§1 Preliminaries

R a ring

$D(R)$ derived category of R-modules

Objects: (cochain) complexes, unbounded, of (arbitrary, right) R-modules

Morphisms: invert quasi-isomorphisms (cochain maps inducing isom. on cohomology)

$D(R)$ is a triangulated category with arbitrary products / coproducts.

$$\text{Mod-}R \subseteq D(R)$$
$$M \longmapsto \cdots \to 0 \to M \to 0 \to \cdots$$

<u>Localizing (colocalizing) subcategories of $D(R)$:</u>

Defn: A full subcategory of $D(R)$ is localizing if it is triangulated (closed under q-isom., shifts and extensions) and coproducts

Consequences:

- Closed under direct summands
  ($X \oplus Y \in \mathcal{L} \Rightarrow X \in \mathcal{L}$)
  [Eilenberg swindle]

$$0 \to Y \oplus X \oplus Y \oplus \cdots \to X \oplus Y \oplus X \oplus Y \oplus \cdots \to X \to 0$$

- Two out three property:
  $$0 \to X \to Y \to Z \to 0$$
  two of X, Y, Z in $\mathcal{L} \Rightarrow$ so is the other.

- If $X_0 \to X_1 \to X_2 \to \cdots$ sequence of chain maps
  $$\{X_i\} \subseteq \mathcal{L} \Rightarrow \varinjlim X_i \in \mathcal{L}$$

• Any bounded above complex

$X := \cdots \to X_{-2} \to X_{-1} \to X_0 \to 0 \to 0 \cdots$

with $\{X_i\} \subseteq \mathcal{L}$ has $X \in \mathcal{L}$

[• all bounded truncations are in $\mathcal{L}$ by taking extensions.
• X is the direct limit of these truncations.]

§2. Generation

If $C \subseteq D(R)$, $\mathrm{Loc}(C)$ is the localizing subcat of $D(R)$ generated by C [similarly $\mathrm{Coloc}(C)$]

Fact: $\mathrm{Loc}(\mathrm{Proj}\text{-}R) = D(R)$
"Projectives generate"

Sketch: • a proj resol of any module is in $\mathrm{Loc}(\mathrm{Proj}\text{-}R)$
• any module is in
• any bounded above complex is in

- every complex is a direct limit of bounded above complexes.

[Similarly, $\mathrm{Coloc}(\mathrm{Inj}\text{-}R) = \mathcal{D}(R)$, "Injectives cogenerate"]

Apparently unnatural questions:

Is $\mathcal{D}(R) = \mathrm{Loc}(\mathrm{Inj}\text{-}R)$ [Do "injectives generate"?]

Is $\mathcal{D}(R) = \mathrm{Coloc}(\mathrm{Proj}\text{-}R)$ [Do "projectives cogenerate"?]

Positive results

- IG for R if $\mathrm{gldim}(R) < \infty$
- " " " " if $\mathrm{id}(P) < \infty$ for all projectives P.
- IG for commutative Noetherian rings [Neeman classified all localizing subcats.]

Negative results

Def^n $\text{Inj-}R^{\perp} = \{ Y \in D(R) :$
$\text{Hom}(I, Y[t]) = 0$
for all inj^s I, $t \in \mathbb{Z} \}$

$\text{Inj-}R^{\perp} = \text{Loc}(\text{Inj-}R)^{\perp}$

So if $0 \neq Y \in \text{Inj-}R^{\perp}$ then
$Y \notin \text{Loc}(\text{Inj-}R)$, so injectives don't generate.

Example $R = k[x_1, x_2, \ldots]$

$k = R/(x_1, x_2, \ldots)$

Then $k \in \text{Inj-}R^{\perp}$, so inj^s don't generate.

§3 Finite dimensional algebras

A is a fin dim algebra

Then for $DA = \text{Hom}_k(A, k)$,

so $\mathrm{Loc}(\mathrm{Inj}\text{-}A) = \mathrm{Loc}(DA)$

Is IG true for A?

<u>Keller (2001)</u> considered this.

<u>Generalized Nakayama Conj</u>

Every indec proj occurs in a min proj resolution of DA.

[Equivalently: for every simple S, $\mathrm{Ext}^i(DA, S) \neq 0$ for some $i \geq 0$]

<u>Keller</u>:

IG for $A \Rightarrow$ GNC for A

[If GNC false, then $S \in DA^{\perp}$.]

<u>Keller</u>: Q: Is there a connection between IG and the finitistic dimension conj?

(Big) $\mathrm{FinDim}(A) = \sup\{\mathrm{pd}(M) : \mathrm{pd}(M) < \infty\}$

FDC: $\mathrm{FinDim}(A) < \infty$.

Thm IG for $A \Rightarrow$ FDC for A.

Sketch: Suppose FDC false.

So we have modules with $\mathrm{pd}(M_n) = n \quad \forall n \geq 0$

$$\bullet \to \bullet \qquad P(M_1)[-1]$$
$$\bullet \to \bullet \to \bullet \qquad P(M_2)[-2]$$
$$\vdots$$
$$\bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \qquad P(M_n)[-n]$$
$$\vdots$$

$$\bigoplus_n P(M_n)[-n] \longrightarrow \prod_n P(M_n)[-n]$$

is a q-isom.

So the quotient $\prod / \bigoplus$ is a bdd below acyclic complex of proj^s, but not contractible.

$$\mathrm{Proj}\text{-}A \xrightarrow{-\otimes_A DA} \mathrm{Inj}\text{-}A$$

is an equivalence of categories.

Apply to $\Pi/\bigoplus$: get non-contractible bded below complex of injectives

$$\mathrm{Hom}_{K(A)}(A, \Pi/\bigoplus[t]) = 0 \quad (\Pi/\bigoplus \text{ acyclic})$$

$$\Rightarrow \mathrm{Hom}_{K(A)}(DA, \Pi/\bigoplus \otimes_A DA[t]) = 0$$

$$\| $$

$$\mathrm{Hom}_{D(A)}(DA, \Pi/\bigoplus \otimes_A DA[t]) = 0$$

So $\underbrace{\Pi/\bigoplus \otimes_A DA}_{\text{non-zero}} \in DA^{\perp}$

§4 Positive Results

- IG for Gorenstein algebras
- IG for comm. FDAs
- IG for A of finite rep^n type.
- IG for monomial algebras (FDC: Green, Kirkman, Kuzmanovich (1991))

§5 Negative results for FDAs

$$IG \iff DA^{\perp} = \{0\}$$

$$FDC \iff D^{+}(A) \cap DA^{\perp} = \{0\}$$

$$IG \text{ for } A \implies PC \text{ for } A^{op}$$