Homological theory of k-idempotent ideal in dualizing varieties

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29 de octubre de 2021

This is a joint work with V. Santiago Vargas (UNAM) and M. L. S. Sandoval Miranda (UAM-I).

1. **Homological theory of k-idempotent ideals in dualizing varieties.**
   Preprint arXiv: 2008.07158v2 (2020)
   L.G. Rodríguez Valdés, M. L. S. Sandoval Miranda, V. Santiago,

Introduction

In classical representation theory one of the objects of study are the artin algebras Λ and its category of finitely generated left Λ-modules $\mathrm{mod}(\Lambda)$. On the other hand, **dualizing varieties** were introduced by Auslander and Reiten as a categorical counterpart for doing representation theory. In fact, Auslander-Reiten theory has its origin from dualizing varieties.

Definition

Let Λ be an artin algebra and I an ideal of Λ. The following are equivalent.

(a) I is an idempotent ideal (that is, $I = I^2$)

(b) There exists an isomorphism for all $M, M' \in \mathrm{mod}(\Lambda/I)$

$$\mathrm{Ext}^1_{\Lambda/I}(M, M') \simeq \mathrm{Ext}^1_{\Lambda}(M, M').$$

Definition

Let Λ be an artin algebra and I an ideal of Λ. It is said that I is a **k-idempotent** ideal if there exists an isomorphism for all $M, M' \in \mathrm{mod}(\Lambda/I)$ $\mathrm{Ext}^i_{\Lambda/I}(M, M') \simeq \mathrm{Ext}^i_{\Lambda}(M, M') \quad \forall 0 \leq i \leq k$.

Given a small preadditive category $\mathcal{C}$ we can think $\mathcal{C}$ as a ring with several objects. So, we can construct its category of left $\mathcal{C}$-modules as follows:

$$\mathrm{Mod}(\mathcal{C}) := (\mathcal{C}, \mathbf{Ab}) := \{F : \mathcal{C} \longrightarrow \mathbf{Ab} \mid F \text{ additive and covariant}\}$$

We recall that

(a) $\mathrm{Mod}(\mathcal{C})$ is a Grothendieck abelian category.

(b) $\{\mathrm{Hom}_{\mathcal{C}}(C, -)\}_{C \in \mathcal{C}}$ is a set of projective generators.

Tensor product

There is a unique (up to isomorphism) functor $\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}^{op}) \times \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$ called the **tensor product**. The abelian group $\otimes_{\mathcal{C}}(A, B)$ is denoted by $A \otimes_{\mathcal{C}} B$ for all $\mathcal{C}^{op}$-modules A and all $\mathcal{C}$-modules B.

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2. For each $\mathcal{C}^{op}$-module A and each $\mathcal{C}$-module B, the functors $A \otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{C}} B$ preserve arbitrary sums.

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2. For each $\mathcal{C}^{op}$-module A and each $\mathcal{C}$-module B, the functors $A \otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{C}} B$ preserve arbitrary sums.
3. For each object C in $\mathcal{C}$ we have $A \otimes_{\mathcal{C}} (C, -) = A(C)$ and $(-, C) \otimes_{\mathcal{C}} B = B(C)$ for all $\mathcal{C}^{op}$-modules A and all $\mathcal{C}$-modules B.

Varieties

Definition

It is said that and additive category $\mathcal{C}$ (preadditive with finite coproducts) is a category with **splitting idempotents** if for each idempotent $e = e^2 \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ there are morphisms $\mu : Y \to X$ and $\rho : X \to Y$ such that $\mu\rho = e$ and $\rho\mu = 1_Y$.

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Definition

Let $\mathcal{C}$ be an additive category with splitting idempotents. Such categories are called **varieties**.

Definition

If $\mathcal{C}$ is an R-variety, then $\mathrm{Mod}(\mathcal{C}) = (\mathcal{C}, \mathbf{Ab})$ is an R-variety, which we identify with the category of covariant functors $(\mathcal{C}, \mathrm{Mod}(R))$.

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The category $(\mathcal{C}, \mathrm{mod}(R))$ is abelian and the inclusion

$$(\mathcal{C}, \mathrm{mod}(R)) \subseteq (\mathcal{C}, \mathrm{Mod}(R)) = \mathrm{Mod}(\mathcal{C})$$

is exact.

