Wide intervals and mutation

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I. Introduction

Let R be a noetherian ring

$$\text{tors-}R = \{\tau = (t, f) \mid \tau \text{ torsion pairs in } \operatorname{mod} R\}$$

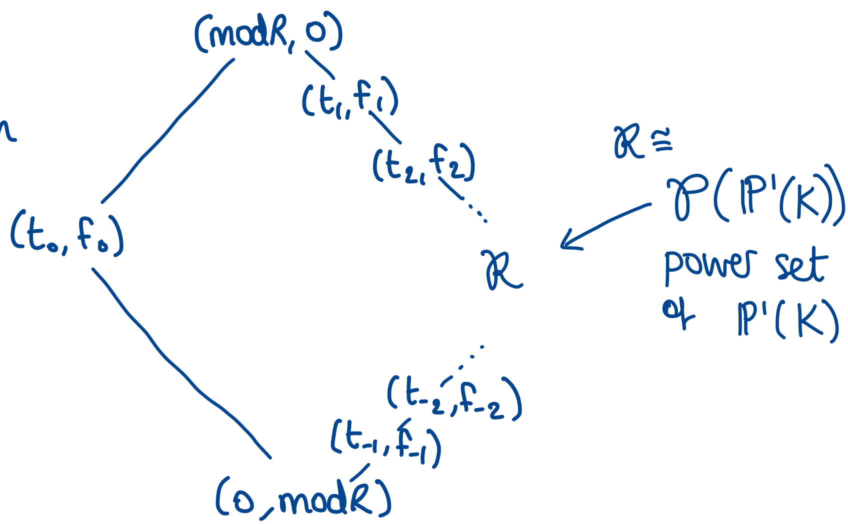
ordered by $(t, f) \geq (u, v) \iff_{\text{def}} t \supseteq u$

Example: K algebraically closed field.

$R = K(\bullet \rightrightarrows \bullet)$ path algebra of kronecker quiver $\bullet \rightrightarrows \bullet$

$\boxed{\text{tors } R:}$

Hasse diagram



Proposition [Zhang-Wei '16, Angeleri Hügel '17]:

$\forall\, \tau \in$ tors-R $\exists$ complex $C_\tau \in D(\mathrm{Mod}R)$ such that

1. $H^i(C_\tau) = 0 \quad \forall i \neq 0,1$
2. $H^0(C_\tau)$ determines τ:

$$\tau = \left({}^{\perp_0}H^0(C_\tau),\ \mathrm{cogen}\, H^0(C_\tau)\right)$$

The complex C_τ is a cosilting complex

Mutation: $C_u \rightsquigarrow C_\tau$

Remarks: $\tau \notin \mathcal{R} \iff H^0(C_\tau)$ finite-dimensional (supp. τ^{-1}-tilting modules)

$\tau \in \mathcal{R} \rightsquigarrow \mathcal{L}_\tau \subseteq \mathbb{P}^1(K)$

$$H^0(C_\tau) \cong \underbrace{\prod_{x \in \mathcal{L}_\tau} S(\infty, x)}_{\text{"x-Prüfer"}} \oplus \underbrace{\prod_{x \notin \mathcal{L}_\tau} S(-\infty, x)}_{\text{"x-adic"}}$$

Definition [Asai-Pfeifer '21]: An interval $[(u,v),(t,f)]$ is called wide if $v \cap t$ is a wide subcategory of mod$R$.

ie closed under kernels, cokernels and extensions

Today: $[u, \tau]$ wide $\iff$ C_τ is a mutation of C_u.

II Wide intervals and HRS-tilts

Definition: Let $\mathcal{C}$ be an abelian or triangulated category. A pair (t, f) of full idempotent complete subcategories of $\mathcal{C}$ is a torsion pair if

① $\mathrm{Hom}_{\mathcal{C}}(t, f) = 0$

② $\mathcal{C} = t * f$

$\mathcal{M}, \mathcal{N} \subseteq \mathcal{C}$

abelian: $\mathcal{M} * \mathcal{N} = \{X \in \mathcal{C} \mid \exists\, 0 \to \underset{\in \mathcal{M}}{M} \to X \to \underset{\in \mathcal{N}}{N} \to 0\}$

triangulated: $\mathcal{M} * \mathcal{N} = \{X \in \mathcal{C} \mid \exists\, \underset{\in \mathcal{M}}{M} \to X \to \underset{\in \mathcal{N}}{N} \to M[1]\}$

