A Proofs for Section 3 (Cn* kernel pseudo-contraction)

Theorem 5 (Cn* kernel pseudo-contraction: representation theorem)**.** *If Cn* satisfies monotonicity, then an operation is a Cn* kernel pseudo-contraction if and only if it satisfies success, inclusion*, core-retainment* and uniformity*.*

Proof. Construction-to-postulates:

- Success: We prove by contradiction. Assume that $\alpha \in \mathrm{Cn}(\mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha))$. Since Cn is compact, there is some non-empty $X \in \mathrm{Ker}[\mathrm{Cn}^*(B) \setminus \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]),\alpha]$. Such X is also an inclusion-minimal subset of $\mathrm{Cn}^*(B)$ implying α, i.e., X must be in $\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]$. Moreover, as $X \subseteq \mathrm{Cn}^*(B) \setminus \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$, we have $\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]) \cap X = \emptyset$, which violates the definition of incision function.
- Inclusion*: Follows directly from the definition.
- Core-retainment*: If the sentence β is such that $\beta \in \mathrm{Cn}^*(B) \setminus \mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha)$, then this implies that $\beta \in \mathrm{Cn}^*(B) \setminus \big(\mathrm{Cn}^*(B) \setminus \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])\big)$, then $\beta \in \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$. Hence, β is an element of some $X \in \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]$. The set $B' := X \setminus \{\beta\}$ is such that $B' \subseteq \mathrm{Cn}^*(B)$, and since X is an α-kernel of $\mathrm{Cn}^*(B)$, we have that $\alpha \in \mathrm{Cn}(B' \cup \{\beta\}) \setminus \mathrm{Cn}(B')$.
- Uniformity*: Suppose that, for all $B' \subseteq \mathrm{Cn}^*(B)$, $\alpha \in \mathrm{Cn}(B')$ if and only if $\beta \in \mathrm{Cn}(B')$. We want to prove that $\mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha) = \mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\beta)$. It is enough to show that $\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha] = \mathrm{Ker}[\mathrm{Cn}^*(B),\beta]$, from which the result follows due to the definition of $\mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}$. We prove by contradiction. Assume that there is some $X \in \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha] \setminus \mathrm{Ker}[\mathrm{Cn}^*(B),\beta]$ (for the opposite case, the proof works by exchanging α and β). Since $X \subseteq \mathrm{Cn}^*(B)$ and X is not a β-kernel of $\mathrm{Cn}^*(B)$, then either $\beta \notin \mathrm{Cn}(X)$ or there is some $X' \subsetneq X$ such that $\beta \in \mathrm{Cn}(X')$. The first case implies that $\alpha \notin \mathrm{Cn}(X)$, which is not possible because X is an α-kernel of $\mathrm{Cn}^*(B)$. The second case is also impossible: since $X' \subseteq \mathrm{Cn}^*(B)$ and $\beta \in \mathrm{Cn}(X')$, we have that $\alpha \in \mathrm{Cn}(X')$, which cannot hold because X is an inclusion-minimal α-implying subset of $\mathrm{Cn}^*(B)$.

Postulates-to-construction: This part of the proof is similar to the proof of the corresponding theorem in (Hansson 1994). We need to show that an operation $\mathrm{c}^{\mathrm{Cn}^*}$ that satisfies success, inclusion*, core-retainment* and uniformity* is a Cn* kernel pseudo-contraction. Let us define $\boldsymbol{f}$ as

$$\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]) := \mathrm{Cn}^*(B) \setminus \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha).$$

We shall prove that **(i)** $\boldsymbol{f}$ is well defined, **(ii)** $\boldsymbol{f}$ is an incision function for $\mathrm{Cn}^*(B)$ and **(iii)** the operation $\mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}$ is equivalent to $\mathrm{c}^{\mathrm{Cn}^*}$:

