HOCHSCHILD CALCULUS of GENTLE ALGEBRAS

0 - Introduction

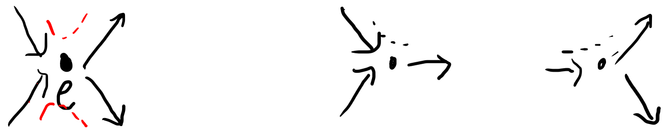
- Homological tools: invariant (derived)

- $HH^*(A)$ is a graded commutative algebra, $\cup$
- $(HH^*(A), \cup, [\ ,\ ]_{Gerst})$ Gerstenhaber alg.
- $HH_*(A)$ cap product $\cap$, $HH_*(A)$ is a $HH^*(A)$-module.
- $HH_*(A) \xrightarrow[B]{} HH_{*+1}(A)$ $\quad B^2 = 0$

AIM OF THE TALK: describe invariants of gentle alg. using all these structures.

1. Locally gentle algebras

- Q



- I is generated by quadratic monomial relations.
- $\dim \underbrace{kQ/I}_{= A}$ $\begin{cases} < \infty & \text{gentle} \\ \infty & \text{locally gentle} \end{cases}$

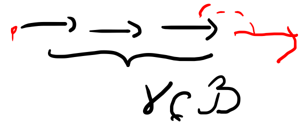
So: quadratic monomial $\Rightarrow$ Koszul

$A^!$ is a locally gentle algebra.

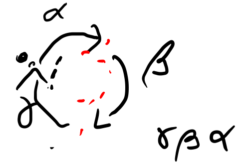
2 - Some useful definitions and terminology

Let $\mathcal{B}$ be a basis of paths of kQ/I obtained from a basis of paths of kQ.

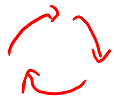
- a path γ is $\mathcal{B}$-maximal if



- $\mathcal{B}$-maximal cycles :



- complete cycles :
- cocomplete cycles :



- primitive cycles (complete and/or cocomplete).

3 - Some invariants

- $n = \#\{\text{primitive complete cycles}\}$
- AAG $\phi_A(0,n)$, $\phi_A(1,n)$
- $\# Q_0$
- gentle case, $\#\{\text{loops}\}$
- $\# Q_1$
- if there are complete cycles, we can see char k
- $\dim A < \infty$?
- $\operatorname{gl\,dim} A < \infty$?

About the resolution and the cycles in the quiver:

· Bardzell resolution (Koszul resolution).

$$\cdots \to A \otimes_E \Gamma_2 \otimes_E A \to A \otimes_E \Gamma_1 \otimes_E A \to A \otimes_E A \xrightarrow{\mu} A \to 0$$

Γ_n : Γ_2 spanned by the relations xy

$\Gamma_1 = kQ_1$ $\quad$ $\Gamma_0 = kQ_0 = E$ $\quad$ Γ_3

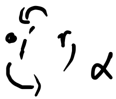
Γ_n $\quad$ $n-2$ superpositions.

4. Brief description of bases of $HH^*(A)$ and of $HH_*(A)$

Goal: you can read cohomology from Q + relations.

$\boxed{HH^0(A) = Z(A)}$

* $1_A = \sum_{e \in Q_0} (e,e)$

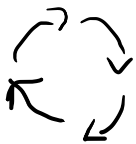


* $(s(\alpha), \alpha) \in \Gamma_0 /\!/ \mathcal{B}$ / α is $\mathcal{B}$-maximal, i.e.:

if $\dim A = \infty$

* For each α primitive complete circuit of period r

$$\langle\langle \alpha \rangle\rangle = \sum_{i=0}^{r-1} \left(s(\mathrm{rot}^i(\alpha)), \mathrm{rot}^i(\alpha)\right)$$

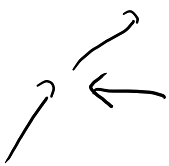


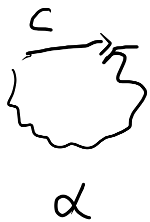
$$\boxed{HH^1(A) = \mathrm{Der}(A) / \mathrm{Inder}(A)}$$

