Variety defined by a Groebner Basis of a Module

E. L. Green, Sibylle Schroll, E. N. M

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This talk is aimed for the beginning graduate student. It is about a work in progress.
Using the theory of Gröbner basis, in KQ and a module over KQ we define variety using the right Gröbner basis of KQ, $\mathcal{B}$ denotes the set of paths in Q. I will describe a small part of our work. Various subjects are not touched in the talk.

(There is a book in Portuguese on the classical theory of Gröbner basis for ideals.)

Admissible Order, " Term order"

We fix an admissible order $\succ$ on $\mathcal{B}$, that is, a well-order compatible with the multiplication in KQ; that is for $p, q, r \in \mathcal{B}$.

1. if $pr \neq 0$, $qr \neq 0$ and $p \succ q$ then $pr \succ qr$,
2. if $rp \neq 0$, $rq \neq 0$ and $p \succ q$ then $rp \succ rq$, and
3. if $prq \neq 0$, then $prq \succeq r$.

For all effects and ideas in this talk we can take the lenght-lexicographic order.

The idempotent projectives

We also fix an index set $\mathcal{I}$. For each $i \in \mathcal{I}$, we associate a vertex w_i in Q. Of course, it is possible that $w_i = w_j$ but $i \neq j$.

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We fix a right projective KQ-module P together with a decomposition $P = \oplus_{i \in \mathcal{I}} w_i KQ$. We view elements of P as $\mathcal{I}$-indexed tuples of the form $(r_i)_{i \in \mathcal{I}}$ where $r_i \in w_i KQ$.

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For $i \in \mathcal{I}$, let e_i be the element $(r_{i,j})_{j \in \mathcal{I}}$ of P such that $r_{i,j} = \delta_{i,j} w_i$ where
$\delta_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$ We call the set of e_i's the *standard generating set of* P.

Induced Order

$\mathcal{B}^* = \cup_{i\in\mathcal{I}}\{e_i p \mid p \in \mathcal{B} \text{ and } w_i p = p\}$.
We extend the admissible order $\succ$ on $\mathcal{B}$ to a well-order $\succ^*$ on $\mathcal{B}^*$ as follows. Fix a well-ordering $>_{\mathcal{I}}$ on $\mathcal{I}$. Let $e_i p, e_j q \in \mathcal{B}^*$. We set $e_i p \succ^* e_j q$ if $p \succ q$ or $p = q$ and $i >_{\mathcal{I}} j$.

The tip

Definition

The tip *of an element* x *in* $P \setminus \{0\}$, denoted by $\mathrm{tip}_{\mathcal{B}^*}(x)$, is p^*, if $x = \sum_{q^* \in \mathcal{B}^*} c_{q^*} q^*$ with $c_{q^*} \in K$, $p^* \in \mathcal{B}^*$ with $c_{p^*} \neq 0$, and $p^* \succeq^* q^*$ if $c_{q^*} \neq 0$. The *coeffcient of the tip of* x, denoted $c_{\mathrm{tip}_{\mathcal{B}^*}(x)}$, is c_{p^*}.

If $X \subset P$, then $\mathrm{tip}(X) = \{\mathrm{tip}(x) \mid x \in X \setminus \{0\}\}$.

The right Gröbner Basis

Definition

Suppose L is a KQ-submodule of P. Then a set $\mathcal{H}^*$ of elements of L is a *right Gröbner basis for L with respect to $\succ^*$* if

$$\langle \text{tip}(\mathcal{H}^*)\rangle_R = \langle \text{tip}(L)\rangle_R.$$

An equivalent definition is that $\mathcal{H}^* \subset L$ is a right Gröbner basis for L if, for every nonzero element x of L, there is some $h \in \mathcal{H}^*$ such that $\text{tip}(x) = \text{tip}(h)p$, for some path $p \in \mathcal{B}$.

Right Parallel Elements

Definition

An element n is *right uniform* if there exists a vertex v in Q such that $n = nv$.

Two nonzero elements m, n, are *right parallel*, denoted $m \|_R n$, if there is a vertex $v \in Q$ such that $mv = m$ and $nv = n$.

