Graph Monoids and Preprojective Roots of Coxeter Groups

$\Gamma = (\Gamma_0, \Gamma_1)$ finite undirected graph

Γ_0 set of vertices

Γ_1 —"— edges

no loops or multiple edges

$\widehat{\ }: \Gamma_0 \longrightarrow \widehat{\Gamma}_0$ bijection

$v \longmapsto \widehat{v}$

$\widehat{\Gamma}_0^*$ free monoid on $\widehat{\Gamma}_0$

the elements = all words $\widehat{X} = \widehat{x}_\ell \ldots \widehat{x}_1$

$x_i \in \Gamma_0$, $\ell \geq 0$

$\widehat{1}$ empty word

operation on $\widehat{\Gamma}_0^*$ = concatenation

$\widehat{1}$ = identity of $\widehat{\Gamma}_0^*$

Cartier & Foata 1969

$\hat{X} R \hat{Y}$ if and only if
$\hat{X} = \hat{U}\hat{x}\hat{y}\hat{V}$ and $\hat{Y} = \hat{U}\hat{y}\hat{x}\hat{V}$, for some
$x, y \in \Gamma_0$, where no edge of Γ joins x and y
and $\hat{U}, \hat{V} \in \hat{\Gamma}_0^*$.

$\approx$ the reflexive and transitive closure of the symmetric binary relation R

Fact. $\approx$ is a congruence:
$\hat{X} \approx \hat{X}' \;\&\; \hat{Y} \approx \hat{Y}' \Rightarrow \hat{X}\hat{Y} \approx \hat{X}'\hat{Y}'$

Def. Graph monoid of Γ is

$$\mathcal{M} = \mathcal{M}_\Gamma = \hat{\Gamma}_0^* / \approx$$

Notation. X = the congruence class of $\hat{X}$,
x congruence class of $\hat{x}$, $x \in \Gamma_0$.

We have

(a) If $x, y \in \Gamma_0$ then: $\hat{x} \approx \hat{y}$ iff $\hat{x} = \hat{y}$, iff $x = y$.

(b) If $\hat{X} = \hat{x}_\ell \dots \hat{x}_1$ then $X = x_\ell \dots x_1$, $x_i \in \Gamma_0$.

(c) If $X = x_\ell \dots x_1$ and $Y = y_m \dots y_1$, $x_i, y_j \in \Gamma_0$, then $X = Y$ iff Y can be obtained from X by a finite number of interchanges of adjacent letters that are vertices not joined by an edge of Γ.

If $X = Y$, then $\ell = m$ and the multiplicity of each $v \in \Gamma_0$ in X and in Y is the same.

The canonical homomorphism $\pi : \hat{\Gamma}_0^* \longrightarrow \hat{\Gamma}_0^*/\approx \; = \mathcal{M}$ strips away the hats.

$$\begin{tikzcd}
v_1 \arrow[r, dash] \arrow[d, dash] & v_2 \arrow[d, dash] \\
v_3 \arrow[r, dash] & v_4
\end{tikzcd}$$

$$v_1 v_2 v_3 v_4 = v_1 v_3 v_2 v_4 \neq$$

$$v_1 v_4 v_3 v_2 = v_4 v_1 v_3 v_2 = v_4 v_1 v_2 v_3 =$$

$$v_1 v_4 v_2 v_3$$

Invariants

$X = x_\ell \ldots x_1$ in $\mathcal{M}$

(a) length $\ell(X) = \ell$

(b) support $\operatorname{Supp} X$ = the set of distinct vertices among the x_j, $1 \le j \le \ell$

(c) multiplicity of $v \in \Gamma_0$ in X, $m_X(v)$ = # of times v appears among the x_j

X is multiplicity-free if $m_X(v) \le 1$, $v \in \Gamma_0$.

Fact. Multiplicity function

$$m : \mathcal{M} \longrightarrow \mathbb{Z}_{\geq 0}^{|\Gamma_0|}$$
$$X \longmapsto (m_X(v))_{v \in \Gamma_0}$$

is a surjective homomorphism of monoids. Here we view $\mathbb{Z}_{\geq 0}^{|\Gamma_0|}$ as an additive monoid.