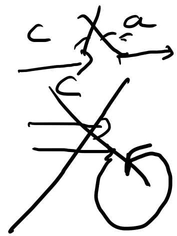
* (c, c) for each $c \in Q_1 \setminus T$
* (c, α) / $c \in Q_1$ and c T-maximal, $\alpha \in \mathcal{B}$, α does not start neither ends with c.
* $(b, s(b))$ / b loop, $b^2 \in I$, char $k = 2$

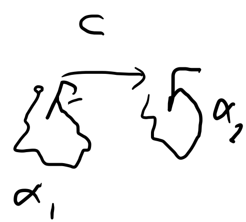
$\dim A = \infty$ [* $(c, c\delta)$ / $\delta \in \mathcal{B}$, $\delta = \delta' c$

T maximal 3





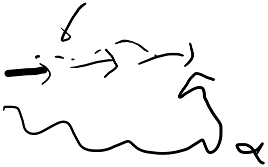




$\alpha = \alpha_2 c \alpha_1$

$\boxed{\begin{array}{c} HH^m(A) \\ m \geq 2 \end{array}}$

- $\langle\langle C \rangle\rangle$ for each C complete circuit of length m (even or char 2)
- (bC, b) for each C complete circuit of length $m-1$ and b the first arrow of C. ($m-1$ even or char 2)
- $(\gamma, \alpha) \in \Gamma_m // \mathcal{B}$, γ Γ-maximal; γ and α do not share neither the first nor the last arrow.



Set $\mathcal{G}$ of generators of the cohomology algebra:

0 * $(s(\alpha), \alpha)$

0 * $\langle\langle \alpha \rangle\rangle$

1 * (c, c) $c \in Q_1 \setminus T$

* $\langle\langle C \rangle\rangle$ $m \geq 2$
* (γ, α) $r \geq 2$

The set $\mathcal{G}$ is minimal except when  

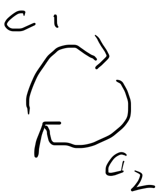
Homology

$\boxed{HH_0(A)}$

- (e, e) for each $e \in Q_0$
- $(c, \Delta(c))$ for each c cocomplete circuit. $\dim A = \infty$
- $(a, \Delta(a))$ for each loop a / $a^2 = 0$ $\quad$ $\mathrm{gldim}\, A = \infty$

$\boxed{HH_1(A)}$

- $[\![c]\!] = \sum_{i=0}^{r-1} \left(\mathrm{rot}^i(c)_{rt}, \mathrm{rot}^i(c)'_{rt}\right)$ for each cocomplete circuit of Q $\quad$ $\dim A = \infty$
- $[\![a]\!] = (s(a), a)$ for each loop a / $a^2 = 0$ $\quad$ $\mathrm{gl\,dim}\, A = \infty$
- (c'_{et}, c_{et}) for each complete cycle c of length 2 and period r / $(-1)^r = 1$.



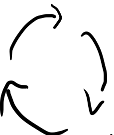
$c_2 c_1$ $\qquad$ $\overset{\shortparallel}{(c_2, c_1)}$

$$\boxed{\begin{array}{c} HH_m(4) \\ m \geq 2 \end{array}}$$

- $[C] = \sum_{i=0}^{r-1} (-1)^{(m+1)i} \big(s(\mathrm{rot}^i(c)), \mathrm{rot}^i(c) \big)$

  c complete circuit of length m, period r / $(-1)^{(m+1)r} = 1$.

- (c'_{et}, c_{et}) c complete circuit of length $m+1$, period r, $(-1)^{mr} = 1$

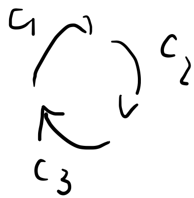


$\tilde{c}$

period 3

length 6

$\underline{c = \tilde{c}^2}$



$(c_3, c_2 c_1)$

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5 - FIRST CONSEQUENCES

PROP 1: TFAE:
(a) $\mathrm{gldim}\, A < \infty$
(b) $\exists\, m_0 \mid HH^m(A) = 0 \ \forall m \geq m_0$
(c) $\nexists$ complete cycles in (Q, I)
(d) $HH_i(A) = 0 \ \forall i \geq 1$ if $\dim A < \infty$
$\forall i \geq 2$ if $\dim A = \infty$

