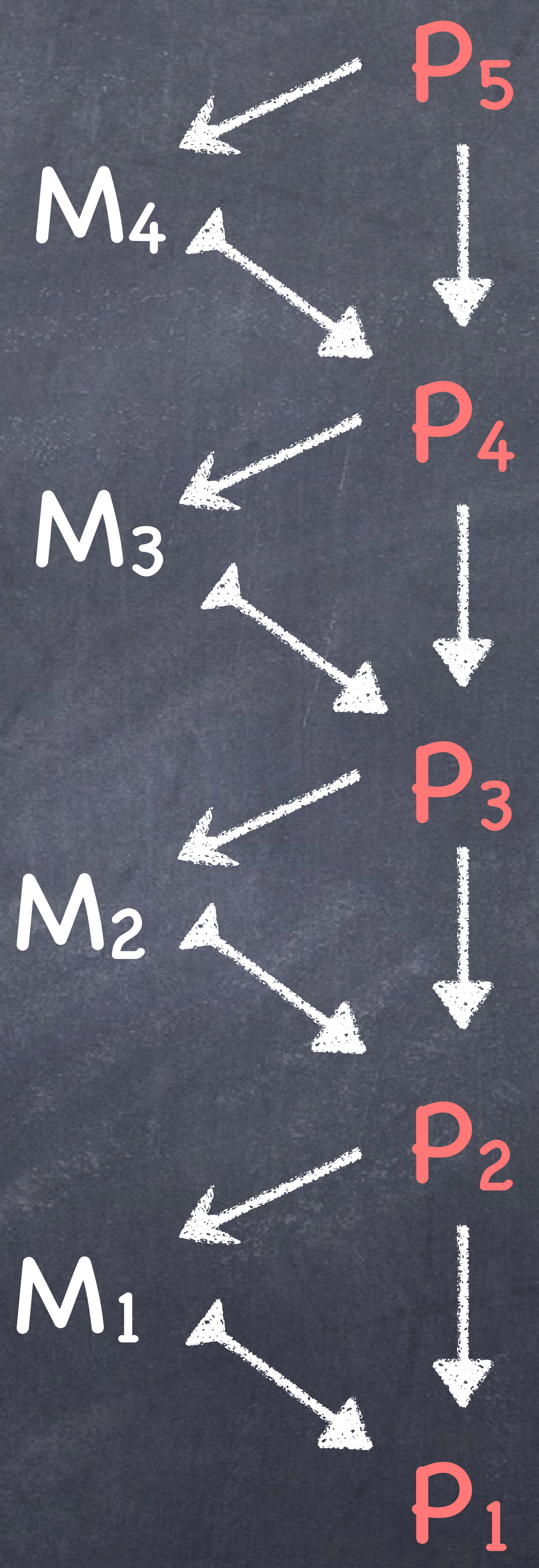
Organization

- Motivation and historical notes.
- Relative strongly Gorenstein projective objects: definition and examples.
- Relative Gorenstein objects of period > 1.
- Properties and relations with periodic objects and relative Gorenstein objects.



Motivation and historical notes

- [1956 – Auslander, Buchsbaum & Serre]: A commutative noetherian local ring R is with residue field k is regular (that is, the minimal number of generators of its maximal ideal is equal to its Krull dimension) iff for every R-module M one has $\mathrm{pd}(M) < \infty$.
- [1969 – Auslander & Bridger]: G-dimension of finitely generated modules over a commutative noetherian ring $\Rightarrow$ Characterization of local Gorenstein rings similar to [1956, ABS].
- [mid 1990s – Enochs, Jenda & Torrecillas]: Gorenstein modules (G-dimension = 0) over arbitrary rings, and not necessarily finitely generated.
- [2007 – Bennis & Mahdou]: Strongly Gorenstein modules.
  An R-module is Gorenstein projective iff it is a direct summand of a strongly Gorenstein projective R-module.
- [2010 – Bennis & Mahdou]: Applications of the strongly Gorenstein modules.

$$\mathrm{gl.Gpd}(R) := \sup \{ \mathrm{Gpd}(M) : M \in R\text{-Mod} \} = \{ \mathrm{Gid}(M) : M \in R\text{-Mod} \} =: \mathrm{gl.Gid}(R)$$

