Higher Auslander-Reiten theory: what is it and how to understand it

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Some conventions and definitions:

- Λ will always denote an Artin algebra.

$$\left(\begin{array}{c}\text{If you prefer, you can consider } \Lambda \text{ as a} \\ \text{finite dimensional algebra over a field.}\end{array}\right)$$

- $\operatorname{mod}\Lambda$ is the category of finitely generated (right) Λ-modules.
- If $\mathcal{C}$ is a subcategory of $\operatorname{mod}\Lambda$, then $\operatorname{add}\mathcal{C}$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of the direct summands of finite direct sums of objects of $\mathcal{C}$.
- If $\mathcal{C}$ has only one object X, then we denote $\operatorname{add}\mathcal{C} = \operatorname{add}X$.
- n will always be a positive integer.

What is higher Auslander–Reiten theory?

Here is a short answer:

Auslander–Reiten theory: (1970s, Maurice Auslander and Idun Reiten.)

- We work with the category $\operatorname{mod}\Lambda$.
- We study almost split sequences in $\operatorname{mod}\Lambda$.

Higher Auslander–Reiten theory: (2000s, Osamu Iyama.)

- We work with a convenient subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$.
- We study n-almost split sequences in $\mathcal{C}$.

(We recover the classical case by taking $n = 1$.)

Higher homological algebra! (2010s)

- $(n+2)$-angulated categories.
- n-abelian and n-exact categories.

(We recover the classical cases by taking $n = 1$.)

We say that a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ is an **almost split sequence** in $\operatorname{mod}\Lambda$ if

- g is right almost split in $\operatorname{mod}\Lambda$,
- f is left almost split in $\operatorname{mod}\Lambda$.

Let $\mathcal{C}$ be a convenient subcategory of $\operatorname{mod}\Lambda$. We say that an exact sequence

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ of length n with terms in $\mathcal{C}$ is an n-**almost split sequence** in $\mathcal{C}$ if

- f_1 is right almost split in $\mathcal{C}$,
- f_{n+1} is left almost split in $\mathcal{C}$,
- $f_i \in \operatorname{rad}_{\mathcal{C}}$ for each $2 \leqslant i \leqslant n$.

How to understand the theory?

Through the functorial approach!

Let Γ be a ring and Ab be the category of abelian groups.

- We can view Γ as a preadditive category $\mathcal{C}$ consisting of only one object X and such that $\mathcal{C}(X, X) = \Gamma$.
- If F is a contravariant additive functor from $\mathcal{C}$ to Ab, then $F(X)$ is a right Γ-module:

$$\gamma \in \Gamma, z \in F(X) \;\; \Rightarrow \;\; z \cdot \gamma = F(\gamma)(z)$$

- The category $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ consisting of the contravariant additive functors from $\mathcal{C}$ to Ab is isomorphic to $\operatorname{Mod}\Gamma$.

"module = functor"

Let $\mathcal{C}$ be a skeletally small preadditive category.

- $\mathcal{C}$ is a "ring with several objects". (Barry Mitchell)
- We call a contravariant additive functor from $\mathcal{C}$ to Ab a $\mathcal{C}$-**module**.
- We denote $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) = \operatorname{Mod} \mathcal{C}$ (the category of $\mathcal{C}$-modules).

"$\mathcal{C}$-module = right $\mathcal{C}$-module".

- A **left** $\mathcal{C}$-**module** is a covariant additive functor from $\mathcal{C}$ to Ab.
- A left $\mathcal{C}$-module is the same as a right $\mathcal{C}^{\mathrm{op}}$-module.
- Hence we have $(\mathcal{C}, \mathrm{Ab}) = \operatorname{Mod} \mathcal{C}^{\mathrm{op}}$ (the category of left $\mathcal{C}$-modules).

- An example of a (right) $\mathcal{C}$-module:

  The contravariant functor $\mathcal{C}(-, X) : \mathcal{C} \to \mathsf{Ab}$ for some $X \in \mathcal{C}$.

- An example of a left $\mathcal{C}$-module:

  The covariant functor $\mathcal{C}(X, -) : \mathcal{C} \to \mathsf{Ab}$ for some $X \in \mathcal{C}$.

In this talk we will also assume that $\mathcal{C}$ is a Krull–Schmidt category, i.e.

- $\mathcal{C}$ is an additive category,
- Every object $X \in \mathcal{C}$ can be written as

$$X \simeq X_1 \oplus \cdots \oplus X_m$$

with the endomorphism ring of each X_i being local.