Definition

Let $\mathcal{C}$ be a Hom-finite R-variety. We denote by $\mathrm{mod}(\mathcal{C})$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ whose objects are the **finitely presented functors**. That is, $M \in \mathrm{mod}(\mathcal{C})$ if and only if, there exists an exact sequence in $\mathrm{Mod}(\mathcal{C})$

$$\mathrm{Hom}_{\mathcal{C}}(C_0, -) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_1, -) \longrightarrow M \longrightarrow 0.$$

Consider the functors

$$\mathbb{D}_{\mathcal{C}^{op}} : (\mathcal{C}^{op}, \mathrm{mod}(R)) \to (\mathcal{C}, \mathrm{mod}(R))$$

$$\mathbb{D}_{\mathcal{C}} : (\mathcal{C}, \mathrm{mod}(R)) \to (\mathcal{C}^{op}, \mathrm{mod}(R)),$$

which are defined as follows:

for any object C in $\mathcal{C}$, $\mathbb{D}(M)(C) = \mathrm{Hom}_R(M(C), I(R/r))$ where r is the Jacobson radical of R, and $I(R/r)$ is the injective envelope of R/r.

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Proposition

The functor $\mathbb{D}_{\mathcal{C}}$ defines a duality

$$\mathbb{D}_{\mathcal{C}} : (\mathcal{C}, \mathrm{mod}(R)) \longrightarrow (\mathcal{C}^{op}, \mathrm{mod}(R)).$$

Definition

An Hom-finite R-variety $\mathcal{C}$ is **dualizing**, if the functor

$$\mathbb{D}_{\mathcal{C}} : (\mathcal{C}, \operatorname{mod}(R)) \to (\mathcal{C}^{op}, \operatorname{mod}(R)) \tag{1}$$

induces a duality between the categories $\operatorname{mod}(\mathcal{C})$ and $\operatorname{mod}(\mathcal{C}^{op})$:

$$\begin{tikzcd}
(\mathcal{C}, \operatorname{mod}(R)) \arrow[r, "\mathbb{D}_{\mathcal{C}}"] & (\mathcal{C}^{op}, \operatorname{mod}(R)) \\
\operatorname{mod}(\mathcal{C}) \arrow[u, hook] \arrow[r] & \operatorname{mod}(\mathcal{C}^{op}) \arrow[u, hook]
\end{tikzcd}$$

Ideals

Definition

Let $\mathcal{C}$ be a preadditive category. An **ideal** $\mathcal{I}$ of $\mathcal{C}$ is an additive subfunctor of $\mathrm{Hom}_{\mathcal{C}}(-,-)$. That is, $\mathcal{I}$ is a class of the morphisms in $\mathcal{C}$ such that:

(a) $\mathcal{I}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B) \cap \mathcal{I}$ is an abelian subgroup of $\mathrm{Hom}_{\mathcal{C}}(A,B)$ for each $A, B \in \mathcal{C}$;

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(b) If $f \in \mathcal{I}(A,B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(C,A)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(B,D)$, then $hfg \in \mathcal{I}(C,D)$.

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$$A \xrightarrow{f\in\mathcal{I}} B$$

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$$\begin{tikzcd} C \arrow[r, "g"] \arrow[rrr, bend right, "hfg\in\mathcal{I}"'] & A \arrow[r, "f\in\mathcal{I}"] & B \arrow[r, "h"] & D \end{tikzcd}$$

Definition

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals in $\mathcal{C}$. The **product of ideals** $\mathcal{I} \cdot \mathcal{J}$ is defined as follows: for each $A, B \in \mathcal{C}$ we set

$$(\mathcal{I} \cdot \mathcal{J})(A, B) := \left\{ \sum_{i=1}^{n} f_i g_i \;\middle|\; f_i \in \mathcal{I}(C_i, B), g_i \in \mathcal{J}(A, C_i) \text{ for some } C_i \in \mathcal{C} \right\}$$

$$
\begin{tikzcd}
 & C_1 \arrow[ddr, "f_1"] & \\
 & C_2 \arrow[dr, "f_2"] & \\
A \arrow[uur, "g_1"] \arrow[ur, "g_2"] \arrow[r] \arrow[dr, "g_{n-1}"'] \arrow[ddr, "g_n"'] & \vdots \arrow[r] & B \\
 & C_{n-1} \arrow[ur, "f_{n-1}"'] & \\
 & C_n \arrow[uur, "f_n"'] &
\end{tikzcd}
$$