Gorenstein projective modules:

$$\begin{tikzcd} P_\bullet : \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] \arrow[dr] & P_0 \arrow[r] & P_{-1} \arrow[r] & P_{-2} \arrow[r] & \cdots \\ & & & M \arrow[u] & & & \end{tikzcd}$$

(exact with each P_k projective)

$$M = Z_0(P_\bullet) \in GP(R)$$

$$\mathrm{Hom}(P_\bullet,Q) : \cdots \longrightarrow \mathrm{Hom}(P_{-2},Q) \longrightarrow \mathrm{Hom}(P_{-1},Q) \longrightarrow \mathrm{Hom}(P_0,Q) \longrightarrow \mathrm{Hom}(P_1,Q) \longrightarrow \mathrm{Hom}(P_2,Q) \longrightarrow \cdots$$

(exact for every Q projective)

Strongly Gorenstein projective modules:

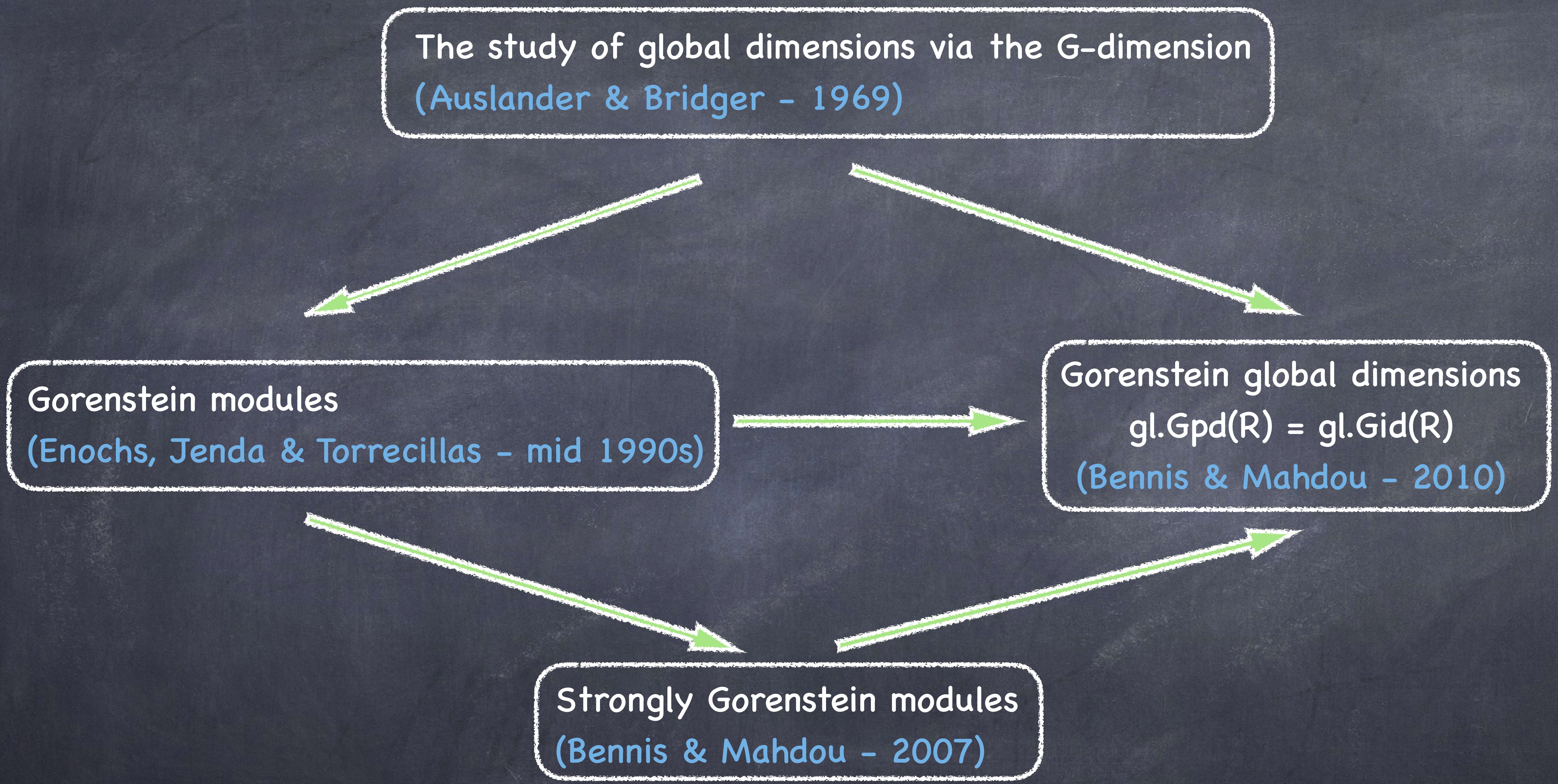
$$\begin{tikzcd}
P_\bullet : \cdots \arrow[r] & P \arrow[r] & P \arrow[r] \arrow[dr] & P \arrow[r] & P \arrow[r] & P \arrow[r] & \cdots \\
 & & & M \arrow[u] & & &
\end{tikzcd}$$