Some facts and definitions:

- A $\mathcal{C}$-module F is **finitely generated** and **projective** if and only if $F \simeq \mathcal{C}(-, X)$ for some $X \in \mathcal{C}$.
- If $F \in \operatorname{Mod} \mathcal{C}$, then we say that F is **finitely presented** if there is an exact sequence

$$\mathcal{C}(-, Y) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-, X) \longrightarrow F \longrightarrow 0$$

  in $\operatorname{Mod} \mathcal{C}$ with $X, Y \in \mathcal{C}$.
- We denote by $\operatorname{mod} \mathcal{C}$ the subcategory of $\operatorname{Mod} \mathcal{C}$ consisting of the finitely presented $\mathcal{C}$-modules.

Why are we doing this?

If Γ is a ring, then we may study $\operatorname{Mod}\Gamma$ to understand Γ.

If $\mathcal{C}$ is a (...) category, then we may study $\operatorname{Mod}\mathcal{C}$ to understand $\mathcal{C}$.

If Λ is an Artin algebra, then we may study $\operatorname{mod}\Lambda$ to understand Λ.

Furthermore, we may study $\operatorname{Mod}(\operatorname{mod}\Lambda)$ to understand $\operatorname{mod}\Lambda$.

- We will give more attention to $\operatorname{mod}(\operatorname{mod}\Lambda)$.
- Auslander–Reiten theory!

More facts and definitions:

- If $F \in \operatorname{Mod} \mathcal{C}$, then we denote its **Jacobson radical** by $\operatorname{rad}_{\mathcal{C}} F$.
- $\operatorname{rad}_{\mathcal{C}} \mathcal{C}(-, X) = \operatorname{rad}_{\mathcal{C}}(-, X)$ for $X \in \mathcal{C}$.
- A $\mathcal{C}$-module F is **simple** if and only if

$$F \simeq \frac{\mathcal{C}(-, Z)}{\operatorname{rad}_{\mathcal{C}}(-, Z)} = S_Z$$

  for some indecomposable $Z \in \mathcal{C}$.
- A left $\mathcal{C}$-module G is **simple** if and only if

$$G \simeq \frac{\mathcal{C}(X, -)}{\operatorname{rad}_{\mathcal{C}}(X, -)} = S^X$$

  for some indecomposable $X \in \mathcal{C}$.

Proposition

An exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ is an almost split sequence in $\operatorname{mod}\Lambda$ if and only if X and Z are indecomposable and if

$$0 \longrightarrow \operatorname{Hom}_\Lambda(-, X) \xrightarrow{\operatorname{Hom}_\Lambda(-,f)} \operatorname{Hom}_\Lambda(-, Y) \xrightarrow{\operatorname{Hom}_\Lambda(-,g)} \operatorname{Hom}_\Lambda(-, Z) \longrightarrow S_Z \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}_\Lambda(Z, -) \xrightarrow{\operatorname{Hom}_\Lambda(g,-)} \operatorname{Hom}_\Lambda(Y, -) \xrightarrow{\operatorname{Hom}_\Lambda(f,-)} \operatorname{Hom}_\Lambda(X, -) \longrightarrow S^X \longrightarrow 0$$

are minimal projective resolutions of S_Z in $\operatorname{Mod}(\operatorname{mod}\Lambda)$ and of S^X $\operatorname{Mod}(\operatorname{mod}\Lambda)^{\mathrm{op}}$, respectively.

Let $\mathcal{C}$ be subcategory of $\operatorname{mod}\Lambda$ such that $\operatorname{add}\mathcal{C} = \mathcal{C}$. We say that an exact sequence

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ of length n with terms in $\mathcal{C}$ is an ***n*-almost split sequence** in $\mathcal{C}$ if X and Z are indecomposable and if

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z) \longrightarrow S_Z \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_n,-)} \mathcal{C}(Y_n,-) \xrightarrow{\mathcal{C}(f_{n+1},-)} \mathcal{C}(X,-) \longrightarrow S^X \longrightarrow 0$$

are minimal projective resolutions of S_Z in $\operatorname{Mod}\mathcal{C}$ and of S^X $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$, respectively.