$$g_i \in \mathcal{J}, \qquad f_i \in \mathcal{I}.$$

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Definition

Let $\mathcal{I}$ be an ideal of $\mathcal{C}$, we set
$\mathrm{Ann}(\mathcal{I}) := \{F \in \mathrm{Mod}(\mathcal{C}) \mid F(f) = 0\ \forall f \in \mathcal{I}(A, B)\ \forall A, B \in \mathcal{C}\}$.

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Let $\mathcal{I}$ be an ideal in a preadditive category $\mathcal{C}$. The **quotient category** $\mathcal{C}/\mathcal{I}$ is defined as follows:

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we set

$$\overline{g} \circ \overline{f} := gf + \mathcal{I}(A,C) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,C).$$

Canonical functor

Let $\mathcal{I}$ be an ideal of $\mathcal{C}$, we have the canonical functor $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ defined as:

- $\pi(A) = A \ \forall A \in \mathcal{C}$ and
- $\pi(f) := \overline{f} = f + \mathcal{I}(A, B) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A, B) \ \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

$\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ induces a functor

$$\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C})$$

given as:

- $\pi_*(G) := G \circ \pi$ for $G \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] \arrow[dr, "G\circ\pi"'] & \mathcal{C}/\mathcal{I} \arrow[d, "G"] \\ & \mathbf{Ab} \end{tikzcd}$$

- $\pi_*(\eta) := \eta \circ \pi$ for $\eta : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(G, H)$.

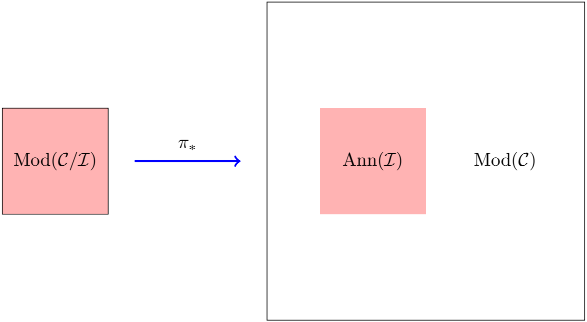

Figura: π_* is full and faithful

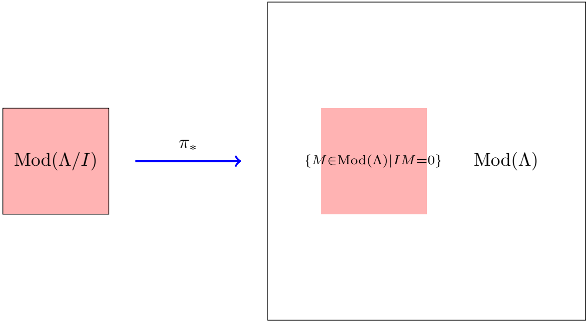
$\mathrm{Mod}(\Lambda/I)$
π_*
$\{M \in \mathrm{Mod}(\Lambda) | IM = 0\}$
$\mathrm{Mod}(\Lambda)$

Figura:

Definition

We define the functor $\frac{\mathcal{C}}{\mathcal{I}}\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ as follows: for $M \in \mathrm{Mod}(\mathcal{C})$ we set $\left(\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} M\right)(C) := \frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} M$ for all $C \in \mathcal{C}/\mathcal{I}$ and $\left(\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} M\right)(\overline{f}) = \frac{\mathcal{C}}{\mathcal{I}}(-,f) \otimes_{\mathcal{C}} M$ for all $\overline{f} = f + \mathcal{I}(C, C') \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, C')$.

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Definition

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Proposition

Let $\mathcal{I}$ be an ideal in $\mathcal{C}$ and $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor. Then we have the following diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*" description] & \mathrm{Mod}(\mathcal{C}) \arrow[l, shift right=6, "\pi^*"'] \arrow[l, shift left=6, "\pi^!"] \end{tikzcd}$$

where (π^*, π_*) and $(\pi_*, \pi^!)$ are adjoint pairs, $\pi^! = \mathrm{Hom}(\frac{\mathcal{C}}{\mathcal{I}}, -)$ and $\pi^* := \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}}$.