$$M = Z_0(P_\bullet) \in \pi\mathrm{GP}(R)$$

(exact with P projective)

$$\mathrm{Hom}(P_\bullet, Q) : \quad \cdots \longrightarrow \mathrm{Hom}(P,Q) \longrightarrow \mathrm{Hom}(P,Q) \longrightarrow \mathrm{Hom}(P,Q) \longrightarrow \mathrm{Hom}(P,Q) \longrightarrow \mathrm{Hom}(P,Q) \longrightarrow \cdots$$

(exact for every Q projective)



Projectives $\subsetneq$ Strongly Gorenstein projectives $\subsetneq$ Gorenstein projectives

Relative periodic (or strongly) Gorenstein objects

[2015 - Pan & Cai] [2021 - Becerril, Mendoza and Santiago]:

Gorenstein (A,B)-projective modules.

$$A_\bullet: \quad \cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A_{-1} \longrightarrow A_{-2} \longrightarrow \cdots$$

$$\begin{tikzcd} A_1 \arrow[dr] & & A_0 \\ & M \arrow[ur] & \end{tikzcd}$$

(exact with each $A_k \in \mathcal{A}$)

$$M = Z_0(A_\bullet) \in \mathcal{GP}_{(\mathcal{A},\mathcal{B})}$$

$$\mathrm{Hom}(A_\bullet,B) : \quad \cdots \longrightarrow \mathrm{Hom}(A_{-2},B) \longrightarrow \mathrm{Hom}(A_{-1},B)$$

$$\longrightarrow \mathrm{Hom}(A_0,B) \longrightarrow \mathrm{Hom}(A_1,B) \longrightarrow \mathrm{Hom}(A_2,B)$$

$$\longrightarrow \cdots$$

(exact for every $B \in \mathcal{B}$)

Examples

1. Gorenstein projective modules: $\mathcal{A} = \mathcal{B} = \text{Proj}(R)$

2. Ding projective modules [2010 - Gillespie]: $\mathcal{A} = \text{Proj}(R)$ $\mathcal{B} = \text{Flat}(R)$

3. AC-Gorenstein projective modules [2014 - Bravo, Gillespie, Hovey]:

$\mathcal{A} = \text{Proj}(R)$ $\mathcal{B} = \text{Level}(R)$

An R-module L is level if $\text{Tor}_1(F,L) = 0$ for every F of type FP_∞.

$$\cdots \longrightarrow R^{(m_2)} \longrightarrow R^{(m_1)} \longrightarrow R^{(m_0)} \longrightarrow F \longrightarrow 0$$