Example

Consider the following quiver

$$
\begin{tikzcd}
 & 2 \arrow[dr, "b"] & & \\
Q = \; 1 \arrow[ur, "a"] \arrow[dr, "c"'] & & 4 \arrow[r, "e"'] & 5 \\
 & 3 \arrow[ur, "d"'] & &
\end{tikzcd}
$$

and M to be the module whose dimension vector is $(1, 1, 1, 1, 0)$ and all the maps are the identity.

We have that it is quotient of P_1 by the submodule L generated by $\{ab - cd, e\}$

Consider the order $a > b > c > d > e$ and in P_1 the induced order then for the submodule L we have $tip(ab - cd) = ab$, in the length lex order.

Right Division

Definition

1. Suppose e_ip and e_jq are elements of $\mathcal{B}^*$, where p and q are paths. Then we say that e_ip *left divides* e_q (written $e_ip \mid_\ell e_jq$,) if $i = j$ and $q = pr$ for some path r.
2. We say that a set $\mathcal{H}^*$ of nonzero elements of P is *right tip-reduced* if, for all nonzero elements $h, h' \in \mathcal{H}^*$ with $h \neq h'$, $\mathrm{tip}(h)$ does not left divide $\mathrm{tip}(h')$.

Reduction

Definition

1. Let $\mathcal{H}^*$ be a set of elements in P and $x = \sum_{p^* \in \mathcal{B}^*} c_{p^*} p^* \in P$, where $c_{p^*} \in K$. We say that x' is a *simple right reduction of* x *by* $\mathcal{H}^*$, denoted $x \to_{\mathcal{H}^*} x'$, if, for some $h \in \mathcal{H}^*$ and $p^* \in \mathcal{B}^*$ with $c_p^* \neq 0$, $p^* = tip(h)s$ for some $s \in \mathcal{B}$; that is, $\mathrm{tip}(h)|_\ell p^*$. In this case, we set

$$x' = x - (c_{p^*}/c_{\mathrm{tip}(h)})\ hs.$$

Reduction

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$$x' = x - \left(c_{p^*}/c_{\mathrm{tip}(h)}\right) hs.$$

2. We say that x is *completely reduced by $\mathcal{H}^*$* if there are no simple reductions of x by $\mathcal{H}^*$. If x is not completely reduced by $\mathcal{H}^*$, then we say that y is a *complete reduction of x by $\mathcal{H}^*$*, denoted $x \Rightarrow_{\mathcal{H}^*} y$, if, for some $n \geq 1$, there is a sequence of simple reductions, $x_i \to_{\mathcal{H}^*} x_{i+1}$, for $1 \leq i \leq n$, with $x_1 = x$ and $x_{n+1} = y$, and y is completely reduced by $\mathcal{H}^*$.

More Reduction

Definition

We say that $\mathcal{H}'^*$ is a *simple right self reduction of* $\mathcal{H}^*$ if there exist nonzero $h, h' \in \mathcal{H}^*$ such that

1. $h \neq h'$,
2. $h = \sum c_{p^*} p^*$ and $h' = \sum c'_{p^*} p^* \in \mathcal{H}^*$
3. there is $q^* \in \mathcal{B}^*$ with $c'_{q^*} \neq 0$ and $\mathrm{tip}(h) \mid_\ell q^*$; i.e. there is a path s in $\mathcal{B}$ such that $q^* = \mathrm{tip}(h)s$,
4. $\mathcal{H}'^*$ consists of removing the element h' from $\mathcal{H}^*$ and replacing it with
$h'' = h' - (c'_{q^*}/c_{\mathrm{tip}(h)})hs$.

We say that $\mathcal{H}^*$ is *completely right self reduced* if there are no simple self reductions of $\mathcal{H}^*$.

Right Gröbner Basis

Theorem

Let $\mathcal{H}^*$ be a right uniform, right tip-reduced generating set of a right submodule L of P. Then $\mathcal{H}^*$ is a right Gröbner basis of L with respect to $\succ^*$. Moreover, L is isomorphic to

$$\bigoplus_{h \in \mathcal{H}^*} hKQ,$$

with isomorphism given by $\varphi \colon \bigoplus_{h \in \mathcal{H}^*} hKQ \to L$ by $\varphi((hs_h)_{h \in \mathcal{H}^*}) = \sum_{h \in \mathcal{H}^*} hs_h$ where each $s_h \in KQ$.