Examples: ① $\mathcal{C} = Ab$ = category of abelian groups

t = torsion groups f = torsion-free groups

② $\mathcal{C} = D(\mathrm{Mod}R)$ or $D^b(\mathrm{mod}R)$

$t = \mathcal{D}^{\leq 0} = \{X^\bullet \in \mathcal{C} \mid H^i(X^\bullet) = 0\ \forall i > 0\}$

$f = \mathcal{D}^{> 0} = \{X^\bullet \in \mathcal{C} \mid H^i(X^\bullet) = 0\ \forall i \leq 0\}$

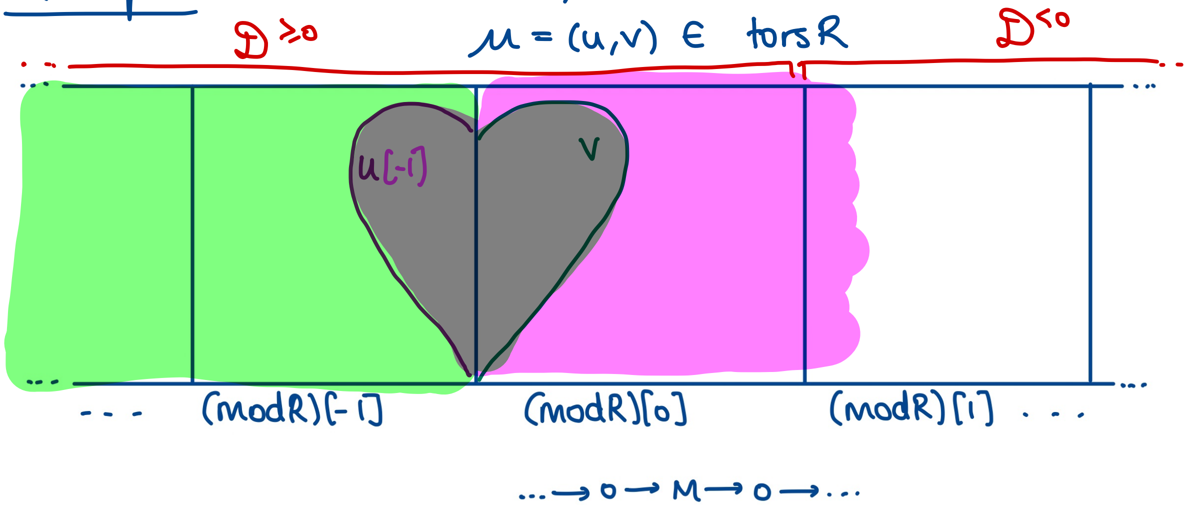
Definition: $\mathcal{C}$ triangulated, $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ a torsion pair. Then $\mathbb{T}$ is a t-structure if $\mathcal{X}[1] \subseteq \mathcal{X}$.

$\Rightarrow \mathcal{H}_{\mathbb{T}} := \mathcal{X}[-1] \cap \mathcal{Y}$ abelian (= the heart) e.g. $\mathbb{T}=(\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0}) \rightsquigarrow \mathcal{H}_{\mathbb{T}} = \mathrm{Mod}R$ or $\mathrm{mod}R$.

<u>Definition/Theorem</u> [Happel-Reiten-Smalø '13]:
$\mathcal{D}$ triangulated (e.g. $D^b(\mathrm{mod}R)$), a t-structure $\mathbb{T}=(\mathcal{X},\mathcal{Y})$ (e.g. $(\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0})$). Let $\tau=(t,f)$ be a torsion pair in $\mathcal{H}_{\mathbb{T}}$ (eg $\tau \in \mathrm{tors}R$).
There exists a t-structure $\mathbb{T}_\tau = (\mathcal{X}_\tau, \mathcal{Y}_\tau)$ where $\mathcal{X}_\tau := t * \mathcal{X}$, $\mathcal{Y}_\tau = \mathcal{Y}[-1] * f$
called the HRS-tilt of $\mathbb{T}$ w.r.t τ.