Examples

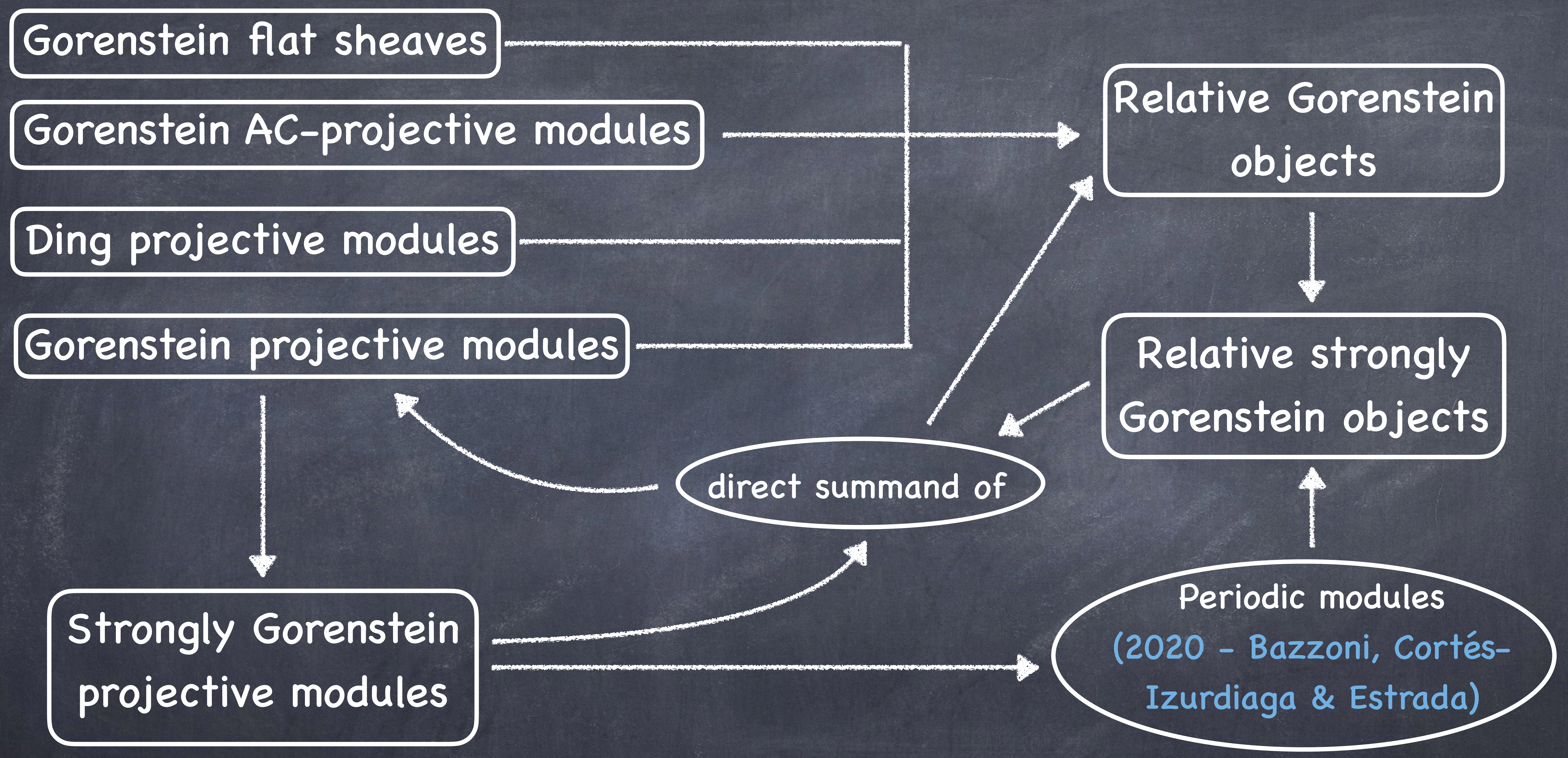
4. Gorenstein flat quasi-coherent sheaves over a noetherian and semi-separated scheme X:

$G \in GF(X)$ if $G = Z_0(F_\bullet)$ where

$$F_\bullet : \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

is an exact complex of flat sheaves such that $F_\bullet \otimes I$ is an exact complex for every injective sheaf I.

$GF(X) = GP_{(\text{flat},\text{flat-cotorsion})}(X)$ [2011 - Murfet & Salarian].


