Preprojective algebras: classical and higher

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Representation Theory of Algebras and Applications
11 June 2021

Let k be an algebraically closed field.

Let Λ be a finite dimensional k-algebra.

Q. What is the job of a representation theorist?

A. To understand all the Λ-modules.

First: find all the (finite dimensional) Λ-modules.

- What are the simple modules?
- Can we build bigger modules?

Next: how do they interact?

Easiest case: Λ is semisimple

(e.g., $\Lambda = \mathbb{C}G$, G finite group)

Then every module is a direct sum of simple modules

e.g., for the regular module we have

$$\Lambda \cong {S_1}^{\oplus d_1} \oplus \cdots \oplus {S_n}^{\oplus d_n}$$

And by Schur's lemma,

$$\mathrm{Hom}_\Lambda \cong \begin{cases} k, & S \cong T \\ 0, & S \ncong T \end{cases}$$

What's the next easiest case?

For general Λ, the regular module is

$$\Lambda \cong {P_1}^{\oplus d_1} \oplus \cdots \oplus {P_n}^{\oplus d_n},$$

a sum of projective modules.

We can replace a module by a projective resolution:

$$\begin{tikzcd} \cdots \arrow[r, "d"] & P^{(2)} \arrow[r, "d"] & P^{(1)} \arrow[r, "d"] & P^{(0)} \arrow[d, two heads] \\ & & & M \end{tikzcd}$$

$d^2 = 0$

Each module M has a projective dimension pdim M : this is the length of its shortest projective resolution.

The global dimension of an algebra is:

$$\operatorname{gldim} \Lambda = \sup \{ \operatorname{pdim} M \} \in \mathbb{N} \cup \infty$$

- gldim $\Lambda = 0 \Leftrightarrow \Lambda$ is semisimple.
- gldim $\Lambda \leq 1 \Leftrightarrow \Lambda$ is hereditary.

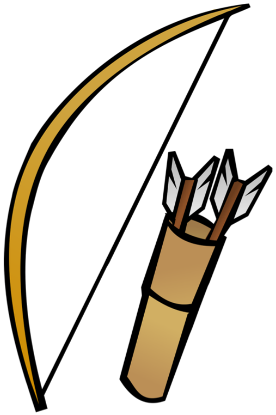
Up to Morita equivalence we can assume Λ is basic, i.e., all simple modules are 1-dimensonal. Then

- Λ is hereditary $\Leftrightarrow \Lambda \cong kQ$ for some quiver Q.

A quiver is a (finite) directed graph.

The path algebra kQ has basis all paths.
Length zero path at vertex i denoted e_i.

Example: $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ (type A_3)



left\right	e_1	e_2	e_3	α	β	$\alpha\beta$
e_1	e_1			α		$\alpha\beta$
e_2		e_2			β	
e_3			e_3			
α		α			$\alpha\beta$	
β			β			
$\alpha\beta$			$\alpha\beta$			

A representation of a quiver assigns vector spaces to vertices, and linear maps to arrows.

There is an equivalence of categories

$$\mathrm{mod}{-}kQ \simeq \mathrm{Rep}(Q)$$

sending $M \in \mathrm{mod}{-}kQ$ to the rep V with $V_i = e_i M$.

Example: $Q = 1 \xrightarrow{\alpha} 2$ (type A_2)

$V: \quad V_1 \xrightarrow{V_\alpha} V_2$

$W: \quad W_1 \xrightarrow{W_\alpha} W_2$

$V \oplus W: \quad V_1 \oplus W_1 \xrightarrow{\begin{pmatrix} V_\alpha & \\ & W_\alpha \end{pmatrix}} V_2 \oplus W_2$

$$\cong$$

$$\left(\mathrm{coim} \xrightarrow{\sim} \mathrm{im}\right)$$
$$\oplus$$
$$\left(\ker \xrightarrow{0} 0\right)$$
$$\oplus$$
$$\left(0 \xrightarrow{0} \mathrm{coker}\right)$$

Indec. rep.s:

$k \xrightarrow{\sim} k \qquad \boxed{\begin{matrix} 1 \\ 2 \end{matrix}}$

$S_1: \quad k \longrightarrow 0 \qquad \boxed{1}$

$S_2: \quad 0 \longrightarrow k \qquad \boxed{2}$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & k \end{tikzcd}$$