Definition

Consider the functor $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. We denote by $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ the i-th right derived functor of $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -)$.

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Property A

Definition

Let $\mathcal{C}$ be a preadditive category. We say that an ideal $\mathcal{I}$ satisfies the **property (A)** if for every $C \in \mathcal{C}$ there exists epimorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$$

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$$\mathrm{Hom}_{\mathcal{C}}(-, Y) \longrightarrow \mathcal{I}(-, C) \longrightarrow 0.$$

Proposition

Let $\mathcal{C}$ be an additive category and $\mathcal{X}$ an additive full subcategory of $\mathcal{C}$. Let $\mathcal{I} = \mathcal{I}_{\mathcal{X}}$ be the ideal of morphisms in $\mathcal{C}$ which factor through some object in $\mathcal{X}$. Then $\mathcal{I}$ satisfies property A if and only if $\mathcal{X}$ is functorially finite in $\mathcal{C}$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal which satisfies property A. Let $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ be the canonical functor, then we can restrict to the finitely presented modules

$$\begin{tikzcd} \mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)*" description] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right, "\pi_1^*"'] \arrow[l, bend left, "\pi_1^!"] \end{tikzcd}$$

Proposition

Let $\mathcal{C}$ be a preadditive category and $\mathcal{I}$ an ideal in $\mathcal{C}$. The following are equivalent.

(a) $\mathcal{I}$ is an idempotent ideal.

Proposition

Let $\mathcal{C}$ be a preadditive category and $\mathcal{I}$ an ideal in $\mathcal{C}$. The following are equivalent.

(a) $\mathcal{I}$ is an idempotent ideal.

(b) There exists an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$

$$\varphi^1_{F,\pi_*(F')} : \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F')).$$

Proposition

Let $F, G \in \mathrm{Mod}(\mathcal{C})$. Then for each $i \geq 0$, there exists canonical morphisms of abelian groups

$$\varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, G) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(G)).$$

Definition

Let $\mathcal{C}$ be a preadditive category and $\mathcal{I}$ an ideal in $\mathcal{C}$.

(a) We say that $\mathcal{I}$ is **k-idempotent** if

$$\varphi^{i}_{F,\pi_*(F')} : \operatorname{Ext}^{i}_{\operatorname{Mod}(\mathcal{C}/I)}(F, F') \longrightarrow \operatorname{Ext}^{i}_{\operatorname{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F'))$$

is an isomorphism for all $F, F' \in \operatorname{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i \leq k$.

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is an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i \leq k$.

(b) We say that $\mathcal{I}$ is **strongly idempotent** if

$$\varphi^{i}_{F,\pi_*(F')} : \mathrm{Ext}^{i}_{\mathrm{Mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^{i}_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F'))$$

is an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i < \infty$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety, $\mathcal{I}$ an ideal which satisfies property A and $1 \leq i \leq k$. The following are equivalent.

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(c) $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ for $1 \leq i \leq k$ and for $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety, $\mathcal{I}$ an ideal which satisfies property A and $1 \leq i \leq k$. The following are equivalent.

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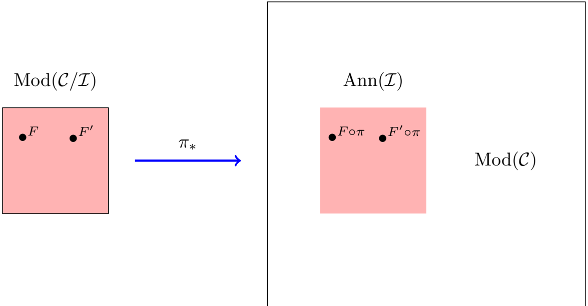
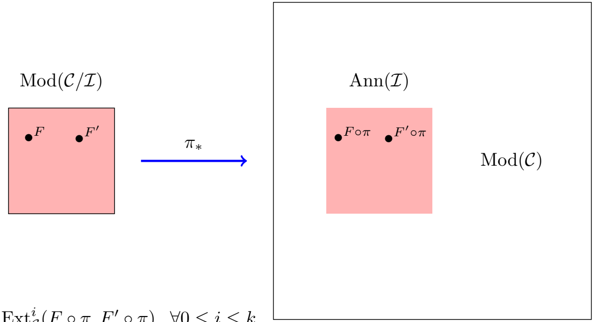
(c) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ for $1 \leq i \leq k$ and for $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(d) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J \circ \pi) = 0$ for $1 \leq i \leq k$ and for each $J \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ which is injective.