Gorenstein flat sheaves
Gorenstein AC-projective modules
Ding projective modules
Gorenstein projective modules
Relative Gorenstein objects
Relative strongly Gorenstein objects
direct summand of
Strongly Gorenstein projective modules
Periodic modules
(2020 - Bazzoni, Cortés-Izurdiaga & Estrada)

Setting: An abelian category $\mathcal{C}$

(not necessarily with enough projectives and injectives)

Definition: Let $(\mathcal{A},\mathcal{B})$ be a pair of (full) subcategories of $\mathcal{C}$.

$$
\begin{tikzcd}[column sep=small]
A_\bullet : \cdots \arrow[r] & A \arrow[r] & A \arrow[rr] \arrow[dr] & & A \arrow[r] & A \arrow[r] & A \arrow[r] & \cdots \\
& & & M \arrow[ur] & & & &
\end{tikzcd}
$$

(exact with A in $\mathcal{A}$)

$$M = Z_0(A_\bullet) \in \pi GP_{(\mathcal{A},\mathcal{B})}$$

$$\mathrm{Hom}(A_\bullet,B) : \quad \cdots \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B)$$

$$\longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B)$$

$$\longrightarrow \cdots$$

(exact for every B in $\mathcal{B}$)

Setting: An abelian category $\mathcal{C}$
(not necessarily with enough projectives and injectives)

Definition: Let $(\mathcal{A},\mathcal{B})$ be a pair of (full) subcategories of $\mathcal{C}$.

Equivalently, there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow M \longrightarrow 0$$

which is $\mathrm{Hom}(-,\mathcal{B})$-acyclic:

$$0 \longrightarrow \mathrm{Hom}(M,B) \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(M,B) \longrightarrow 0$$

is exact for every B in $\mathcal{B}$.

Examples

1. Strongly Gorenstein projective modules: A = B = Proj(R)

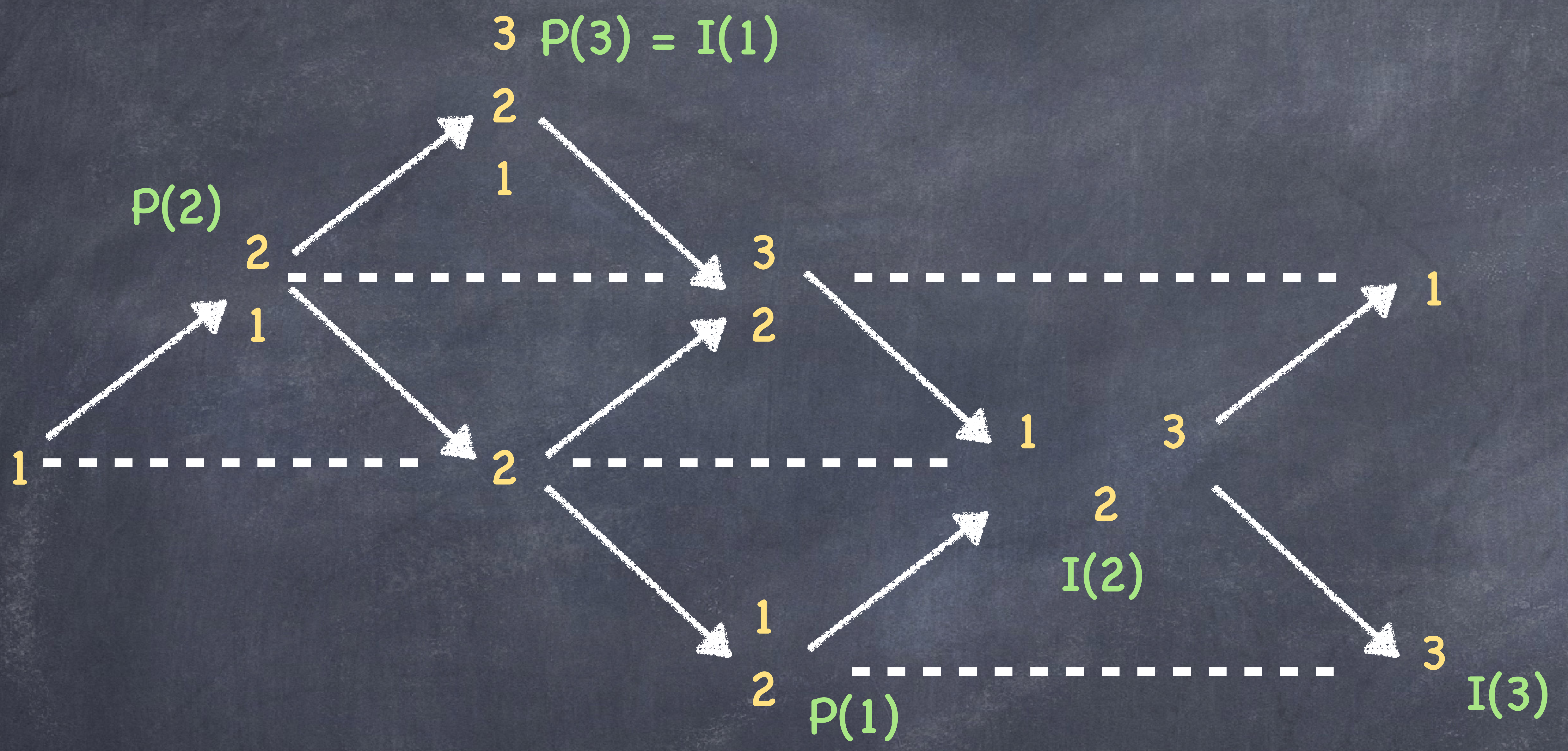
2. Consider the following quiver with relations:

