Topics on the Igusa-Todorov functions

Gustavo Mata

July 30th

One of the most important conjectures in the Representation Theory of Artin Algebras is the finitistic dimension conjecture.

It states that the supremum of the projective dimensions for the f. g. modules with finite projective dimension over an Artin algebra is finite.

In an attempt to prove the finitistic dimension conjecture, Igusa and Todorov defined two functions from the finitely generated modules over an Artin algebra to the natural numbers, which generalizes the notion of projective dimension.

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We fix the following notation

- A is an Artin algebra.
- $J \subset A$ is the Jacobson ideal of A.
- $\mathrm{Mod}A$ and $\mathrm{mod}A$ are the right A-modules and the finitely generated right A-modules, respectively.
- $\mathcal{S}(A)$, $\mathcal{P}(A)$ and $\mathcal{I}(A)$ are the simple, projective and injective A-modules, respectively.
- $\mathrm{ind}A$ are the indescomposable A-modules.
- Given $M \in \mathrm{mod}A$ we denote by:
  - $\mathrm{pd}(M)$ its projective dimension.
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  - $\Omega(M)$ its syzygy.
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The following version of Fitting Lemma is used to define the Igusa-Todorov functions:

Lemma (Fitting Lemma)

Let R be a noetherian ring. Consider $M \in \mathrm{Mod}R$ and $f \in End_R(M)$. Then, $\forall\ X \subset M$, such that $X \in \mathrm{mod}R$ there is a non-negative integer

$$\eta_f(X) = \min\{k \in N : f|_{f^m(X)} : f^m(X) \stackrel{\cong}{\to} f^{m+1}(X), \forall m \geq k\}.$$

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Definition

Let $K_0(A)$ be the abelian group generated by all symbols $[M]$, with $M \in \mathrm{mod}A$, modulo the relations

1. $[M] - [M_1] - [M_2]$ *if* $M \cong M_1 \oplus M_2$,
2. $[P]$ *for each projective module* P.

Let $\bar{\Omega} : K_0(A) \to K_0(A)$ be the group endomorphism induced by Ω. That is

$$\bar{\Omega}([M]) = [\Omega(M)].$$

If $M \in \mathrm{mod}A$, then $\langle \mathrm{add}M \rangle$ denotes the subgroup of $K_0(A)$ generated by the classes of indecomposable (non projective) summands of M.

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Observe that

- *for a finitely generated subgroup $G \subset K_0(A)$, the map $\bar{\Omega}_{|G} : G \to \bar{\Omega}(G)$ is an isomorphism if and only if $\mathrm{rk}(G) = \mathrm{rk}(\bar{\Omega}(G))$*
- *In general $\bar{\Omega}(\langle \mathrm{add} M \rangle) \neq \langle \mathrm{add} \Omega(M) \rangle$.*

Given a finitely generated subgroup G of $K_0(A)$, we define $\eta_{\bar{\Omega}}(G)$ as in the Fitting Lemma.

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Definition (Igusa, Todorov)

The Igusa-Todorov function ϕ_A of $M \in \mathrm{mod}A$ is defined as

$$\phi(M) = \eta_{\bar{\Omega}}(\langle \mathrm{add} M \rangle).$$

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The Igusa-Todorov function ψ_A of $M \in \mathrm{mod}A$ is defined as

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Proposition (Igusa, Todorov)

Given $M, N \in \operatorname{mod}A$.

- *If* $\operatorname{pd}(M) < \infty$, *then* $\phi(M) = \psi(M) = \operatorname{pd}(M)$.
- *If* $M \in \operatorname{ind}A$ *and* $\operatorname{pd}(M) = \infty$, *then* $\phi(M) = \psi(M) = 0$.
- $\phi(M) \leq \phi(M \oplus N)$ *and* $\psi(M) \leq \psi(M \oplus N)$.
- $\phi(M^k) = \phi(M)$ *and* $\psi(M^k) = \psi(M)$ *if* $k \in \mathbb{Z}^+$.

Proposition (Huard, Lanzilotta, Mendoza)

If $M \in \operatorname{mod}A$, *then*

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Consider the radical square zero algebra $A = \frac{\Bbbk Q}{J^2}$, with Q as follows

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow n \circlearrowleft$$

If $\mathcal{S}(A) = \{S_1, \dots, S_n\}$, we have:

$$\Omega(S_i) = \begin{cases} S_{i+1} \text{ if } i = 1, \dots, n-1, \\ S_n \text{ if } i = n. \end{cases}$$

Let us compute $\phi(S_1 \oplus S_n)$.

$$\begin{aligned}
&\langle \mathrm{add}(S_1 \oplus S_2) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}, \\
&\bar{\Omega}(\langle \mathrm{add}(S_1 \oplus S_2) \rangle) \cong \langle \mathrm{add}(S_2 \oplus S_n) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}, \\
&\vdots \\
&\bar{\Omega}^{n-2}(\langle \mathrm{add}(S_1 \oplus S_2) \rangle) \cong \langle \mathrm{add}(S_{n-1} \oplus S_n) \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \\
&\bar{\Omega}^{n-1}(\langle \mathrm{add}(S_1 \oplus S_2) \rangle) \cong \langle \mathrm{add}(S_n \oplus S_n) \rangle \cong \mathbb{Z} \\
&\bar{\Omega}^{n-1+k}(\langle \mathrm{add}(S_1 \oplus S_2) \rangle) \cong \langle \mathrm{add}(S_n \oplus S_n) \rangle \cong \mathbb{Z}, \forall\ k \geq 0.
\end{aligned}$$

Hence $\phi(S_1 \oplus S_n) = n - 1$.