Corollary

If L is a finitely generated KQ-submodule of P and $\mathcal{H}^*$ is a finite (right uniform) generating set for L, then there is a finite sequence of simple reductions starting with the elements of the generating set $\mathcal{H}^*$ so that the resulting set is a (right uniform) right tip-reduced generating set of L. In particular, this gives a right Gröbner basis.

Fundamental Lemma

Proposition

Fundamental Lemma

Suppose that L is a submodule of P.
Let $\mathcal{H}^*$ be a right Gröbner basis of L. Then:
1. the complete reduction by $\mathcal{H}^*$ is unique; that is if $x \in P$, $x \Rightarrow_{\mathcal{H}^*} y$, and $x \Rightarrow_{\mathcal{H}^*} y'$, then $y = y'$, Moreover, $x - y \in L$.

2. As KQ_0-modules,

$$P \cong L \bigoplus \mathrm{Span}_K(\mathrm{nontip}(L)).$$

Consequence of the fundamental lemma

Proposition

The set $\mathcal{H}^*_{\mathcal{T}^*} = \{h_t = t - n_t \mid t \in \mathcal{T}^*\}$ *is right uniform and right tip-reduced. Moreover,* $\mathcal{T}^* = \mathrm{tip}(\mathcal{H}^*_{\mathcal{T}^*})$ *and* $\mathcal{H}^*_{\mathcal{T}^*}$ *is the right tip reduced Gröbner basis of* L.

Starting with the tips

Up to now we start with a submodule L of P and choose $\mathcal{T}^*$ to be the unique minimal generating set of the submodule $\langle \mathrm{tip}(L) \rangle_R$. Another approach is to start with a right tip-reduced subset $\mathcal{T}^*$ of $\mathcal{B}^*$ and construct submodules L of P such that the reduced right Gröbner basis of L has tip-set $\mathcal{T}^*$.

Denote by $\mathcal{N}^* = \mathcal{B} \setminus \langle \mathcal{T}^* \rangle_R$ and for $t \in \mathcal{T}^*$ let

$$\mathcal{N}_t^* = \{n \in \mathcal{N}^* \mid n \|_R t \text{ and } t \succ^* n\}$$

For $t \in \mathcal{T}^*$, let $\varphi_t \colon \mathcal{N}_t^* \to K$ so that $\varphi_t(n) = 0$ for all but a finite number of n's. Define

$$\mathcal{H}_\varphi^* = \{h_t = t - \sum_{n \in \mathcal{N}_t^*} \varphi_t(n) n \mid t \in \mathcal{T}^*\}.$$

The next result classifies the submodules of P whose reduced right Gröbner bases have tip set $\mathcal{T}^*$.

Submodules having Gröbner bases with tip set $\mathcal{T}^*$

Theorem

Let $\mathcal{T}^*$ be a right tip-reduced subset of $\mathcal{B}^*$, $\mathcal{L}$ be the set of submodules L of P such that $\mathcal{T}^*$ is the tip set of the reduced right Gröbner basis of L, and Φ be the set whose elements are sets of maps

$$\Phi = \{\varphi_t \colon \mathcal{N}_t^* \to K \mid t \in \mathcal{T}^* \text{ and for each } t, \varphi_t(n) = 0 \text{ for almost all } n \in \mathcal{N}_t^*\}.$$

There is a one-to-one correspondence between $\mathcal{L}$ and Φ. Moreover, given $\varphi \in \Phi$, $\mathcal{H}_\varphi^*$ is the reduced right Gröbner basis of the submodule of P corresponding to φ, that is $\mathcal{H}_\varphi^*$ is the reduced right Gröbner basis of $\langle \mathcal{H}_\varphi^* \rangle_R$.

The ambient space

We define the following set:

$$D(\mathcal{T}^*) \text{ to be the disjoint union } \bigcup_{t \in \mathcal{T}^*} \mathcal{N}_t^*.$$

For simplicity in this talk assume that $D(\mathcal{T}^*)$ is a finite set.