$$\mathrm{Ext}^i_{\mathcal{C}/I}(F, F') \simeq \mathrm{Ext}^i_{\mathcal{C}}(F \circ \pi, F' \circ \pi), \ \forall 0 \leq i \leq k$$

Figura: Definition of k-idempotent



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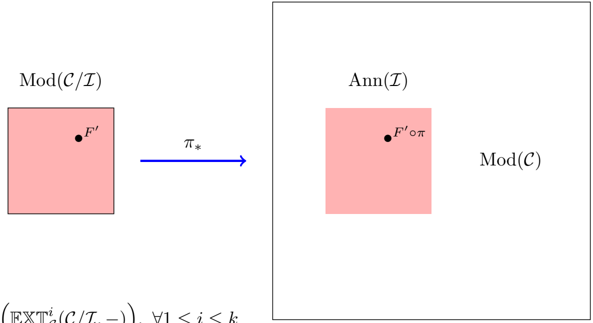

$$\mathrm{Ann}(\mathcal{I}) \subseteq \mathrm{Ker}\Big(\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, -)\Big),\ \forall 1 \leq i \leq k$$

Figura: Equivalent condition of k-idempotent

Let Λ be an artin algebra and P a finitely generated projective Λ-módule. In this case we consider the **trace ideal**

$$I = \mathrm{Tr}_P(\Lambda) := \sum_{f \in \mathrm{Hom}(P,\Lambda)} \mathrm{Im}(f) \subseteq \Lambda.$$

It is well known that I is two sided ideal of Λ and $I^2 = I$.

Definition

Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a family in $\mathrm{Mod}(\mathcal{C})$. For each $N \in \mathrm{Mod}(\mathcal{C})$ de define a $\mathcal{C}$-submodule of N, denoted by $\mathrm{Tr}_{\mathcal{F}}(N)$ as follows: for $X \in \mathcal{C}$ we define

$$\mathrm{Tr}_{\mathcal{F}}(N)(X) = \sum_{\{f \in \mathrm{Hom}(F,N) \mid F \in \mathcal{F}\}} \mathrm{Im}(f_X).$$

In the case $\mathcal{F} = \{F\}$ is just one object, we will write $\mathrm{Tr}_F N$.

Definition

Let $\mathcal{C}$ be a preadditive category and $\mathcal{F} = \{F_i\}_{i \in I}$ a family of objects in $\mathrm{Mod}(\mathcal{C})$.
For each $C \in \mathcal{C}$ consider the $\mathcal{C}$-submodule $\mathrm{Tr}_{\mathcal{F}}(\mathrm{Hom}_{\mathcal{C}}(C, -))$ of $\mathrm{Hom}_{\mathcal{C}}(C, -)$.
We define the subfunctor $\mathrm{Tr}_{\mathcal{F}}\mathcal{C}$ of $\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{Ab}$ as follows:

$$(\mathrm{Tr}_{\mathcal{F}}\mathcal{C})(C, C') := \mathrm{Tr}_{\mathcal{F}}(\mathrm{Hom}_{\mathcal{C}}(C, -))(C')$$

for all $C, C' \in \mathcal{C}$. This ideal will be called **trace ideal**. In the case that $\mathcal{F} = \{P\}$ with P a projective $\mathcal{C}$-module, we will write $\mathrm{Tr}_P\mathcal{C}$.

Let $\mathcal{C}$ be a preadditive category and let P be a projective $\mathcal{C}$-module. Then $\mathcal{I} := \mathrm{Tr}_P\mathcal{C}$ defines an idempotent ideal of $\mathcal{C}$ (i.e $\mathcal{I}^2 = \mathcal{I}$).

Let $\mathcal{C}$ be a preadditive category and let P be a projective $\mathcal{C}$-module. Then $\mathcal{I} := \mathrm{Tr}_P\mathcal{C}$ defines an idempotent ideal of $\mathcal{C}$ (i.e $\mathcal{I}^2 = \mathcal{I}$).

Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -)$ a finitely generated projective $\mathcal{C}$-module and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Then $\mathcal{I}$ satisfies property A.

Corollary

Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P \mathcal{C}$. Then we can restrict the diagram of a previous result to the finitely presented modules

$$\begin{tikzcd} \mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)*"] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right, "\pi_1^*"'] \arrow[l, bend left, "\pi_1^!"] \end{tikzcd}$$

Definition

Let $\mathcal{C}$ be a dualizing R-variety and $P \in \mathrm{mod}(\mathcal{C})$ be a projective module. For each $0 \leq k \leq \infty$ we define $\mathbb{P}_k$ to be the full subcategory of $\mathrm{mod}(\mathcal{C})$ consisting of the $\mathcal{C}$-modules X having a projective resolution

$$\cdots P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with $P_i \in \mathrm{add}(P)$ for $0 \leq i \leq k$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. For $1 \leq k \leq \infty$ and $X \in \mathrm{mod}(\mathcal{C})$, the following are equivalent.

(a) $X \in \mathbb{P}_k$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P \mathcal{C}$. For $1 \leq k \leq \infty$ and $X \in \mathrm{mod}(\mathcal{C})$, the following are equivalent.

(a) $X \in \mathbb{P}_k$.

(b) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \ldots, k$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P \mathcal{C}$. For $1 \leq k \leq \infty$ and $X \in \mathrm{mod}(\mathcal{C})$, the following are equivalent.

(a) $X \in \mathbb{P}_k$.

(b) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \ldots, k$.

(c) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) = 0$ for all $J \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ injective and $i = 0, \ldots, k$.

We recall that given an abelian category $\mathcal{A}$ and $\mathcal{B}$ a subcategory of $\mathcal{A}$ we define the perpendicular category

$$^{\perp_i}\mathcal{B} = \{A \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0,\ \ \forall B \in \mathcal{B},\ \forall\ 0 \leq i \leq k\}$$

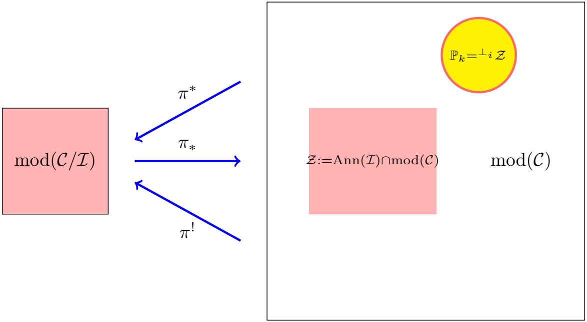

Figura: $\mathbb{P}_k = {}^{\perp_i}\mathcal{Z}$ with $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$.

We also have the dual notion of $\mathbb{P}_k$.

Definition

Let $\mathcal{C}$ be a dualizing R-variety, $P \in \operatorname{mod}(\mathcal{C})$ a projective module and $J := I_0(\frac{P}{\operatorname{rad}(P)}) \in \operatorname{mod}(\mathcal{C})$ the injective envelope of $\frac{P}{\operatorname{rad}(P)}$. For each $0 \leq k \leq \infty$ we define $\mathbb{I}_k$ to be the full subcategory of $\operatorname{mod}(\mathcal{C})$ consisting of the $\mathcal{C}$-modules Y having an injective coresolution $0 \longrightarrow Y \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots$ with $J_i \in \operatorname{add}(J)$ for $0 \leq i \leq k$.

Proposition

Let $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{mod}(\mathcal{C})$ be a projective module, $\mathcal{I} = \operatorname{Tr}_P \mathcal{C}$ and $1 \leq k \leq \infty$. Let $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical projection. The following conditions are equivalent for $Y \in \operatorname{mod}(\mathcal{C})$.

(a) $Y \in \mathbb{I}_k$.

Proposition

Let $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ be a projective module, $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$ and $1 \leq k \leq \infty$. Let $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical projection. The following conditions are equivalent for $Y \in \mathrm{mod}(\mathcal{C})$.

(a) $Y \in \mathbb{I}_k$.

(b) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$ for all $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \ldots, k$.

Proposition

Let $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ be a projective module, $\mathcal{I} = \mathrm{Tr}_P \mathcal{C}$ and $1 \leq k \leq \infty$. Let $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical projection. The following conditions are equivalent for $Y \in \mathrm{mod}(\mathcal{C})$.