Proposition

An exact sequence

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ of length n with terms in $\mathcal{C}$ is an n-almost split sequence in $\mathcal{C}$ if and only if

- f_1 is right almost split in $\mathcal{C}$,
- f_{n+1} is left almost split in $\mathcal{C}$,
- $f_i \in \operatorname{rad}_{\mathcal{C}}$ for each $2 \leqslant i \leqslant n$,
- The sequences

  $$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z)$$

  and

  $$0 \longrightarrow \mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_n,-)} \mathcal{C}(Y_n,-) \xrightarrow{\mathcal{C}(f_{n+1},-)} \mathcal{C}(X,-)$$

  are exact in $\operatorname{Mod}\mathcal{C}$ and in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$, respectively.

It would be interesting to have the following property:

"If

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an exact sequence in $\operatorname{mod} \Lambda$ of length n with terms in $\mathcal{C}$, then

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z)$$

and

$$0 \longrightarrow \mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_n,-)} \mathcal{C}(Y_n,-) \xrightarrow{\mathcal{C}(f_{n+1},-)} \mathcal{C}(X,-)$$

are exact in $\operatorname{Mod} \mathcal{C}$ and $\operatorname{Mod} \mathcal{C}^{\mathrm{op}}$, respectively."

To have this property it is sufficient to suppose that

$$\operatorname{Ext}^i_\Lambda(X,Y) = 0$$

for every $X, Y \in \mathcal{C}$ and $0 < i < n$.

Proposition

Suppose that $\mathrm{Ext}^i_\Lambda(X, Y) = 0$ for every $X, Y \in \mathcal{C}$ and $0 < i < n$. Then an exact sequence

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ of length n with terms in $\mathcal{C}$ is an n-almost split sequence in $\mathcal{C}$ if and only if

- f_1 is right almost split in $\mathcal{C}$,
- f_{n+1} is left almost split in $\mathcal{C}$,
- $f_i \in \mathrm{rad}_{\mathcal{C}}$ for each $2 \leqslant i \leqslant n$.

We say that $\mathcal{C}$ **has n-almost split sequences** if

- For every indecomposable and nonprojective $Z \in \mathcal{C}$ there is an n-almost split sequence

$$0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

in $\mathcal{C}$.

- For every indecomposable and noninjective $X \in \mathcal{C}$ there is an n-almost split sequence

$$0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

in $\mathcal{C}$.

Which subcategories of $\operatorname{mod}\Lambda$ have n-almost splits sequences?

Well, if we impose some conditions on $\mathcal{C}$, then we can guarantee that it has n-almost split sequences.

Theorem

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$ such that $\operatorname{add}\mathcal{C} = \mathcal{C}$. If $\mathcal{C}$ satisfies the conditions below, then it has n-almost split sequences.

1. $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ for every $X, Y \in \mathcal{C}$ and $0 < i < n$,
2. $\mathcal{C}$ is functorially finite in $\operatorname{mod}\Lambda$,
3. If $X \in \operatorname{mod}\Lambda$ satisfies that $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ for every $Y \in \mathcal{C}$ and $0 < i < n$, then $X \in \mathcal{C}$,
4. If $Y \in \operatorname{mod}\Lambda$ satisfies that $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ for every $X \in \mathcal{C}$ and $0 < i < n$, then $Y \in \mathcal{C}$.

Such a subcategory is called an n-**cluster tilting subcategory of** $\operatorname{mod}\Lambda$.

What do these properties mean?

2. implies that $\operatorname{mod}\mathcal{C}$ and $\operatorname{mod}\mathcal{C}^{\mathrm{op}}$ are abelian categories.
3. implies* that $\mathcal{P}(\Lambda) \subseteq \mathcal{C}$ and that $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}) \leqslant n+1$.
4. implies* that $\mathcal{I}(\Lambda) \subseteq \mathcal{C}$ and that $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}^{\mathrm{op}}) \leqslant n+1$.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$ such that $\operatorname{add}\mathcal{C} = \mathcal{C}$ and define

$$ {}^{\perp n}\mathcal{C} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, Y) = 0 \text{ for all } 0 < i < n \text{ and } Y \in \mathcal{C}\}, $$
$$ \mathcal{C}^{\perp n} = \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, Y) = 0 \text{ for all } 0 < i < n \text{ and } X \in \mathcal{C}\}. $$

Then to say that $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ means that $\mathcal{C}$ is functorially finite in $\operatorname{mod}\Lambda$, ${}^{\perp n}\mathcal{C} = \mathcal{C}$ and $\mathcal{C} = \mathcal{C}^{\perp n}$.