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Definition

Write elements of as $D(\mathcal{T}^*)$-tuples $\mathbf{c} = (c_{t,n})$ where $t \in \mathcal{T}^*$, $n \in \mathcal{N}_t^*$, and $c_{t,n} \in K$. We define $\mathcal{A}(\mathcal{T}^*)$ to be the $\mathbf{c} = (c_{t,n}) \in K^{D(\mathcal{T}^*)}$

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Given $\mathbf{c} = (c_{t,n}) \in \mathcal{A}(\mathcal{T}^*)$, let $\mathcal{H}^*(\mathbf{c})$ denote the set

$$\mathcal{H}^*(\mathbf{c}) = \{h_t = t - \sum_{n \in \mathcal{N}_t^*} c_{t,n} n \mid t \in \mathcal{T}^*\}$$

and

$L(\mathbf{c})$ be the submodule of P generated by the elements of $\mathcal{H}^*(\mathbf{c})$.

Existence Theorem I

Theorem

Let $\mathcal{T}^*$ be a right tip-reduced subset of $\mathcal{B}^*$. If L is a submodule of P with $\mathcal{T}^*$ the tip-set of the reduced right Gröbner basis of L, then there is some $\mathbf{c} \in \mathcal{A}(\mathcal{T}^*)$ such that $L = L(\mathbf{c})$.

Correspondence Theorem

Theorem

Let $\mathcal{T}^*$ be a right tip-reduced subset of $\mathcal{B}^*$. The following sets are in one-to-one correspondence with each other.

1. $\mathcal{A}(\mathcal{T}^*)$.
2. $\{\mathcal{H}^*(\mathbf{c}) \mid \mathbf{c} \in \mathcal{A}(\mathcal{T}^*)\}$.
3. $\{L(\mathbf{c}) \mid \mathbf{c} \in \mathcal{A}(\mathcal{T}^*)$ *with* $\mathcal{H}^*(\mathbf{c})$ *the reduced right Gröbner basis of* $L(\mathbf{c})\}$.

Corollary

The only point $(\mathbf{c})$ *in* $\mathcal{A}(\mathcal{T}^*)$ *such that* $L(\mathbf{c})$ *has right tip-reduced right Gröbner basis* $\mathcal{T}^*$ *is the point* $\mathbf{0} = (\dots, 0, 0, \dots)$*, that is* $L(\mathbf{0})$ *is the only submodule of* P *whose right tip reduced right Gröbner is* $\mathcal{T}^*$*.*

We freely refer to the module $L(\mathbf{c})$ as a point $(\mathbf{c})$ in affine space $\mathcal{A}(\mathcal{T}^*)$.

Some preserved properties

Proposition

Let $\mathcal{T}^*$ be a right tip-reduced subset of $\mathcal{B}^*$ and let $(\mathbf{c})$ and $(\mathbf{c}')$ be points in $\mathcal{A}(\mathcal{T}^*)$. Then

1. $\mathrm{tip}(L(\mathbf{c})) = \mathrm{tip}(L(\mathbf{c}'))$
2. *The tip-sets of the reduced Gröbner bases of $L(\mathbf{c})$ and $L(\mathbf{c}')$ are both equal to $\mathcal{T}^*$.*
3. *The dimension vectors of $P/L(\mathbf{c})$ and $P/L(\mathbf{c}')$ are equal.*

The definition of $\mathcal{V}_I(\mathcal{T}^*)$

Let $I \subset J$ be denote a two-sided Let M be a right KQ/I-module and fix the presentation and the Kernel, on the following exact sequence.

$$0 \to L(M) \stackrel{inc}{\to} P \to M \to 0$$

The definition of $\mathcal{V}_I(\mathcal{T}^*)$

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By the Correspondence Theorem we have that to every $(\mathbf{c}) \in \mathcal{A}(\mathcal{T}^*)$ corresponds a unique right tip-reduced set $\mathcal{H}^*(\mathbf{c})$ and a unique submodule $L(\mathbf{c})$ of P whose reduced right Gröbner is $\mathcal{H}^*(\mathbf{c})$ where $\mathrm{tip}(\mathcal{H}^*(\mathbf{c})) = \mathcal{T}^*$

Definition

Set

$$\mathcal{V}_I(\mathcal{T}^*) = \{(\mathbf{c}) \in \mathcal{A}(\mathcal{T}^*) \mid P/L(\mathbf{c}) \text{ is a } KQ/I\text{-module.}\}$$

Remark

If L is a submodule of P and $\mathcal{T}^*$ is the tip-set of the reduced right Gröbner basis of L then by Theorem of correspondence there is some $(\mathbf{c}) \in \mathcal{A}(\mathcal{T}^*)$ such that $L = L(\mathbf{c})$. If, in addition, I annihilates P/L, then $(\mathbf{c}) \in \mathcal{V}_I(\mathcal{T}^*)$.