(i) We have to show that the result of $\boldsymbol{f}$ is always the same for a fixed input. Suppose that α_1 and α_2 are such that $\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha_1] = \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha_2]$. Consider any $B' \in \mathrm{Cn}^*(B)$. If $\alpha_1 \in \mathrm{Cn}(B')$, from compactness of Cn we can consider a finite inclusion-minimal α_1-implying set $B'' \subseteq B'$, i.e. $B'' \in \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha_1]$, thus $B'' \in \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha_2]$ and $\alpha_2 \in \mathrm{Cn}(B'')$, which implies (due to monotonicity of Cn) that $\alpha_2 \in \mathrm{Cn}(B')$. Similarly, $\alpha_2 \in \mathrm{Cn}(B')$ implies $\alpha_1 \in \mathrm{Cn}(B')$. Hence, we have that $\alpha_1 \in \mathrm{Cn}(B')$ if and only if $\alpha_2 \in \mathrm{Cn}(B')$. By uniformity*, $\mathrm{c}^{\mathrm{Cn}^*}(B,\alpha_1) = \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha_2)$, and from the definition of $\boldsymbol{f}$ we conclude that $\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha_1]) = \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha_1])$.

(ii) To prove that $\boldsymbol{f}$ is an incision function, we must show that *(1)* $\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]) \subseteq \bigcup \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]$ and *(2)* $\boldsymbol{f}(\mathrm{Ker}[B,\alpha]) \cap X \neq \emptyset$ for every $X \in \mathrm{Ker}[B,\alpha] \setminus \{\emptyset\}$.

(1) Consider any $\beta \in \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$. We will show that $\beta \in \bigcup \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]$. From the definition of $\boldsymbol{f}$, we have that $\beta \in \mathrm{Cn}^*(B) \setminus \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)$. Since $\mathrm{c}^{\mathrm{Cn}^*}$ satisfies core-retainment*, there must be some $B' \subseteq \mathrm{Cn}^*(B)$ such that $\alpha \in \mathrm{Cn}(B' \cup \{\beta\}) \setminus \mathrm{Cn}(B')$. By compactness of Cn, there is some (finite) inclusion-minimal α-implying subset $B'' \subseteq B' \cup \{\beta\}$ such that $\alpha \in \mathrm{Cn}(B'')$, and β must be in B'' because Cn is monotonic and $\alpha \notin \mathrm{Cn}(B')$. As $B' \subseteq \mathrm{Cn}^*(B)$ and $\beta \in \mathrm{Cn}^*(B)$, the set B'' is a also subset of $\mathrm{Cn}^*(B)$. Hence, $B'' \in \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]$, and since $\beta \in B''$, we conclude that $\beta \in \bigcup \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]$.

(2) Take any non-empty $X \in \mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]$. By the definition of kernel, it must be the case that $\alpha \notin \mathrm{Cn}(\emptyset)$ and $\alpha \in \mathrm{Cn}(X)$. From success, we have that $\alpha \notin \mathrm{Cn}(\mathrm{c}^{\mathrm{Cn}^*}(B,\alpha))$, which implies (due to inclusion of Cn) that $\alpha \notin \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)$. Then, $X \not\subseteq \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)$, and there must be some sentence $\beta \in X \setminus \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)$. Since $X \subseteq \mathrm{Cn}^*(B)$, we have that $\beta \in \mathrm{Cn}^*(B) \setminus \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)$, that is, $\beta \in \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$. This concludes the proof that $\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]) \cap X \neq \emptyset$.

(iii) We can rewrite $\mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha)$ as:

$$\begin{aligned}
\mathrm{kc}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha) &= \mathrm{Cn}^*(B) \setminus \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha]) \\
&= \mathrm{Cn}^*(B) \setminus \big(\mathrm{Cn}^*(B) \setminus \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)\big) \\
&= \mathrm{Cn}^*(B) \cap \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha) \\
&= \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha).
\end{aligned}$$

□

Observation 9. *If Cn* satisfies subclassicality and inclusion, then any operation that satisfies core-retainment* also satisfies logical core-retainment.*

Proof. Assuming that Cn* satisfies inclusion and subclassicality, if $\mathrm{c}^{\mathrm{Cn}^*}$ satisfies core-retainment*, then for every $\beta \in B \setminus \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)$, we have $\beta \in \mathrm{Cn}^*(B) \setminus \mathrm{c}^{\mathrm{Cn}^*}(B,\alpha)$ (from inclusion of Cn*). Because of core-retainment*, there must be some $B' \subseteq \mathrm{Cn}^*(B)$ such that $\alpha \in \mathrm{Cn}(B' \cup \{\beta\}) \setminus \mathrm{Cn}(B')$. Since Cn* is subclassical, such B' is also a subset of $\mathrm{Cn}(B)$. Hence, $\mathrm{c}^{\mathrm{Cn}^*}$ satisfies logical core-retainment. □