$$Q:\quad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] & 3 \arrow[l, "\gamma"'] \end{tikzcd} \qquad \alpha\beta = 0 = \beta\alpha$$

Let k be a field and Λ the path algebra over k given by Q.

[2019 - Zhang & Xiong]

Auslander-Reiten quiver:



Remark:

$\pi GP_{P(X)} \neq \pi gp(\Lambda)$

$X = \mathrm{add}\left(1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right)$ is a Frobenius subcategory of $\mathrm{mod}(\Lambda)$

$A = B = P(X)$ (projective objects in X)

3. Relative strongly Gorenstein injectives:

A = finitely presented R-modules

B = injective R-modules

Proposition: Every strongly (fp,inj)-Gorenstein injective R-module is absolutely pure (a.k.a. FP-injective).

Let M in $\pi\mathrm{GI}_{(\mathrm{fp,inj})}$:

$\Longrightarrow$ $0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0$ exact and pure exact

$\Longrightarrow$ $0 \longrightarrow M \longrightarrow E \longrightarrow E \longrightarrow \cdots$ pure exact injective coresolution

$\Longrightarrow$ $\mathrm{Ext}^1(F, M) = 0$ for every F finitely presented.

strongly (fp,inj)-Gorenstein injectives $\subseteq$ absolutely pures

Proposition: The following are equivalent:

(a) R is a (left) noetherian ring.

(b) absolutely pures = injectives.

(c) every absolutely pure R-module is strongly (fp,inj)-Gorenstein injective.

proof:

(a) $\Longleftrightarrow$ (b) [1970 - Megibben].

(b) $\Longrightarrow$ (c) Clear.

(c) $\Longrightarrow$ (b) Suppose M is an absolutely pure R-module.

(c) $\Longrightarrow 0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0$ exact and pure exact

$\Longrightarrow \cdots \longrightarrow E \longrightarrow E \longrightarrow E \longrightarrow \cdots$ exact complex with abs. pure cycles (i.e. M)

[2020 - Bazzoni, Cortés-Izurdiaga & Estrada]

$\Downarrow$

$\cdots \longrightarrow E \longrightarrow E \longrightarrow E \longrightarrow \cdots$ is an injective complex (i.e., exact with injective cycles)

∎

Relative Gorenstein objects with period > 1

Definition: Let $(\mathcal{A},\mathcal{B})$ be a pair of (full) subcategories of $\mathcal{C}$.

1. There exists an exact sequence

$$0 \longrightarrow M \longrightarrow A_m \longrightarrow \cdots \longrightarrow A_1 \longrightarrow M \longrightarrow 0$$

(M is $\mathcal{A}$-periodic with period m) $M \in \pi_m(\mathcal{A})$

2. The previous sequence is $\mathrm{Hom}(-,\mathcal{B})$-acyclic.

(M is m-periodic $(\mathcal{A},\mathcal{B})$-Gorenstein projective) $M \in \pi GP_{(\mathcal{A},\mathcal{B},m)}$

Properties and relations with periodic objects and relative Gorenstein objects.

Proposition (relations): Let $(\mathcal{A},\mathcal{B})$ be a hereditary pair (i.e., $\mathrm{Ext}^i(A,B) = 0$ for every A in $\mathcal{A}$, B in $\mathcal{B}$ and $i > 0$).