Remark

If L is a submodule of P and $\mathcal{T}^*$ is the tip-set of the reduced right Gröbner basis of L then by Theorem of correspondence there is some $(\mathbf{c}) \in \mathcal{A}(\mathcal{T}^*)$ such that $L = L(\mathbf{c})$. If, in addition, I annihilates P/L, then $(\mathbf{c}) \in \mathcal{V}_I(\mathcal{T}^*)$.

Again by Theorem of correspondence and the definition of $\mathcal{V}_I(\mathcal{T}^*)$, the converse holds; that is, if $(\mathbf{c}) \in \mathcal{V}_I(\mathcal{T}^*)$ then $PI \subset L(\mathbf{c})$ and $\mathcal{T}^*$ is the tip-set of the reduced right Gröbner basis of the submodule $L(\mathbf{c})$ of P.

Algebraic Variety

Theorem

Let $\mathcal{T}^*$ be a right tip-reduced subset of $\mathcal{B}^*$ and I a two-sided ideal of KQ. Then $\mathcal{V}_I(\mathcal{T}^*)$ is an affine algebraic variety in $\mathcal{A}(\mathcal{T}^*)$.

Existence Theorem II

Theorem

Let I be a two-sided ideal of KQ. Every KQ/I-module corresponds to a point in a variety $\mathcal{V}_I(\mathcal{T}^*)$ for some $\mathcal{T}^*$.

Some extreme cases

Let I be an ideal in KQ. Assume that KQ/I is finite dimensional, In the following cases the variety contained the module M is just a point.

1. If M be an indecomposable projective KQ/I-module.
2. If M is a simple module.

I will finish given some examples

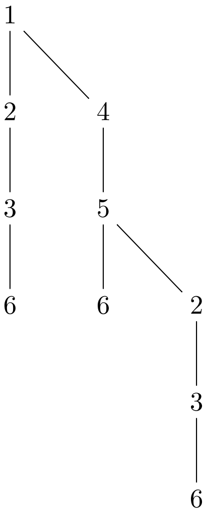
Example one

Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[d, "c"] & 2 \arrow[r, "f"] & 3 \arrow[d, "g"] \\
4 \arrow[r, "d"] & 5 \arrow[u, "b"] \arrow[r, "e"] & 6
\end{tikzcd}$$

Let M be the KQ-module such that the vector space at each vertex is K and each linear map is the identity map.

Then the projective cover is $P = v_1 KQ$


1
2
4
3
5
6
6
2
3
6

Let $L = \ker(P \to M)$ where $P \to M$ be a KQ-projective cover of M. Then $L = v_2KQ \oplus v_6KQ$. Let $\mathcal{T}^* = \{cdb, afg\}$. Taking $a > b > c > \cdots$ we obtain $h_{cbd} = cdb - Xa$ and $h_{afg} - Ycde$. Let I be the two-sided ideal $\langle cbdfg \rangle$. It is immediate that $cbdfg$ is the right reduced Gröbner basis of I.

Right reducing $cdbef$ by $\{h_1, h_2\}$ we get $cdbfg$ reduces to $Xafg$. Continuing, $Xafg$ reduces to $XYcde$ which is completely reduced. We conclude that $\mathcal{V}_I(\mathcal{T}^*)$ is zeros of XY.

