AN INTRODUCTION TO REPRESENTATION THEORY.

1. Lecture 1. Basic facts and algebras and their representations.

1.1. **What is representations theory?** Representation theory studies abstract algebraic structures by representing their elements as structures in linear algebras, such as vectors spaces and linear transformations between them.

$$\left\{ \begin{array}{c} \text{abstract algebraic} \\ \text{structures} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{concrect objects in linear algebra} \\ \text{which ''respect'' abstract structure} \end{array} \right\}$$

Abstract algebraic structures can be very different. The following structures we will study on our seminars:

- groups;
- associative algebras;
- Lie algebras;
- quivers;
- posets.

On the other hand objects in linear algebra usually are:

- vector (unitary) spaces;
- transformations between them.

Why is it interesting?

There are basically several reasons. A representation makes an abstract algebraic object more concrete by describing its elements by matrices and the algebraic operations in terms of matrix addition and matrix multiplication. Hence representation theory provides a powerful tool to reduce problems in abstract algebra to problems in linear algebra (a subject of which is well understood). If a vector space is infinite-dimensional (Hilbert space) then representation theory injects methods of functional analysis into the (for example) group theory (if one studies representations of groups). So this theory provides a bridge between different areas of mathematics.

What are typical questions?

The typical question is:

to classify all representations of a given abstract algebraic structure.

For this one defines what are *simple* representation and what are *isomorphic* representations. In some cases it is possible to show the any representation is (in some sence) a sum of simple ones. Hence the main question reduces to the following

to classify all *simple* (up to *isomorphism*) representations.

What are typical methods?

Roughly speaking, studying the representations of "any" algebraic structure can be reduced to studying the representations of certain associative algebra. For example

- repr. of groups $\Longleftrightarrow$ repr. of group algebras;
- repr. of Lie algebras $\Longleftrightarrow$ repr. of universal enveloping algebras;
- repr. of quivers $\Longleftrightarrow$ repr. of path algebras;
- repr. of posets $\Longleftrightarrow$ repr. of incidence algebras.
- so on...

So, roughly speaking, representation theory studies representations of associative algebras.

Studying the representations of a given algebra is more or less the same as studying modules over this algebra. So the theory of modules plays an important role in representation theory. And in this sense the results in theory of modules are the results in representation theory.

Today I will recall basic facts about associative algebras and will introduce basic concepts about their representations.

1.2. **Basic facts about associative algebras.** Let k be a field. We will always assume that k is algebraically closed. Our basic field will be the field of complex numbers $\mathbb{C}$, but we will also consider fields of characteristic p — the algebraic closure F_p of the finite field F_p.

Definition 1. An associative algebra over k is a vector space A over k together with a bilinear map $A \times A \to A$, $(a, b) \to ab$, such that $(ab)c = a(bc)$.

Definition 2. A unit in an associative algebra A is an element $1 \in A$ such that $1a = a1 = a$.

Proposition 1. If a unit exists, it is unique.

Proof. Let $1, 1'$ be two units. Then $1 = 11' = 1'$. □

Example 1. Some examples of algebras over k:

(1) $A = k$;
(2) $A = k[x_1, \dots, x_n]$ — the algebra of polynomials in variables $x_1, \dots, x_n$;
(3) $A = \text{End}V$ – the algebra of endomorphisms of a vector space V over k (i.e., linear maps from V to itself). The multiplication is given by composition of operators;

(4) The free algebra $A = k\langle x_1, \dots, x_n\rangle$. A basis of this algebra consists of words in letters $x_1, \dots, x_n$, and multiplication in this basis is simply concatenation of words;
(5) The group algebra $A = k[G]$ of a group G. Its basis is $\{a_g, g \in G\}$, with multiplication law $a_g a_h = a_{gh}$.

An algebra A is *commutative* if $ab = ba$ for all $a, b \in A$.

Question 1. Which algebras in preceding examples are comutative?

Definition 3. A homomorphism of algebras $f : A \to B$ is a linear map such that $f(xy) = f(x)f(y)$ for all $x, y \in A$, and $f(1) = 1$.

1.2.1. *Ideal and Quotients.* A *left ideal* of an algebra A is a subspace $I \subset A$ such that $aI \subset I$ for all $a \in A$. Similarly, a *right ideal* of an algebra A is a subspace $I \subset A$ such that $Ia \subset I$ for all $a \in A$. A two-sided ideal is a subspace that is both a left and a right ideal.

Example 2. Some examples of ideals

(1) If A is any algebra, 0 and A are two-sided ideals. An algebra A is called *simple* if 0 and A are its only two-sided ideals;
(2) If $\varphi : A \to B$ is a homomorphism of algebras, then ker is a two-sided ideal of A.
(3) If S is any subset of an algebra A, then the two-sided ideal *generated* by S is denoted $\langle S\rangle$ and is the span of elements of the form asb, where $a, b \in A$ and $s \in S$. Similarly we can define $\langle S\rangle_l = \text{span}\{as\}$ and $\langle S\rangle_r = \text{span}\{sb\}$ the left, respectively right, ideal generated by S.