$\mathcal{H}_\tau$ heart

<u>Example</u>: $\mathcal{D} = D^b(\mathrm{mod}R)$, $R = K(\bullet \rightrightarrows \bullet)$, $\mathbb{T} = (\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0})$



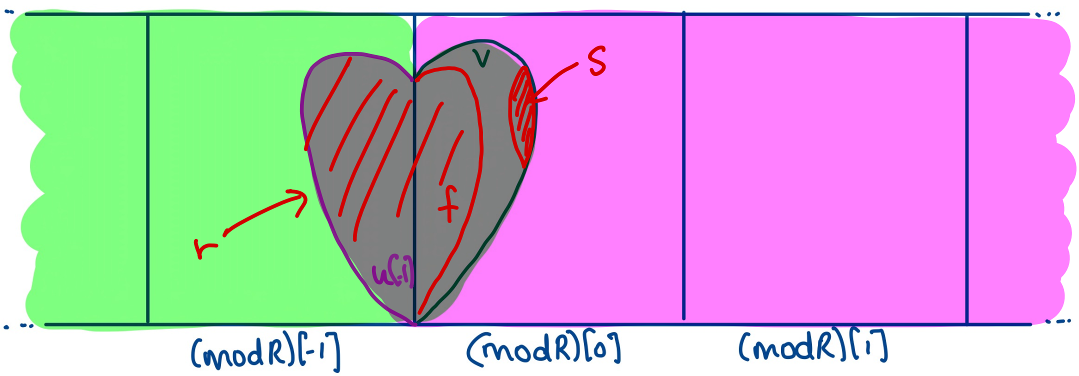
<u>Question</u>: Suppose we have $\mu \leq \tau$ in $\operatorname{tors} R$.
How are $\mathbb{T}_\mu$ related to $\mathbb{T}_\tau$?
What about $[\mu, \tau]$ wide?

<u>Proposition</u> [ALSV]: $\mathbb{T}$ t-structure, $\mu \leq \tau$ torsion pairs in $\mathcal{H}_{\mathbb{T}}$. Then

① $\exists$ torsion pair $\sigma = (s, r)$ in $\mathcal{H}_\mu$ where $s = t \cap v$ and $r = f * (u[-1])$

② $\mathbb{T}_\tau = (\mathbb{T}_\mu)_\sigma$

<u>Picture</u>:



③ $[\mu, \tau]$ is a wide interval $\Longleftrightarrow$ $s = t \cap v$ is a <u>Serre subcategory</u> of $\mathcal{H}_\mu$.

$\forall\ 0 \to X \to Y \to Z \to 0$ in $\mathcal{H}_\mu$
$Y \in s \Longleftrightarrow X, Z \in s$.

Definition: A torsion pair $(\mathcal{T},\mathcal{F})$ in $\operatorname{Mod} R$ is called **cosilting** if $\mathcal{F} = \varinjlim \mathcal{F}$.

$\operatorname{Cosilt} R := \{ (\mathcal{T},\mathcal{F}) \mid \text{cosilting torsion pair} \}$ ordered by $\preccurlyeq$.

Theorem [Crawley-Boevey '94]: $\exists$ order-preserving bijection

$$\operatorname{tors} R \xrightarrow{\sim} \operatorname{Cosilt}\text{-}R$$

$$\tau = (t,f) \longmapsto \vec{\tau} = (\varinjlim t, \varinjlim f)$$

Notation: Let $\tau \in \operatorname{tors} R$. Denote the HRS-tilt $\mathbb{T}_{\vec{\tau}}$ in $D(\operatorname{Mod} R)$ by $\vec{\mathbb{T}}_\tau$ with heart $\vec{\mathcal{H}}_\tau$

Theorem [Saorín '17]: Let $\tau \in \operatorname{tors} R$. Have

$\mathbb{T}_\tau$ in $D^b(\operatorname{mod} R)$ with heart $\mathcal{H}_\tau$ and

$\vec{\mathbb{T}}_\tau$ in $D(\operatorname{Mod} R)$ with heart $\vec{\mathcal{H}}_\tau$

($D^b(\operatorname{mod} R) \subseteq D(\operatorname{Mod} R)$, $\mathcal{H}_\tau \subseteq \vec{\mathcal{H}}_\tau$)

Then $\vec{\mathcal{H}}_\tau$ is locally coherent Grothendieck category with $\operatorname{fp} \vec{\mathcal{H}}_\tau = \mathcal{H}_\tau$. $\varinjlim \mathcal{H}_\tau = \vec{\mathcal{H}}_\tau$