Example 2

Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[rrr, bend left, "h"] \arrow[rrr, bend right, "f"'] & 2 \arrow[r, "b"] & 3 \arrow[r, "c"] & 4 \arrow[r, "d"] \arrow[rr, bend right, "g"'] & 5 \arrow[r, "e"] & 6
\end{tikzcd}$$

$P = v_1 KQ$

$\mathcal{T}^* = \{abc,\ fde\}$

$a > b > c > \cdots > f > g$ with length-lex

$\mathcal{H}^* = \{abc - Xf - Zh, fde - Yfg - Uhde - Vhg\}$

$I = \langle abcde \rangle$

$abcde \rightarrow_{\mathcal{H}^*} Xfde + Zhde$

$Xfde \rightarrow_{\mathcal{H}^*} X(Yfg + Uhde + Vhg) + Zhde$.
To obtain an element $(X, Y, Z.U, V)$,in $\mathcal{V}_I(\mathcal{T}^*)$
we need that $XY = 0, XU + Z = 0$,and $XV = 0$.

so $\mathcal{V}_I(\mathcal{T}^*) =$ the zero locust of $(XY, XU + Z, XV)$ in K^5.

Example 3

Let Q be the quiver:

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[rr, "c"] & & 3 \end{tikzcd}$$

Consider the representation

$$M_\lambda = \begin{tikzcd} & 2 \arrow[dr, "1"] & \\ 1 \arrow[ur, "1"] \arrow[rr, "\lambda"] & & 3 \end{tikzcd} \qquad \lambda \in K$$

The KQ projective cover in KQ is $v_1 KQ$ and the kernel is the projective simple, KQv_3, Observe that in this case $M_\lambda \ncong M'_\lambda$ if $\lambda \neq \lambda'$.

Let $\mathcal{T}^* =< ab >$, then $\mathcal{V}(\mathcal{T}^*) = K$, $L =< ab - \lambda c >$

Example 4

Let Q be the following quiver:

$$Q = \begin{tikzcd} & 2 \arrow[dr, "b"] & & \\ 1 \arrow[ur, "a"] \arrow[dr, "c"'] & & 4 \arrow[r, "e"'] & 5 \\ & 3 \arrow[ur, "d"] & & \end{tikzcd}$$

Also for each $0 \neq \lambda \in K$ the module associated with the representation:

$$M_\lambda = Q = \begin{tikzcd} & 2 \arrow[dr, "b"] & & \\ k \arrow[ur, "1"] \arrow[dr, "1"'] & & 4 \arrow[r, "0"'] & 5 \\ & 3 \arrow[ur, "\lambda"] & & \end{tikzcd}$$

The first KQ syzygy of M is isomorphic to the direct sum $S_5 \oplus P(4)$

$A = KQ/I$ where $I =< abe >$

Here, as usual in these examples, we consider the degree lexicographer order with $a > b > c > d > e$.

Take $\mathcal{T}^* = \{cde, ab\}$ and in this case $\mathcal{V}(\mathcal{T}^*) = K$

Observe that $M_\lambda \equiv M_{\lambda'}$ implies $\lambda = \lambda'$.

Let Q be the following quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "a_1"] \arrow[r, "b_1"] \arrow[r, bend right, "c_1"] & 2 \arrow[r, bend left, "a_2"] \arrow[r, "b_2"] \arrow[r, bend right, "c_2"] & 3 \arrow[r, bend left, "a_3"] \arrow[r, "b_3"] \arrow[r, bend right, "c_3"] & 4 \end{tikzcd}$$

Let $P = vKQ$ where v is the vertex 1 of Q and let $\mathcal{B}^*$ be the basis of P. Let $\succ$ be the length lexicographical order on $\mathcal{B}$ the basis of paths of KQ such that $a_1 \succ a_2 \succ a_3 \succ b_1 \ldots \succ c_3$ and also call $\succ$ the induced order on the basis $\mathcal{B}^*$ of P. We start by defining two subsets $\mathcal{S}$ and $\mathcal{S}'$of $\mathcal{B}^*$. Namely, let

$$\mathcal{S} = \text{ the set of all paths of length 2 except } \{a_1c_2, c_1a_2, c_1b_2, c_1c_2\}$$

$$\mathcal{S}' = \text{ the set of all paths of length 3 except } \{a_1c_2c_3, c_1a_2c_3, c_1b_2c_3, c_1c_2c_3\}$$

Let I be the two-sided monomial ideal generated by

$$\mathcal{G} = \mathcal{S} \cup \mathcal{S}' \cup \{a_1 c_2 c_3\}.$$

Then $\mathcal{G}$ is the right reduced Gröbner of I
and since $PI = I$, $\mathcal{G}$ also is the reduced right Gröbner $\mathcal{G}^*$ of PI, that is $\mathcal{G} = \mathcal{G}^*$.