Then:

$$\boxed{\pi\mathrm{GP}_{(\mathcal{A},\mathcal{B},m)} = \mathrm{GP}_{(\mathcal{A},\mathcal{B})} \cap \pi_m(\mathcal{A})}$$

Proposition (characterizations): Let (A,B) be a hereditary pair with A closed under finite coproducts, and consider the following conditions:

1. M is m-periodic (A,B)-Gorenstein projective.

$$\eta : 0 \longrightarrow M \xrightarrow{f_m} A_m \xrightarrow{f_{m-1}} \cdots \longrightarrow A_1 \xrightarrow{f_0} M \longrightarrow 0$$

2. $\exists\ \eta$ exact s.t. $\mathrm{Ext}^{i+1}(M,B) = \mathrm{Ext}^{i+2}(M,B) = \ldots = \mathrm{Ext}^{i+m}(M,B) = 0$ for some $i \geq 0$.

3. $\exists\ \eta$ exact s.t. $\displaystyle\bigoplus_{k=2}^{m+1} \mathrm{Im}(f_k) \in \pi\mathrm{GP}_{(A,B,1)}$.

4. $\exists\ \eta$ exact s.t. $\displaystyle\bigoplus_{k=2}^{m+1} \mathrm{Im}(f_k) \in \mathrm{GP}_{(A,B)}$.

$$\begin{tikzcd} 1. \arrow[r, leftrightarrow] & 2. \arrow[d] \\ 4. \arrow[u] & 3. \arrow[l] \end{tikzcd}$$

(A,B) is a GP-admissible pair

[2021 - Becerril, Mendoza & Santiago]

$(\mathcal{A},\mathcal{B})$ is a GP-admissible pair:

1. $(\mathcal{A},\mathcal{B})$ is hereditary.
2. $A \to M \to 0$ for every M in $\mathcal{C}$.
3. $\mathcal{A}$ is closed under extensions.
4. $\mathcal{A}$ and $\mathcal{B}$ are closed under finite coproducts.
5. $\mathcal{A} \cap \mathcal{B}$ is a relative cogenerator in $\mathcal{A}$:

   for every A in $\mathcal{A}$: $0 \to A \to W \to A' \to 0$, with $W \in \mathcal{A} \cap \mathcal{B}$ and $A' \in \mathcal{A}$.

Proposition (a relative version for a result of Bennis and Mahdou):

Let $(\mathcal{A},\mathcal{B})$ be a GP-admissible pair in $\mathcal{C}$ (AB4) with $\mathcal{A}$ closed under (arbitrary) coproducts. Then, M is $(\mathcal{A},\mathcal{B})$-Gorenstein projective iff it is a direct summand of a m-periodic $(\mathcal{A},\mathcal{B})$-Gorenstein projective object.

Definition: Let $(\mathcal{A},\mathcal{B})$ be a pair of (full) subcategories of $\mathcal{C}$.
M is acyclic m-periodic $(\mathcal{A},\mathcal{B})$-Gorenstein projective if there exists an exact sequence