PROP 2: TFAE: (a) $\dim A < \infty$
(b) $\dim HH^0(A) < \infty$
(c) $\dim HH^1(A) < \infty$
(d) $\nexists$ cocomplete cycles in (Q, I)
(e) (condition in terms of homology)

COR: $HH^*(A)$ is finite dimensional $\Leftrightarrow$ $\mathrm{gldim}\, A < \infty$ and $\dim A < \infty$

PROP 3: A gentle,

(a) Let $N = \#\{\text{primitive complete circuits in } (Q,I)\}$

$\forall m > N$, $\dim HH^m(A) \leq 2N$

(b) If $\exists\, m > |Q_1|$ / $HH^m(A) \neq 0$, then there are infinitely many $i \in \mathbb{N}$ / $HH^i(A) \neq 0$

PROP 4: The presentation of the algebra $(HH^*(A), \cup)$ is monomial quadratic (but the generators are in $\neq$ degrees)

[whenever we can choose a maximal tree T such that $Q_1 \setminus T$ intersects each cycle only once

PROP 5: The graded Jacobian radical $\operatorname{rad} HH^*(A)$ is $\langle HH^+(A) \cup \{\underbrace{(s(\alpha), \alpha)}_{\in HH^0(A)}\} \rangle$

so $HH^*(A) / \operatorname{rad} HH^*(A) \cong k[x_1, \dots, x_m] / (x_i x_j,\ 1 \leq i < j \leq m)$

where $m = \#\{$primitive cocomplete cycles in the locally gentle presentation of $A\}$

In particular, m is a derived invariant of A.

COR: A gentle. Let $T(A) = \dfrac{\operatorname{rad} HH^*(A)}{\operatorname{rad}^2 HH^*(A)}$, This is a fin. dim. graded v.s. and its Hilbert-Poincaré series

$$h_{T(A)}(t) = \sum_{m \geq 0} \dim T^m(A) t^m \in \mathbb{Z}[t]$$ is a derived invariant.

PROP 6: A gentle, $\#\{$loops in $Q\}$ is a derived invariant.

We get it from $HH_0(A)$, using also that $\# Q_0$ is a derived inv.

6 - MORE INVARIANTS

We need the Gerstenhaber bracket

How to compute it?

It is known (Ladkani, Redondo-Román) that

$$\dim HH^1(A) = 1 - \big(\overbrace{|Q_0| - |Q_1|}^{\chi(Q)}\big) + \underbrace{\phi_A(1,1) + \overbrace{\phi_A(0,1)}^{\text{char } k = 2}}_{\dim [HH^1(A),\, HH^1(A)]}$$

$$1 - |Q_0| + |Q_1| = \dim \frac{HH^1(A)}{(HH^1(A))'}$$

$\Rightarrow$ $|Q_1|$ is a derived invariant.

Generalising :

PROP: 1st case: char $k=0$ or $\mathrm{gldim}\, A<\infty$

$$\dim HH^n(A) \Big/ [HH^1(A), HH^n(A)] = \begin{cases} 1-\chi(Q) & n=1 \\ 0 & n\neq 1 \end{cases}$$

2nd case: char $k=p$ and $\mathrm{gldim}\, A=\infty$

$$\dim HH^n \Big/ [HH^1, HH^n] = \begin{cases} 0 & n=0 \\ 1-\chi(Q) & n=1 \\ 0 & (n,p)=1 \\ \#\{c \,/\, \lg(c) \,/\, m\} . & n=pm \end{cases}$$

In part, if char $k=0$, then $\dim HH^n(A) =$

Agutte $\dim HH^n(A) = \underbrace{\phi(1,n)}_{(\gamma,\alpha)} + \underbrace{\sum_{r/m} \phi(0,r)}$

How to recover char k? If there are complete cycles:

Step 1: From $HH^1(A)$ and $\phi(1,1)$, we can recognize if char $k = 2$

Step 2:

Step 3:

Step 4:

$\phi(0,n)$ $\qquad$ $\phi(1,n)$ $\qquad$ $\underbrace{\phi(m,n)}$ $\quad m \geq 2$

$HH^1(TA)$

$HH^\bullet(A)$

$HH^*(TA)$