Example: $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ (type A_3)

$$k \to 0 \to 0 \qquad \boxed{1}$$

$$0 \to k \to 0 \qquad \boxed{2}$$

$$0 \to 0 \to k \qquad \boxed{3}$$

$$k \xrightarrow{\sim} k \to 0 \qquad \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}$$

$$0 \to k \to k \qquad \boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}$$

$$k \xrightarrow{\sim} k \xrightarrow{\sim} k \qquad \boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$$

Example: category of indecomposables in type A_3

$$
\begin{tikzcd}[column sep=small]
 & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
\underline{3} \arrow[ur] & & \underline{2} \arrow[ll, dashed, "\tau"'] \arrow[ur] & & 1 \arrow[ll, dashed, "\tau"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & P_1 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \tau^{-1}P_2 \arrow[dr] & \\
P_3 \arrow[ur] \arrow[rr, dash, dashed] & & \tau^{-1}P_3 \arrow[ur] \arrow[rr, dash, dotted] & & \tau^{-2}P_3
\end{tikzcd}
$$

Here, maps are "indec":

$3 \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ factors as $3 \longrightarrow \begin{smallmatrix}2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix}$.

For any f.d. algebra Λ, there is a function

$$\tau\colon \mathrm{Ind}(\Lambda) \to \mathrm{Ind}(\Lambda) \cup \{0\}$$

called the Auslander-Reiten translate, with $\tau M = 0$ iff M is projective. It has a (partial) inverse

$$\tau^{-}\colon \mathrm{Ind}(\Lambda) \to \mathrm{Ind}(\Lambda) \cup \{0\}$$

We say M is "preprojective" if $\tau^r M$ is projective, for some $r \geq 0$.

Theorem [PA, Ga]: For $\Lambda = kQ$, every module is preprojective $\Leftrightarrow$ kQ is of finite representation type.

Question [Gelfand-Ponomarev]: is there an algebra Π

- which has $\Lambda = kQ$ as a subalgebra, and
- where the regular Π-module restricts to the direct sum of all preprojective Λ-modules?

Construction [based on GP, 1979]: let $\bar{Q}$ be the doubled quiver of Q, so for each arrow $\alpha: i \to j$ in Q we add another arrow $\alpha^*: j \to i$.

Then define

$$\Pi = k\bar{Q} / \left(\sum_{\alpha \in Q} \alpha\alpha^* - \alpha^*\alpha \right)$$

Example: $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

$\overline{Q} = 1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3$

$\Gamma = \alpha\alpha^* - \alpha^*\alpha + \beta\beta^* - \beta^*\beta$

$e_1\Gamma = \alpha\alpha^* = 0$

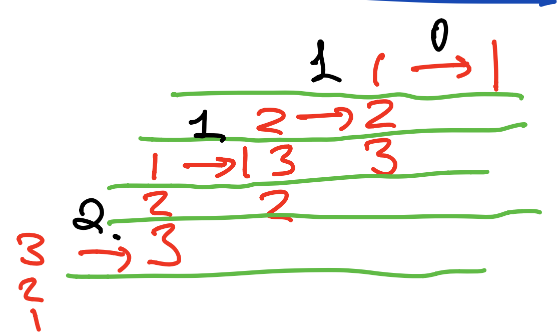
$e_2\Gamma = -\alpha^*\alpha + \beta\beta^* = 0$

$e_3\Gamma = -\beta^*\beta = 0$

Basis for Π:

$e_1,\ e_2,\ e_3,$
$\alpha,\ \beta,\ \alpha^*,\ \beta^*,$
$\alpha\beta,\ \alpha^*\alpha,\ \beta^*\alpha^*$

$$\pi\Pi = \begin{array}{c}1\\2\\3\end{array} \quad \begin{array}{c}2\\1\ \ 3\\2\end{array} \quad \begin{array}{c}3\\2\\1\end{array}$$



Idea [Baer-Geigle-Lenzing]: construct Π directly (instead of writing explicit presentation).