(a) $Y \in \mathbb{I}_k$.

(b) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$ for all $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \ldots, k$.

(c) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(Q), Y) = 0$ for all $Q \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ projective and $i = 0, \ldots, k$.

We recall that given an abelian category $\mathcal{A}$ and $\mathcal{B}$ a subcategory of $\mathcal{A}$ we define the perpendicular category

$$\mathcal{B}^{\perp_i} = \{A \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0, \ \forall B \in \mathcal{B}, \ \forall \ 0 \leq i \leq k\}$$

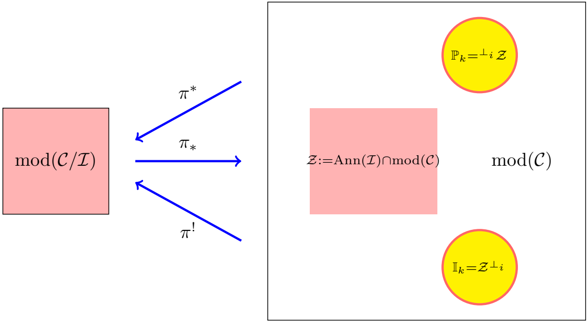

Figura: $\mathbb{P}_k = {}^{\perp_i}\mathcal{Z}$ and $\mathbb{I}_k = \mathcal{Z}^{\perp_i}$

Definition

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Then the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "i_*=i_!" description] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{C} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

is called a **recollement**, if the additive functors $i^*, i_* = i_!, i^!, j_!, j^! = j^*$ and j_* satisfy the following conditions:

(R1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples, i.e. (i^*, i_*), $(i_!, i^!)$ $(j_!, j^!)$ and (j^*, j_*) are adjoint pairs;

(R2) $j^* i_* = 0$;

(R3) $i_*, j_!, j_*$ are full embedding functors.

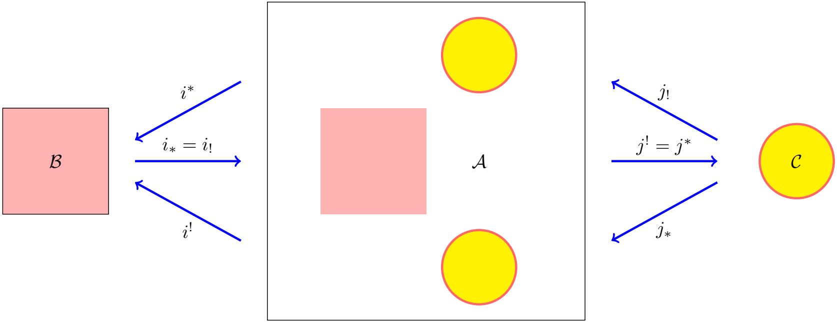

Figura: Recollement

Proposition

Let Λ an artin algebra and $P \in \operatorname{mod}(\Lambda)$ a finitely generated projective. Consider the trace ideal $\operatorname{Tr}_P(\Lambda) = I$. Then, there exist a recollement

$$\begin{tikzcd}[column sep=huge]
\operatorname{mod}(\Lambda/I) \arrow[r, "\pi_*" description] & \operatorname{mod}(\Lambda) \arrow[l, bend right, "\Lambda/I\otimes_\Lambda -"'] \arrow[l, bend left, "\operatorname{Hom}_\Lambda(\Lambda/I,-)"] \arrow[r, "\operatorname{Hom}_\Lambda(P,-)" description] & \operatorname{mod}(R_P) \arrow[l, bend right, "P\otimes_{R_P} -"'] \arrow[l, bend left, "\operatorname{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

where $R_P = \operatorname{End}(P)^{op}$ and $P^* = \operatorname{Hom}_\Lambda(P, \Lambda)$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$, $\mathcal{B} = \mathrm{add}(C)$ and $R_P = \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$. Then, there exist a recollement

$$\begin{tikzcd}[column sep=huge] \mathrm{mod}(\mathcal{C}/\mathcal{I}_{\mathcal{B}}) \arrow[r, "\pi_*" description] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right, "\mathcal{C}/\mathcal{I}_{\mathcal{B}}\otimes_{\mathcal{C}}-"'] \arrow[l, bend left, "\mathcal{C}(\mathcal{C}/\mathcal{I}_{\mathcal{B}},-)"] \arrow[r, "\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)" description] & \mathrm{mod}(R_P) \arrow[l, bend right, "P\otimes_{R_P}-"'] \arrow[l, bend left, "\mathrm{Hom}_{R_P}(P^*,-)"] \end{tikzcd}$$

where $\mathcal{I}_{\mathcal{B}}$ is the ideal of morphisms in $\mathcal{C}$ which factor through objects in $\mathcal{B}$. Moreover, we have that $\mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_P\mathcal{C}$.