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Definition

Let A be an Artin algebra, and $\mathcal{C}$ a full subcategory of $\mathrm{mod}A$*. We define*

- $\phi\dim(\mathcal{C}) = \sup\{\phi(M) : M \in \mathrm{Ob}\mathcal{C}\}$, *and*
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Examples of algebras of Ω^1-finite representation type are:

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If $A = \frac{\mathbb{k}Q}{J^k}$, then $\phi\dim(A) \leq f_k(n)$ where

$$f_k(m) = \begin{cases} 0 & \text{if } m = 0, \\ 2\left(\frac{m-1}{k}\right) + 1 & \text{if } m \equiv 1 \pmod{k}, \\ 2\left(\frac{m-2}{k}\right) + 2 & \text{if } m \equiv 2 \pmod{k}, \\ 2\left\lceil \frac{m-2}{k} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

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Algebra with infinite ϕ-dimension

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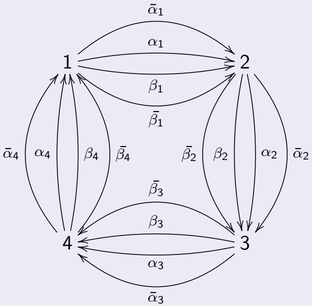

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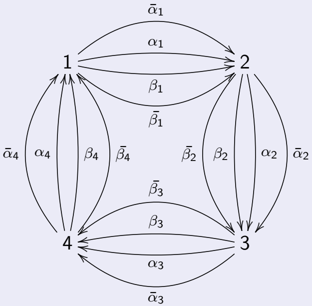

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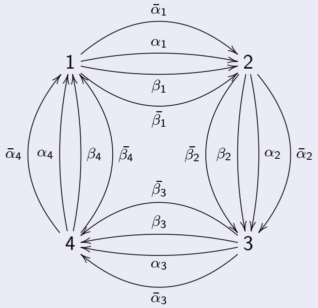

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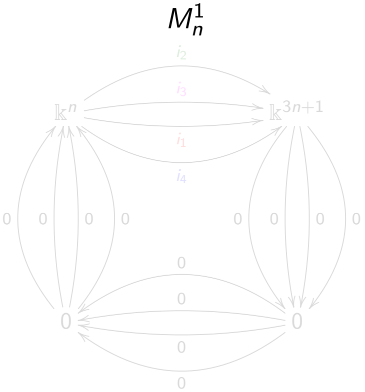

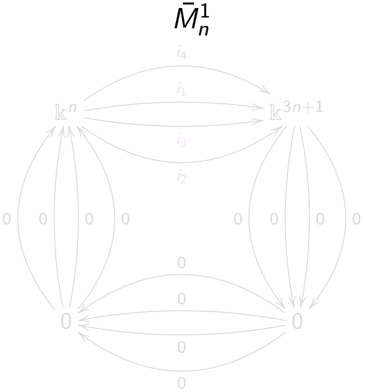

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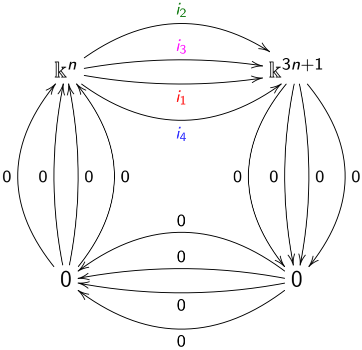

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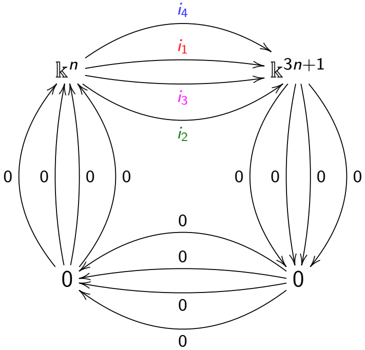

with the maps $i_m : \mathbb{k}^n \to \mathbb{k}^{3n+1}$, for $m \in \{1, 2, 3, 4\}$, are defined by:

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where $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_{3n+1}\}$ are the canonical bases of $\mathbb{k}^n$ and $\mathbb{k}^{3n+1}$, respectively.

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Then, $\forall i \in \{1, 2, 3, 4\}$, we have :

- $M_n^i \ncong \bar{M}_n^i$ for $n \geq 2$ and
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We can compute the syzygies of the previous modules for $n \geq 2$ and $i \in \mathbb{Z}_4$.

- $\Omega(M_n^i) = M_{n-1}^{i+1} \oplus S_{i+2}^{7n+2}$,
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Definition (Wei)

Let A be an Artin algebra and $n \in \mathbb{N}$. Then A is said to be (n)-**Igusa-Todorov** *if there exists an A-module V such that for any A-module M there is an exact sequence*

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Question: Are all Artin algebras Igusa-Todorov?

Let $\Lambda(V)$ be the exterior algebra of a vector space V over a field $\mathbb{k}$.

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If $\mathbb{k}$ is an uncountable field, $\Lambda(\mathbb{k}^m)$ is not Igusa-Todorov for $m \geq 3$.

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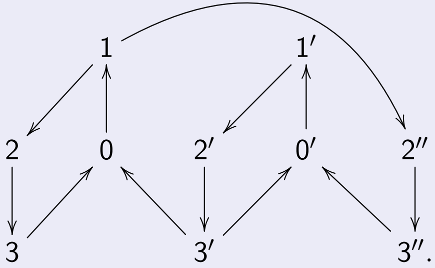

We have:

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¡Gracias!
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Thank you!