Proposition

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$ such that $\operatorname{add}\mathcal{C} = \mathcal{C}$. If $\mathcal{C}$ is functorially finite in $\operatorname{mod}\Lambda$, then the following are equivalent:

1. $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$.
2. $\mathcal{C} = \mathcal{C}^{\perp n}$ and $\mathcal{P}(\Lambda) \subseteq \mathcal{C}$.
3. ${}^{\perp n}\mathcal{C} = \mathcal{C}$ and $\mathcal{I}(\Lambda) \subseteq \mathcal{C}$.

If $M \in \operatorname{mod} \Lambda$ satisfies that $\operatorname{add} M$ is an n-cluster tilting subcategory of $\operatorname{mod} \Lambda$, then we say that M is an **n-cluster tilting module over Λ**.

However, we have the following result:

Proposition

If $M \in \operatorname{mod} \Lambda$, then $\operatorname{add} M$ is functorially finite in $\operatorname{mod} \Lambda$.

Therefore, for $M \in \operatorname{mod} \Lambda$ the following are equivalent:

1. M is an n-cluster tilting module over Λ.
2. ${}^{\perp n}(\operatorname{add} M) = \operatorname{add} M$ and $\operatorname{add} M = (\operatorname{add} M)^{\perp n}$.

An example: (Gustavo Jasso / Laertis Vaso)

Consider the quiver

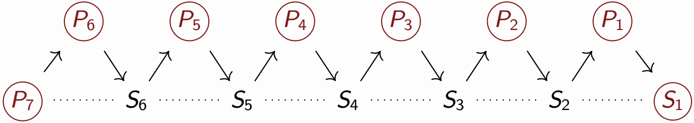
$$Q: \qquad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5 \xrightarrow{\alpha_5} 6 \xrightarrow{\alpha_6} 7$$

and take $\Lambda = KQ/(\operatorname{rad} KQ)^2$, where K is a field. Then the Auslander-Reiten quiver of Λ is given by:

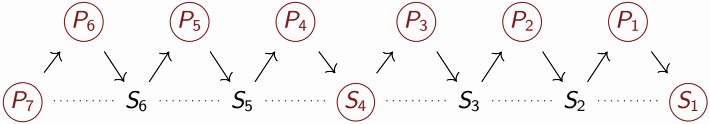
$$\begin{array}{ccccccccccccc} & & P_6 & & & & P_5 & & & & P_4 & & & & P_3 & & & & P_2 & & & & P_1 & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ P_7 & \cdots\cdots & & & S_6 & \cdots\cdots & & & S_5 & \cdots\cdots & & & S_4 & \cdots\cdots & & & S_3 & \cdots\cdots & & & S_2 & \cdots\cdots & & & S_1 \end{array}$$

- Obviously, $\operatorname{mod}\Lambda$ is a 1-cluster tilting subcategory of $\operatorname{mod}\Lambda$.
- Moreover, $\operatorname{mod}\Lambda$ has the following n-cluster tilting subcategories:

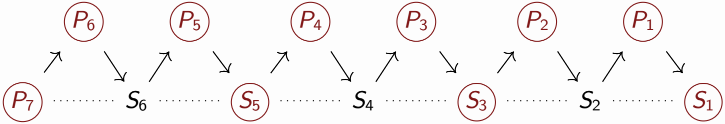
$n = 6$:


P6 P5 P4 P3 P2 P1
P7 S6 S5 S4 S3 S2 S1

$n = 3$:


P6 P5 P4 P3 P2 P1
P7 S6 S5 S4 S3 S2 S1

$n = 2$:


P6 P5 P4 P3 P2 P1
P7 S6 S5 S4 S3 S2 S1

Some open problems:

Problem (1)

When does $\operatorname{mod}\Lambda$ contain an n-cluster tilting subcategory (module)?

Problem (2)

Is every n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ with $n \geqslant 2$ of finite type?

We can verify that if $1 \leqslant \operatorname{gl.dim}\Lambda < \infty$ and if $\operatorname{mod}\Lambda$ has an n-cluster tilting subcategory, then $n \leqslant \operatorname{gl.dim}\Lambda$.

Problem (3)

If $\operatorname{gl.dim}\Lambda = \infty$, then are there only finitely many positive integers n such that $\operatorname{mod}\Lambda$ admits an n-cluster tilting module?

Thank you!