Proposition 11. *If $\boldsymbol{f}$ is smooth and Cn* satisfies inclusion, then the Cn* kernel pseudo-contraction $\mathrm{c}_{\boldsymbol{f}}^{\mathrm{Cn}^*}$ satisfies relative closure.*

Proof. Let $\mathrm{c}_{\boldsymbol{f}}^{\mathrm{Cn}^*}$ be a Cn* kernel pseudo-contraction, where $\boldsymbol{f}$ is smooth. We will prove, by contradiction, that $\mathrm{c}_{\boldsymbol{f}}^{\mathrm{Cn}^*}$ satisfies relative closure. Assuming that this does not hold, there must be some sentence $\beta \in \Big(B \cap \mathrm{Cn}(\mathrm{c}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha))\Big) \setminus \mathrm{c}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha)$. Then, $\beta \in B \setminus \mathrm{c}_{\boldsymbol{f}}^{\mathrm{Cn}^*}(B,\alpha)$, i.e., $\beta \in B$ but $\beta \notin \mathrm{Cn}^*(B) \setminus \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$. Since Cn* satisfies inclusion, β must be in $\mathrm{Cn}^*(B)$, so β must be in $\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$. Let $B' := B \setminus \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$. From the definition of β, we know that $\beta \in \mathrm{Cn}(B')$. As β is in both $\mathrm{Cn}(B')$ and $\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$, the set $\mathrm{Cn}(B') \cap \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$ is non-empty. From the smoothness of $\boldsymbol{f}$, the set $B' \cap \boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$ must not be empty. This is a contradiction, because the definition of B' implies that it cannot contain any element of $\boldsymbol{f}(\mathrm{Ker}[\mathrm{Cn}^*(B),\alpha])$. □

Proposition 12. *If Cn* satisfies subclassicality, an operation that satisfies inclusion* and core-retainment* also satisfies vacuity*.*

Proof. Let c be an operation satisfying inclusion* and core-retainment*, and assume that Cn* is subclassical. Let B be a set of sentences and α be any sentence such that $\alpha \notin \mathrm{Cn}(B)$. We want to prove that $\mathrm{c}(B,\alpha) = \mathrm{Cn}^*(B)$. Inclusion* gives us $\mathrm{c}(B,\alpha) \subseteq \mathrm{Cn}^*(B)$, so we only have to show that $\mathrm{c}(B,\alpha) \supseteq \mathrm{Cn}^*(B)$, i.e. the set $\mathrm{Cn}^*(B) \setminus \mathrm{c}(B,\alpha)$ is empty. We will prove by contradiction. Assume that there is some $\beta \in \mathrm{Cn}^*(B) \setminus \mathrm{c}(B,\alpha)$. As c satisfies core-retainment*, there must be some $B' \subseteq \mathrm{Cn}^*(B)$ such that $\alpha \in \mathrm{Cn}(B' \cup \{\beta\}) \setminus \mathrm{Cn}(B')$. Since $B' \subseteq \mathrm{Cn}^*(B)$ and $\beta \in \mathrm{Cn}^*(B)$, we have $B' \cup \{\beta\} \subseteq \mathrm{Cn}^*(B)$, and from subclassicality of Cn*, this implies that $B' \cup \{\beta\} \subseteq \mathrm{Cn}(B)$. Because Cn satisfies monotonicity, we have $\mathrm{Cn}(B' \cup \{\beta\}) \subseteq \mathrm{Cn}(\mathrm{Cn}(B))$, and using idempotence of Cn, we obtain $\mathrm{Cn}(B' \cup \{\beta\}) \subseteq \mathrm{Cn}(B)$. This is a contradiction, because $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ but $\alpha \notin \mathrm{Cn}(B)$. □

B Proofs for Section 6 (Correspondence between Belief Revision and Repairs in Description Logics)

Remark 21 ((Matos et al. 2019))**.** *If $\Phi \subseteq B$, then the maximal α-non-implying subsets of B with respect to Φ contain all of the elements of Φ, i.e., $X \supseteq \Phi$ for every $X \in \mathrm{MaxNon}(B,\alpha,\Phi)$.*