Key fact: for $\Lambda = kQ$, inverse AR translate is a functor

$$\tau^{-}\colon \mathrm{mod}{-}kQ \to \mathrm{mod}{-}kQ$$

$\mathrm{Hom}_{\Lambda}(\Lambda, M) \xrightarrow{\sim} M$

Definition [BGL, 1987]: Let $\Lambda = kQ$ and let

$$\Pi = \bigoplus_{r\geq 0} \mathrm{Hom}_{\Lambda}(\Lambda, \tau^{-r}\Lambda) \cong \bigoplus_{r\geq 0} \tau^{-r}\Lambda$$

with composition $g * f = \tau^{-r}(g)f$.

$$\begin{tikzcd} \Lambda \arrow[r, "f"] \arrow[rrr, bend right] & \tau^{-r}\Lambda = \tau^{-r}\Lambda \arrow[rr, "\tau^{-r}(g)"] & & \tau^{-r-s}\Lambda \end{tikzcd}$$

Later, Ringel and Crawley-Boevey showed that these defintions give isomorphic algebras.

So call them "the" preprojective algebra of $\Lambda = kQ$.

$\Pi = k\bar{Q}/I$ has a grading by path length.

$\Pi = \bigoplus_{r\geq 0} \mathrm{Hom}_\Lambda(\Lambda, \tau^{-r}\Lambda)$ has a grading by r.

These gradings do not correspond.

To get the second grading on $k\bar{Q}/I$, set

$$\deg(\alpha) = 0, \qquad \deg(\alpha^*) = 1.$$

Auslander-Reiten theory works for all algebras, but it's most powerful when gldim $\Lambda \leq 1$.

Iyama developed a generalisation which can be more useful for algebras with higher global dimension.

Definition [Iyama]: $\tau_d = \Omega^{d-1}\tau$ and $\tau_d^- = \tau^-\Omega^{1-d}$, where ΩM is the syzygy of M.

We say M is "d-preprojective" if $\tau_d^r M$ is projective, for some $r \geq 0$.

Example: $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $\Lambda = kQ/(\alpha\beta)$.

$\operatorname{gldim} \Lambda = 2$, $\quad d = 2$.

$$\Lambda = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3$$

2-AR quiver:

$$\begin{tikzcd}
 & & P_1 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[drr] & & \\
 & P_2 = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[ur] & & & \\
P_3 = 3 \arrow[ur] & & & & 1 = \tau_2^- P_3
\end{tikzcd}$$

S_2 is <u>not</u> 2-preprojective.

Fact: if gldim $\Lambda \leq d$ then τ_d^- is a functor on the category of d-preprojective Λ-modules.

Definition [IO, ...]: The $(d+1)$-preprojective algebra of Λ is:

$$\Pi = \bigoplus_{r \geq 0} \mathrm{Hom}_\Lambda(\Lambda, \tau_d^{-r}\Lambda)$$

Question: can we give an explicit presentation?

$$\Pi =?= k\bar{Q}/(\bar{R})$$

gldim $\Lambda = d$.

Suppose $\Lambda = kQ/R$. We want $\Pi = k\bar{Q}/(\bar{R})$.

Strategy to find $\bar{Q}$:

- Compute projective resolution of each simple Λ-module S_i.

$$\begin{tikzcd} P^{d,i} \arrow[r] & \cdots \arrow[r] & P^{1,i} \arrow[r] & P^{0,i} \arrow[d, two heads] \\ & & & S_i \end{tikzcd}$$

- For each summand of d^{th} term isomorphic to P_j, add an arrow $j \to i$ to the quiver Q.

Proposition [G-Iyama, Thibault] The quiver of Π is $\bar{Q}$.

Example: $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $\Lambda = kQ/(\alpha\beta)$.

$$\Lambda = \underbrace{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}_{P_1} \oplus \underbrace{\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}}_{P_2} \oplus \underbrace{3}_{P_3}$$

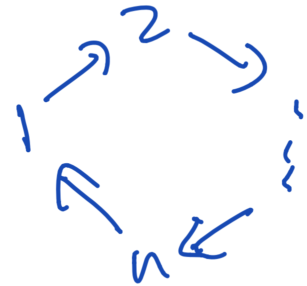
$$\begin{tikzcd} 3 \arrow[r] & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[d] \\ & & 1 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 3 \arrow[r] & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[d] \\ & & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 3 \arrow[d] \\ & & 3 \end{tikzcd}$$

So $\overline{Q} =$
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

Definition: a "superpotential" for a quiver is a sum of cycles up to (super)cyclic equivalence

So for a cycle

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{n-1}} n \xrightarrow{\alpha_n} 1$$



we have

$$\alpha_1 \alpha_2 \ldots \alpha_n = (-1)^{n+1} \alpha_2 \ldots \alpha_n \alpha_1$$

For paths p, q we can take the cyclic derivative

$$\partial_p(pq) = q$$

$\partial_p(p'q) = 0$ if $p \neq p'$.