Let A be an algebra and I a two-sided ideal in A. Then A/I is the set of (additive) cosets of I. Let $\pi : A \to A/I$ be the quotient map. We can define multiplication in A/I by $\pi(a)\pi(b) := \pi(ab)$. This is well defined. Indeed, if $\pi(a) = \pi(a')$ then

$$\pi(a'b) = \pi(ab + (a'a)b) = \pi(ab) + \pi((a'a)b) = \pi(ab),$$

because $(a'a)b \in Ib \subset I = \ker\pi$, as I is a right ideal; similarly, if $\pi(b) = \pi(b)$ then

$$\pi(ab') = \pi(ab + a(b'b)) = \pi(ab) + \pi(a(b'b)) = \pi(ab),$$

because $a(b'b) \in aI \subset I = \ker\pi$, as I is also a left ideal. Thus, A/I is an algebra.

1.3. **Representations.**

Definition 4. A representation of an algebra A is a vector space V together with a homomorphism of algebras $\rho : A \to \text{End}V$.

Example 3. Here is some examples of representatins:

(1) $V = 0$.

(2) $V = A$, and $\rho : A \to \mathrm{End}A$ is defined as follows: $\rho(a)$ is the operator of left multiplication by a, so that $\rho(a)b = ab$ (the usual product). This representation is called the regular representation of A.
(3) $A = k$. Then a representation of A is simply a vector space over k.
(4) $A = k\langle x_1, \ldots, x_n\rangle$. Then a representation of A is just a vector space V over k with a collection of arbitrary linear operators $\rho(x_1), \ldots, \rho(x_n) : V \to V$.

Definition 5. A subrepresentation of a representation V of an algebra A is a subspace WV which is invariant under all the operators $\rho(a) : V \to V$, $a \in A$; i.e. $\rho(a)(w) \in W$ for all $w \in W$ and $a \in A$.

Example 4. 0 and V are always subrepresentations.

Definition 6. A representation $V \neq 0$ of A is irreducible (or simple) if the only subrepresentations of V are 0 and V.

Definition 7. Let V_1, V_2 be two representations of an algebra A. A homomorphism (or intertwining operator) $\varphi : V_1 \to V_2$ is a linear operator which commutes with the action of A, i.e., $\varphi(av) = a\varphi(v)$ for any $v \in V_1$. A homomorphism is said to be an isomorphism of representations if it is an isomorphism of vector spaces. The set (space) of all homomorphisms of representations $V_1 \to V_2$ is denoted by $\mathrm{Hom}_A(V_1, V_2)$.

We will now prove our first result Schurs lemma. Although it is very easy to prove, it is fundamental in the whole subject of representation theory.

Proposition 2. (Schurs lemma) Let V_1, V_2 be representations of an algebra A over any field k (which need not be algebraically closed). Let $\varphi : V_1 V_2$ be a nonzero homomorphism of representations. Then:

(i) If V_1 is irreducible then φ is injective;
(ii) If V_2 is irreducible then φ is surjective.

Thus, if V_1 and V_2 are irreducible then φ is an isomorphism.

Proof. (i) The kernel K of φ is a subrepresentation of V_1. Since $\varphi \neq 0$, this subrepresentation cannot be V_1. So by irreducibility of V_1 we have $K = 0$. (ii) The image I of φ is a subrepresentation of V_2. Since $\varphi \neq 0$, this subrepresentation cannot be 0. So by irreducibility of V_2 we have $I = V_2$. □

Corollary 3. *(Schurs lemma for algebraically closed fields) Let V be a finite dimensional irreducible representation of an algebra A over an algebraically closed field k, and $\varphi : V \to V$ is an intertwining operator. Then $\varphi = \lambda I$ for some $\lambda \in k$ (a scalar operator).*

Corollary 4. *Let A be a commutative algebra. Then every irreducible finite dimensional representation V of A is 1-dimensional.*

Example 5. Here is some basic examples

- $A = k$. Since representations of A are simply vector spaces, $V = A$ is the only irreducible representation.

- $A = k[x]$. Since this algebra is commutative, the irreducible representations of A are its 1-dimensional representations. As we discussed above, they are defined by a single operator $\rho(x)$. In the 1-dimensional case, this is just a number from k. So all the irreducible representations of A are $V_\lambda = k, \lambda \in k$, in which the action of A defined by $\rho(x) = \lambda$. Clearly, these representations are pairwise non-isomorphic.
- The group algebra $A = k[G]$, where G is a group. A representation of A is the same thing as a representation of G, i.e., a vector space V together with a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Home work.