Partial order

$X, Y \in \mathcal{M}$ $\quad X \preceq Y$ if $Y = ZX$, $Z \in \mathcal{M}$

$(\mathcal{M}, \preceq)$ satisfies the descending chain condition

A poset P is a lattice if, for all $a, b \in P$, there exist
the least upper bound (join) $a \vee b$
and the greatest lower bound (meet) $a \wedge b$.
A lattice is distributive if

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

Fact. $(\mathcal{M}, \preceq)$ is not a lattice.

$$
\begin{tikzcd}
v_1 \arrow[r, dash] \arrow[d, dash] & v_2 \arrow[d, dash] \\
v_3 \arrow[r, dash] & v_4
\end{tikzcd}
$$

$v_3, v_4 \in \mathcal{M}$

There is no common upper bound for v_3 and v_4.

Suppose $v_3 \leq X$ and $v_4 \leq X$. Then

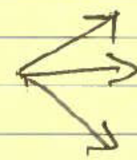
$X = \dots v_3 = \dots v_4$.

$$\dots v_3 \dots v_4 = \dots v_3 v_4 \dots \neq \dots v_4 v_3 \dots$$

Bernstein, Gelfand, Ponomarev 1973

Λ an acyclic orientation of Γ

(Γ, Λ) an acyclic quiver

 sink  source

If x is a sink in (Γ, Λ), denote by $x \cdot \Lambda$ the orientation of Γ in which the direction of each arrow of (Γ, Λ) not incident to x is preserved, but the direction of each arrow incident to x is reversed

Fact. $x \cdot \Lambda$ is acyclic.

Def. $X \in \mathcal{M}$ is Λ-admissible if

$X = x_e \dots x_1$ where x_1 is a sink in (Γ, Λ), x_2 is a sink in $(\Gamma, x_1 \cdot \Lambda)$, x_3 is a sink in $(\Gamma, x_2 \cdot (x_1 \cdot \Lambda))$ etc.

<u>Fact</u>. Let $X = x_\ell \cdots x_1 = y_m \cdots y_1$ in $\mathcal{M}$.

If x_1 is a sink in (Γ, Λ),

x_2 —— " —— $(\Gamma, x_1 \cdot \Lambda)$,

x_3 —— " —— $(\Gamma, x_2 \cdot (x_1 \cdot \Lambda))$,

etc., then $m = \ell$ and

y_1 is a sink in (Γ, Λ),

y_2 " —— " —— $(\Gamma, y_1 \cdot \Lambda)$,

y_3 —— " —— $(\Gamma, y_2 \cdot (y_1 \cdot \Lambda))$, etc.

Denote by $\mathcal{M}(\Lambda)$ the set of all Λ-admissible elements of $\mathcal{M}$.

<u>Note</u>: $\mathcal{M}(\Lambda)$ is not a submonoid of $\mathcal{M}$.

$$\begin{tikzcd} v_1 & v_2 \arrow[l] \\ v_3 \arrow[u] & v_4 \arrow[l] \arrow[u] \end{tikzcd} \qquad \Lambda$$

$$\begin{tikzcd} v_1 \arrow[r] \arrow[d] & v_2 \\ v_3 & v_4 \arrow[l] \arrow[u] \end{tikzcd} \qquad v_1 \cdot \Lambda$$

$$\begin{tikzcd} v_1 \arrow[r] & v_2 \\ v_3 \arrow[u] \arrow[r] & v_4 \arrow[u] \end{tikzcd} \qquad v_3 \cdot (v_1 \cdot \Lambda)$$

$$\begin{tikzcd} v_1 & v_2 \arrow[l] \arrow[d] \\ v_3 \arrow[u] \arrow[r] & v_4 \end{tikzcd} \qquad v_2 \cdot (v_3 \cdot (v_1 \cdot \Lambda))$$

$$\begin{tikzcd} v_1 & v_2 \arrow[l] \\ v_3 \arrow[u] & v_4 \arrow[l] \arrow[u] \end{tikzcd} \qquad v_4 \cdot (v_2 \cdot (v_3 \cdot (v_1 \cdot \Lambda))) = \Lambda$$

v_1, v_3v_1, v_2v_1, $v_2v_3v_1 = v_3v_2v_1$, $v_4v_3v_2v_1$

are Λ-admissible

$v_4v_1v_2v_3$ is not Λ-admissible

$v_1v_2v_3v_4$ —— " —— " ——

<u>Note</u>. $m(v_4v_1v_2v_3) = m(v_1v_2v_3v_4) = (1,1,1,1)$ but $v_4v_1v_2v_3 \neq v_1v_2v_3v_4$. Hence the surjective homomorphism of monoids $m: \mathcal{M} \longrightarrow \mathbb{Z}^4_{\geqslant 0}$ is not injective.