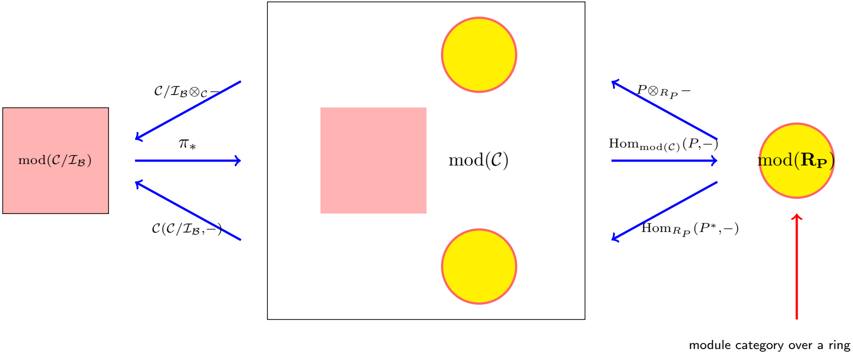

Figura: The recollement of finitely presented functors.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The map

$$\Phi^n_{X,Y} : \mathrm{Ext}^n_{\mathrm{mod}(\mathcal{C})}(X, Y) \longrightarrow \mathrm{Ext}^n_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y)\Big)$$

is an isomorphism for all $n \geq 0$, provided one of the three following conditions holds:

(a) $X \in \mathbb{P}_i$, $Y \in \mathbb{I}_j$ and $n \leq i + j$,

Proposition

Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The map

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is an isomorphism for all $n \geq 0$, provided one of the three following conditions holds:

(a) $X \in \mathbb{P}_i$, $Y \in \mathbb{I}_j$ and $n \leq i + j$,

(b) $X \in \mathrm{mod}(\mathcal{C})$ and $Y \in \mathbb{I}_{n+1}$,

Proposition

Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The map

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is an isomorphism for all $n \geq 0$, provided one of the three following conditions holds:

(a) $X \in \mathbb{P}_i$, $Y \in \mathbb{I}_j$ and $n \leq i + j$,

(b) $X \in \mathrm{mod}(\mathcal{C})$ and $Y \in \mathbb{I}_{n+1}$,

(c) $X \in \mathbb{P}_{n+1}$ and $Y \in \mathrm{mod}(\mathcal{C})$.

Corollary

Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The following conditions hold.

(a) If $X \in \mathbb{P}_{\infty}$ then $\mathrm{pd}(X) = \mathrm{pd}_{R_P}\Big((P, X)\Big)$.

Corollary

Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The following conditions hold.

(a) If $X \in \mathbb{P}_{\infty}$ then $\mathrm{pd}(X) = \mathrm{pd}_{R_P}\Big((P, X)\Big)$.

(b) If $X \in \mathbb{I}_{\infty}$ then $\mathrm{id}(X) = \mathrm{id}_{R_P}\Big((P, X)\Big)$.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety with cokernels and consider $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. If $\mathbb{P}_1 = \mathbb{P}_\infty$ or $\mathbb{I}_1 = \mathbb{I}_\infty$ then $\mathrm{gl.dim}(R_P) \leq 2$. In particular, R_P is a quasi-hereditary algebra.

Proposition

Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Consider the functor $\pi_* : \mathrm{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C})$. If $\mathcal{I}(C', -)$ is projective for all $C' \in \mathcal{C}$, we have a full embedding

$$\mathrm{D}^b(\pi_*) : \mathrm{D}^b(\mathrm{mod}(\mathcal{C}/\mathcal{I})) \longrightarrow \mathrm{D}^b(\mathrm{mod}(\mathcal{C}))$$

between its bounded derived categories.

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