Corollary [ALSV, Herzog, Krause]: Suppose $\mu = (u,v) \preccurlyeq \tau = (t,f)$ in $\operatorname{tors} R$. TFAE:

1. $[\mu,\tau]$ is a wide interval

② $S = T \cap V$ is a Serre subcategory of $\mathcal{H}_\mu$

③ $\vec{\sigma} = (\varinjlim S, \varinjlim R)$ is a hereditary torsion pair in $\vec{\mathcal{H}}_\mu$

$\sigma = (S, R) = (T \cap V, U[-1] * F)$

III Mutation of cosilting

Definition: An object C in $D(\mathrm{Mod}R)$ is called cosilting if $\Pi_C = ({}^{\perp_0}C[\leq 0], {}^{\perp_0}C[>0])$ is a t-structure.

${}^{\perp_0}C[\leq 0] = \{X \in D(\mathrm{Mod}R) \mid \mathrm{Hom}(X, C[i]) = 0 \ i \leq 0\}$

Remark: We will assume all cosilting objects are pure-injective.

Example: Let $\tau \in \mathrm{tors}\, R$. Then $\vec{\Pi}_\tau = \Pi_C$ for some cosilting complex $C \in D(\mathrm{Mod}R)$ with $H^i(C) = 0 \ \forall i \neq 0, 1$.
+ converse

Theorem/definition [ALSV]: Suppose C is a cosilting complex in $D(\mathrm{Mod}R)$, let $\mathcal{E} \subseteq \mathrm{Prod}\, C = \{\oplus$ and of products $C\}$ s.t. $\mathcal{E} = \mathrm{Prod}\, \mathcal{E}$. If there exists an $\mathcal{E}$-cover $E_0 \xrightarrow{\Phi} C$, then consider $E_1 \longrightarrow E_0 \xrightarrow{\Phi} C \longrightarrow E_1[1]$. Then $E_1 \oplus E_0$ is a cosilting complex called the right mutation of C w.r.t. $\mathcal{E}$.

Theorem [ALSV]: Let Π_C, $\Pi_{C'}$ be cosilting t-structures. Then TFAE:

1. C' is a right mutation of C
2. $\Pi_{C'}$ is the HRS-tilt of Π_C w.r.t. a hereditary torsion pair.

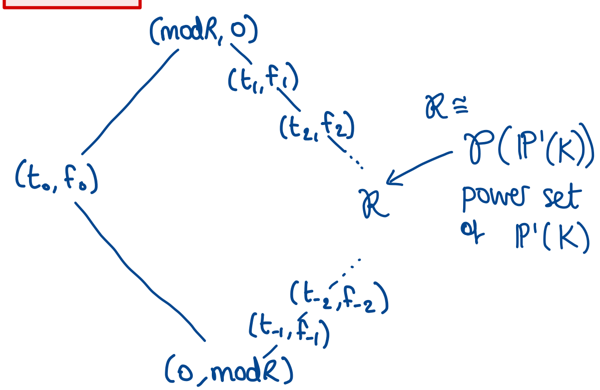
Corollary: Let $\tau = (t, f) \geq \mu = (u, v)$ in tors-R. Then TFAE:

1. $\tau \geq \mu$ is a wide interval
2. $\vec{\Pi}_\tau = \Pi_{C_\tau}$, $\vec{\Pi}_\mu = \Pi_{C_\mu}$ and C_τ is a right mutation of C_μ.

Example: K algebraically closed field.

$R = K(\bullet \rightrightarrows \bullet)$ path algebra of kronecker quiver $\bullet \rightrightarrows \bullet$

$\boxed{\text{tors } R:}$



τ, μ in $\mathcal{R}$

$$C_\tau = \prod_{x \in \mathcal{L}_\tau} S(x, \infty) \oplus \prod_{x \notin \mathcal{L}_\tau} S(x, -\infty)$$

$$C_\mu = \prod_{x \in \mathcal{L}_\mu} S(x, \infty) \oplus \prod_{x \notin \mathcal{L}_\mu} S(x, -\infty)$$

$$\tau \leq \mu \iff \mathcal{L}_\tau \supseteq \mathcal{L}_\mu$$

"mutation = swapping Prüfers for adics"