<u>Prop.</u> Let $X, Y \in \mathcal{M}(\Lambda)$. Then $X \leq Y$ iff $m(X) \leq m(Y)$, where the partial order on $\mathbb{Z}^{|\Gamma_0|}_{\geqslant 0}$ is componentwise.

Hence $X = Y$ iff $m(X) = m(Y)$, i.e., $m|\mathcal{M}(\Lambda) : \mathcal{M}(\Lambda) \longrightarrow \mathbb{Z}^{|\Gamma_0|}_{\geqslant 0}$ is injective, but need not be surjective.

$(+)$-admissible sequences BGP

A sequence $x_1, \dots, x_\ell$ of vertices of (Γ, Λ) is $(+)$-admissible if $X = x_\ell \dots x_1$ belongs to $\mathcal{M}(\Lambda)$.

<u>Fact</u>. Suppose $X = x_\ell \dots x_1$ and $Y = y_m \dots y_1$ belong to $\mathcal{M}(\Lambda)$. If $X = Y$, then

$$F^+_{x_\ell} \dots F^+_{x_1} = F^+_{y_m} \dots F^+_{y_1},$$ [redacted]here F^+_v, where v is a sink, is a $(+)$-reflection functor.

<u>Def</u>. A representation V of (Γ, Λ) is preprojective if there exists an $X = x_\ell \dots x_1$ in $\mathcal{M}(\Lambda)$ for which $F^+_{x_\ell} \dots F^+_{x_1} V = 0$. (BGP called such a V $(+)$-irregular).

<u>Question</u>. Let V be an indecomposable preprojective representation of (Γ, Λ). What are all elements of $\mathcal{M}(\Lambda)$ for which the associated composite of $(+)$-reflection functors annihilates V? For Γ connected.

Fact. [KP] (a) Let V be an indecomposable preprojective representation of (Γ, Λ). Then the set of elements of $\mathcal{M}(\Lambda)$ for which the associated composite of $(+)$-reflection functors annihilates V has a unique minimal element X_V. Moreover, the element X_V is principal (to be defined) and the associated word in the Weyl group of Γ is reduced.

(b) Let W be an indecomposable preprojective representation of (Γ, Λ). Then $V \cong W$ if and only if $X_V = X_W$.

(c) Let X be a principal element of $\mathcal{M}(\Lambda)$. Then $X = X_V$ for some indecomposable preprojective representation V of (Γ, Λ) if and only if the associated word in the Weyl group of Γ is reduced.

Reachability order on the set of vertices of an acyclic quiver

If $x, y \in \Gamma_0$, then $x \leq y$ iff there is a path $x \to \dots \to y$. Denote the obtained poset by (Γ_0, Λ).

<u>Def.</u> A subset S of a poset $P = (P, \leq)$ is a filter if $x \in S$ and $x \leq y$ imply $y \in S$. A principal filter of P generated by $x \in P$ is

$$\langle x \rangle = \{ y \in P \mid x \leq y \}.$$

If S is an arbitrary subset of P, then the filter of P generated by S is

$$\langle S \rangle = \bigcup_{x \in S} \langle x \rangle.$$

Prop. (a) If $X \in \mathcal{M}(\Lambda)$ then $\operatorname{Supp} X$ is a filter of the poset (Γ_0, Λ).

(b) For each filter Θ of the poset (Γ_0, Λ), there exists a unique multiplicity-free Λ-admissible element $X \in \mathcal{M}$ satisfying $\Theta = \operatorname{Supp} X$.

$$\begin{tikzcd} v_1 & v_2 \arrow[l] \\ v_3 \arrow[u] & v_4 \arrow[l] \arrow[u] \end{tikzcd}$$

(Γ, Λ)

Filter of (Γ_0, Λ)	Multiplicity-free element of $\mathcal{M}(\Lambda)$
ϕ	1
$\langle v_1 \rangle$	v_1
$\langle v_2 \rangle$	$v_2 v_1$
$\langle v_3 \rangle$	$v_3 v_1$
$\langle v_2 \rangle \cup \langle v_3 \rangle$	$v_3 v_2 v_1$
$\langle v_4 \rangle$	$v_4 v_3 v_2 v_1$

Def. Two vertices of a graph are adjacent if they are joined by an edge. If Θ is a subset of Γ_0, a vertex is adjacent to Θ if the vertex is not in Θ but is adjacent to a vertex in Θ. Denote by $\mathrm{adj}(\Theta)$ the set of vertices adjacent to Θ.