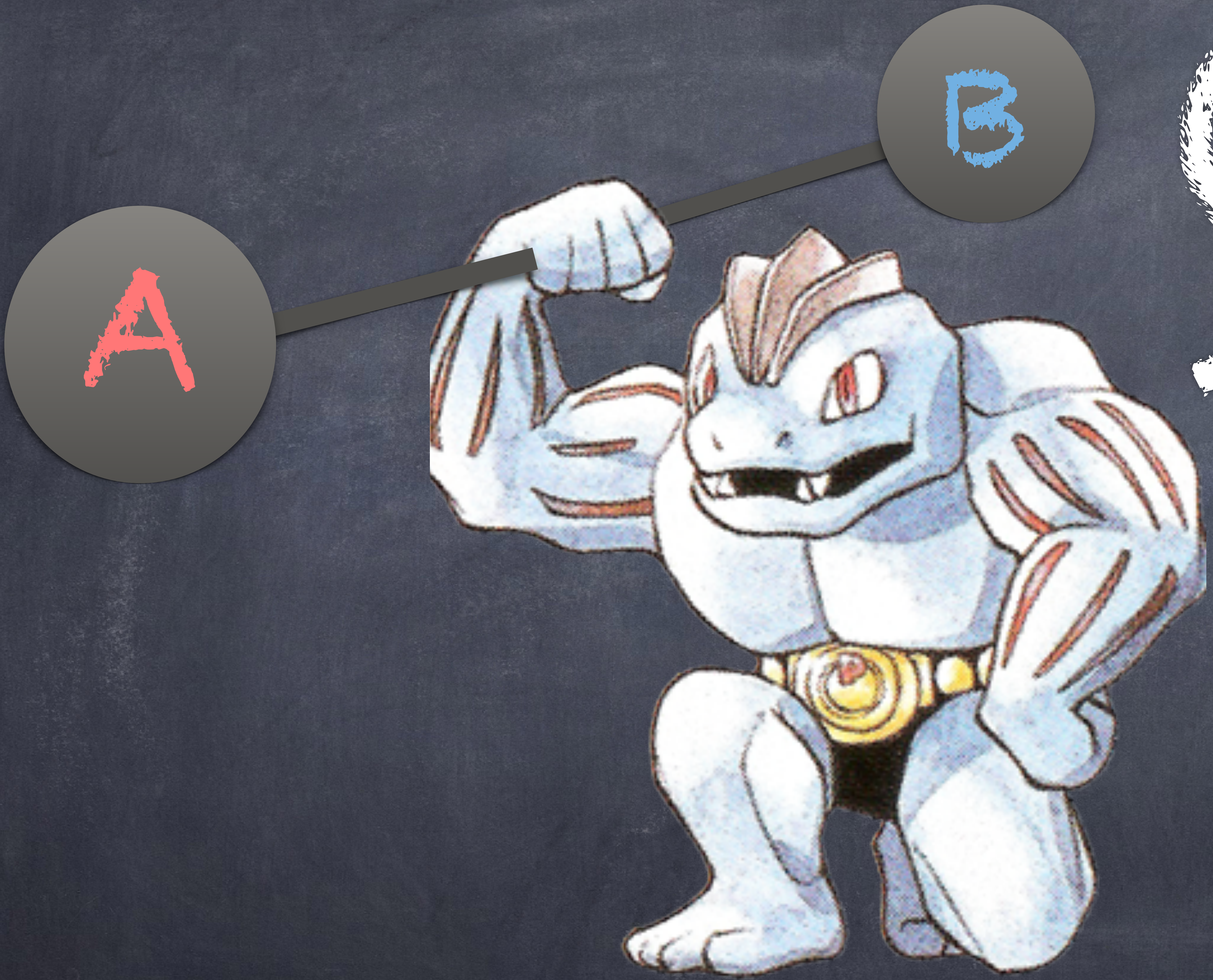
$$0 \longrightarrow M \longrightarrow A_m \longrightarrow \cdots \longrightarrow A_1 \longrightarrow M \longrightarrow 0$$

which is $\mathrm{Hom}(\mathcal{A},-)$-acyclic and $\mathrm{Hom}(-,\mathcal{B})$-acyclic. $\quad M \in \pi\mathrm{GP}^{acy}_{(\mathcal{A},\mathcal{B},m)}$

Theorem (a relative version for a result of Zhao and Huang):

Let $(\mathcal{A},\mathcal{B})$ be a pair of full subcategories of $\mathcal{C}$ s.t. $\mathrm{Ext}^1(A,B) = 0$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and let $m, n \in \mathbb{Z}_{>0}$ with $\gcd(m,n) \neq \min\{m,n\}$. If $\mathrm{Ext}^i(A,A') = 0$ for every $A \in \mathcal{A}$, $A' \in \mathcal{A}$ and $0 \leq i \leq \min\{m,n\}$ and $\mathcal{A}$ is closed under extensions and kernels of epimorphisms, then:

$$\boxed{\pi\mathrm{GP}^{acy}_{(\mathcal{A},\mathcal{B},m)} \cap \pi\mathrm{GP}^{acy}_{(\mathcal{A},\mathcal{B},n)} = \pi\mathrm{GP}^{acy}_{(\mathcal{A},\mathcal{B},\gcd(m,n))}}$$



Muito obrigado