In this example let

$$\mathcal{T}^* = \mathcal{S} \cup \mathcal{S}' \cup \{a_1 c_2, c_1 a_2 c_3, c_1 b_2 c_3\}.$$

Let $\mathbf{X} = (X_{t,n})$, for $t \in \mathcal{T}^*$ and $n \in \mathcal{N}_t^*$, be variables and let

$$\mathcal{H}^*(\mathbf{X}) = \{h_t = t - \sum_{n \in \mathcal{N}_t^*} X_{t,n} n \mid t \in \mathcal{T}^*\}.$$

We may assume $X_{t,n} = 0$ for all $t \in \mathcal{S} \cup \mathcal{S}' \subset PI \cap \mathcal{T}^*$ and hence $h_t = t$, for all $t \in \mathcal{S} \cup \mathcal{S}'$.

We have

$$\begin{array}{lcl} \mathcal{N}^*_{a_1 c_2} & = & \{c_1 a_2, c_1 b_2, c_1 c_2\} \\ \mathcal{N}^*_{c_1 a_2 c_3} = \mathcal{N}^*_{c_1 b_2 c_3} & = & \{c_1 c_2 c_3\} \end{array}$$

So the only linear combinations with possible non-zero coefficients in $\mathcal{H}^*(\mathbf{X})$ are

$$h_{a_1c_2} = a_1c_2 - X_{a_1c_2,c_1a_2}c_1a_2 - X_{a_1c_2,c_1b_2}c_1b_2 - X_{a_1c_2,c_1c_2}c_1c_2$$

$$h_{c_1a_2c_3} = c_1a_2c_3 - X_{c_1a_2c_3,c_1c_2c_3}c_1c_2c_3$$

and

$$h_{c_1b_2c_3} = c_1b_2c_3 - X_{c_1b_2c_3,c_1c_2c_3}c_1c_2c_3.$$

For simplicity, let

$$\begin{array}{rcl} X_1 & = & X_{a_1c_2,c_1a_2} \\ X_2 & = & X_{a_1c_2,c_1b_2} \\ X_3 & = & X_{a_1c_2,c_1c_2} \\ X_4 & = & X_{c_1a_2c_3,c_1c_2c_3} \\ X_5 & = & X_{c_1b_2c_3,c_1c_2c_3} \end{array}$$

Let $L(\mathbf{X}) = \langle \mathcal{H}^*(\mathbf{X}) \rangle_R$. In order for $L(\mathbf{X})$ to specialise to $L(\mathbf{c})$ such that $(\mathbf{c}) \in \mathcal{V}_I(\mathcal{T}^*)$, every generator starting at v in the reduced right Gröbner basis of PI must completely reduce to zero by $\mathcal{H}^*(\mathbf{c})$.

But all the paths $t \in \mathcal{S} \cup \mathcal{S}'$ are such that $h_t = t$ and thus they are in $L(\mathbf{X})$. The only path in $\mathcal{G}^*$ that is not in $\mathcal{T}^*$ is $a_1c_2c_3$ and we need to completely reduce it by $\mathcal{H}^*(\mathbf{X})$.

We have a simple reduction using $h_{a_1 c_2}$ and

$$\begin{aligned} a_1 c_2 c_3 &\to_{\mathcal{H}^*(\mathbf{X})} X_1 c_1 a_2 c_3 + X_2 c_1 b_2 c_3 + X_3 c_1 c_2 c_3 \\ &\to_{\mathcal{H}^*(\mathbf{X})} X_1 X_4 c_1 c_2 c_3 + X_2 X_5 c_1 c_2 c_3 + X_3 c_1 c_2 c_3 \end{aligned}$$

where the latter is equal to $(X_1 X_4 + X_2 X_5 + X_3) c_1 c_2 c_3$. The zero locus of this polynomial give rise to the points $(\mathbf{c}) \in \mathcal{V}_I(\mathcal{T}^*)$.

Muito Obrigado,
Muchas Gracias,
Thank you very much
Saúde a todos