Prop. (a) If $1 \neq X \in \mathcal{M}(\Lambda)$, there exists a unique decomposition $X = X_r \cdots X_1$ where $1 \neq X_i \in \mathcal{M}(\Lambda)$ is multiplicity-free for all i and the filters $F_i = \mathrm{Supp}\, X_i$ of (Γ_0, Λ) satisfy $F_{i+1} \cup \mathrm{adj}(F_{i+1}) \subset F_i$, $1 \leq i < r$.

(b) If $F_1, \ldots, F_r$, $r > 0$, are nonempty filters of (Γ_0, Λ) satisfying $F_{i+1} \cup \mathrm{adj}(F_{i+1}) \subset F_i$, $1 \leq i < r$, denote by $X_i \in \mathcal{M}(\Lambda)$ the unique multiplicity-free element for which $F_i = \mathrm{Supp}\, X_i$, $i = 1, \ldots, r$. Then $1 \neq X_r \cdots X_1 \in \mathcal{M}(\Lambda)$.

$$\begin{tikzcd} v_1 & v_2 \arrow[l] \\ v_3 \arrow[u] & v_4 \arrow[l] \arrow[u] \end{tikzcd} \qquad (\Gamma, \Lambda)$$

$$\langle v_1 \rangle \cup \operatorname{adj}(\langle v_1 \rangle) = \langle v_1 \rangle \cup \{v_2, v_3\} \subset \langle v_4 \rangle$$

$$\underbrace{v_1}_{X_2} \underbrace{v_4 v_3 v_2 v_1}_{X_1} \in \mathcal{M}(\Lambda) \qquad r = 2$$

Thm. $\mathcal{M}(\Lambda)$ is a distributive lattice.

Def. Let $F_1 \supset \dots \supset F_r$, $r > 0$, be a sequence of filters of (Γ_0, Λ) where F_r is a principal filter and $\langle F_{i+1} \cup \mathrm{adj}(F_{i+1}) \rangle = F_i$, $1 \leq i < r$. Let X_i be the unique multiplicity-free element of $\mathcal{M}(\Lambda)$ satisfying $F_i = \mathrm{Supp}\, X_i$. Then $X = X_r \dots X_1$ is called a principal element of $\mathcal{M}(\Lambda)$ of size r. Denote by $\mathcal{P}(\Lambda)$ the set of principal elements of $\mathcal{M}(\Lambda)$.

$$\begin{tikzcd} v_1 & v_2 \arrow[l] \\ v_3 \arrow[u] & v_4 \arrow[l] \arrow[u] \end{tikzcd} \qquad (\Gamma, \Lambda)$$

$$\underset{\substack{\shortparallel \\ F_4}}{\langle v_1 \rangle} \subset \underset{\substack{\shortparallel \\ F_3}}{\langle v_2 \rangle \cup \langle v_3 \rangle} \subset \underset{\substack{\shortparallel \\ F_2}}{\langle v_4 \rangle} \subset \underset{\substack{\shortparallel \\ F_1}}{\langle v_4 \rangle}$$

$$\underbrace{v_1}_{X_4} \underbrace{v_3 v_2 v_1}_{X_3} \underbrace{v_4 v_3 v_2 v_1}_{X_2} \underbrace{v_4 v_3 v_2 v_1}_{X_1}$$

a principal element of size 4

$$\underset{\substack{\shortparallel \\ F_2}}{\langle v_3 \rangle} \subset \underset{\substack{\shortparallel \\ F_1}}{\langle v_4 \rangle}$$

$$\underbrace{v_3 v_1}_{\substack{\shortparallel \\ X_2}} \underbrace{v_4 v_3 v_2 v_1}_{\substack{\shortparallel \\ X_1}}$$

a principal element of size 2

Def. A finite subset $\{S_1, \dots, S_m\}$ of $\mathcal{B}(\Lambda)$ is independent if whenever $S_1 \vee \dots \vee S_m = T_1 \vee \dots \vee T_q$ with $T_j \in \mathcal{B}(\Lambda)$, $1 \leq j \leq q$, then $m \leq q$.

Prop. (a) A subset $\{S_1, \dots, S_m\}$ of $\mathcal{B}(\Lambda)$ is independent if and only if it is an antichain of the poset $(\mathcal{M}, \preceq)$.
(b) For each $1 \neq X \in \mathcal{M}(\Lambda)$, there exists a unique independent subset $\{S_1, \dots, S_m\}$ of $\mathcal{B}(\Lambda)$ satisfying $X = S_1 \vee \dots \vee S_m$.