Examples:

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\ 4 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"] \end{tikzcd}$$

$$\begin{aligned} W = \alpha\beta\gamma\delta &= -\beta\gamma\delta\alpha \\ &= \gamma\delta\alpha\beta \\ &= -\delta\alpha\beta\gamma \end{aligned}$$

$$\begin{tikzcd} 1 \arrow[rr, "\alpha"] & & 2 \arrow[dl, "\beta"] \\ & 3 \arrow[ul, "\gamma"] & \end{tikzcd}$$

$$\begin{aligned} W = \alpha\beta\gamma &= \beta\gamma\alpha \\ &= \gamma\alpha\beta \end{aligned}$$

$$\partial_\alpha W = \beta\gamma, \quad \partial_\beta W = \gamma\alpha, \quad \partial_\gamma = \alpha\beta$$

Suppose $\Lambda = kQ/R$ with R homogeneous (deg ≥ 2).

Then Λ is graded by path length.

We say Λ is "Koszul" if, in each projective resolution of a simple module, all the maps have degree 1.

Let $V_n = kQ_n =$ vector space of paths of length n.

Let $K_d = V_{d-2}R \cap V_{d-3}\, RV_1 \cap \cdots \cap RV_{d-2}$.

Choose a basis $\mathcal{B}$ of K_d. Then each $b \in \mathcal{B}$ corresponds to an arrow added to Q to get $\bar{Q}$.

Define a superpotential:

$$W = \sum_{b \in \mathcal{B}} bb^*$$

Theorem [G-Iyama, Thibault for d-RI]

If $\Lambda = kQ/R$ is Koszul, of global dimension d, then

$$\Pi = k\bar{Q}/(R + \boxed{\partial_p W} \mid p \in V_{d-1})$$

If moreover $\underline{\mathrm{Ext}^i_\Lambda(\Lambda^*, \Lambda) = 0}$ for $\underline{1 < i < d}$, then:

$$\Pi = \underline{k\bar{Q}/(\partial_p W \mid p \in V_{d-1})}$$

The condition holds for all d-hereditary algebras.
This includes algebras with a d-cluster tilting module.

Question: is the Koszul assumption necessary?

Example: $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and $\Lambda = kQ/(\underline{\alpha\beta}, \underline{\beta\gamma})$.

$\alpha\beta\gamma$

$\delta = (\alpha\beta\gamma)^*$

$$\overline{Q} = \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\ 4 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"] \end{tikzcd}$$

$W = \alpha\beta\gamma\delta$

$\mathrm{gldim}\,\Lambda = 3 = d.$ $\quad d-1 = 2.$

$$\left(\begin{array}{ll} \partial_{\alpha\beta} W = \gamma\delta & \partial_{\gamma\delta} = \alpha\beta \\ \partial_{\beta\gamma} W = -\delta\alpha & \partial_{\delta\alpha} = -\beta\gamma. \end{array} \right)$$

$\Pi = k\overline{Q}$

When Λ is d-hereditary, we compute projective resolutions of all simple Π-modules.

Two cases:

- d-RF: simple Π-modules have p.dim $= d + 1$
- d-RI: simple Π-modules are periodic of "twisted period" $d + 1$

Reference:

"Higher preprojective algebras, Koszul algebras, and superpotentials", J. G. and O. Iyama,

Compositio Mathematica (2020), 156(12), 2588-2627

Thank you for listening!

$$
\begin{tikzcd}[column sep=huge]
\Lambda \arrow[r, leftrightarrow, "\text{koszul}", "\text{dual}"'] \arrow[d, rightsquigarrow] & \Lambda^{!} \arrow[d, rightsquigarrow] \\
\Pi(\Lambda) \arrow[r, leftrightarrow, "\text{in some}", "\text{examples, quadratic dual,}"'] & \mathrm{STriv}(\Lambda^{!}) = \Lambda^{!} \oplus (\Lambda^{!})^{*}
\end{tikzcd}
$$

higher preprojective

higher zigzag