Proof. If there is some $X \in \mathrm{MaxNon}(B,\alpha,\Phi)$ such that $X \not\supseteq \Phi$, then the set $Y = X \cup \Phi$ is such that $X \subset Y \subseteq B$, and since $\Phi \cup Y = \Phi \cup X$, we have that $\alpha \notin \mathrm{Cn}(\Phi \cup Y) = \mathrm{Cn}(\Phi \cup X)$, violating the definition of MaxNon. □

Remark 23 ((Matos et al. 2019))**.** *The minimal α-implying subsets of B with respect to Φ do not contain elements of Φ, i.e., $X \cap \Phi = \varnothing$ for every $X \in \mathrm{MinImp}(B,\alpha,\Phi)$.*

Proof. If there is some $X \in \mathrm{MinImp}(B,\alpha,\Phi)$ such that $X \cap \Phi \neq \varnothing$, then the set $Y = X \setminus \Phi$ is such that $Y \subset X$, and since $\Phi \cup Y = \Phi \cup X$, we have that $\alpha \in \mathrm{Cn}(\Phi \cup Y) = \mathrm{Cn}(\Phi \cup X)$, which contradicts the definition of MinImp. □

Theorem 27 (Partial meet base contraction $\implies$ Classical repair (Matos et al. 2019))**.** *Under the conditions of Proposition 26, if $\boldsymbol{g}$ is such that $\mathcal{O}_s \subseteq X$ for every $X \in \boldsymbol{g}(\mathrm{Rem}[\mathcal{O},\alpha])$, then the operation $\mathrm{Rep}_{\boldsymbol{g}}$ defined as $\mathrm{Rep}_{\boldsymbol{g}}(\mathcal{O},\alpha) = \mathrm{pmc}_{\boldsymbol{g}}(\mathcal{O},\alpha) \setminus \mathcal{O}_s$ yields a classical repair.*

Proof. Let $\mathcal{O}' = \mathrm{Rep}_{\boldsymbol{g}}(\mathcal{O},\alpha)$. Since $\boldsymbol{g}$ only selects α-remainders including $\mathcal{O}_s$, we have that $\mathcal{O}_s \subseteq \mathrm{pmc}_{\boldsymbol{g}}(\mathcal{O},\alpha)$, which implies that $\mathcal{O}_s \cup \mathcal{O}' = \mathrm{pmc}_{\boldsymbol{g}}(\mathcal{O},\alpha)$. Hence, from the inclusion postulate, we have that $\mathcal{O}_s \cup \mathcal{O}' \subseteq \mathcal{O}$, and monotonicity of Cn gives $\mathrm{Cn}(\mathcal{O}_s \cup \mathcal{O}') \subseteq \mathrm{Cn}(\mathcal{O})$. This is sufficient to show that the result of $\mathrm{Rep}_{\boldsymbol{g}}$ is a repair. From the inclusion postulate, we have that $\mathrm{pmc}_{\boldsymbol{g}}(\mathcal{O},\alpha) \subseteq \mathcal{O}$, which proves that $\mathcal{O}' = \mathrm{pmc}_{\boldsymbol{g}}(\mathcal{O},\alpha) \setminus \mathcal{O}_s \subseteq \mathcal{O}_r$. Therefore, $\mathrm{Rep}_{\boldsymbol{g}}$ yields a classical repair. □

Lemma 35. *Consider a general partial meet pseudo-contraction defined as in Definition 31. For every sentence φ in $B \setminus \bigcap \boldsymbol{g}(\mathrm{MaxNon}(B,\alpha))$, there is a set X such that $X \in \boldsymbol{g}'(\mathrm{MaxNon}(\mathrm{Cn}^{**}(B,\alpha),\alpha))$ and $\varphi \notin X$.*

Proof. Assume that $\varphi \in B \setminus \bigcap \boldsymbol{g}(\mathrm{MaxNon}(B,\alpha))$. We need to show that there is some set X such that in $X \in \boldsymbol{g}'(\mathrm{MaxNon}(\mathrm{Cn}^{**}(B,\alpha),\alpha))$ and $\varphi \notin X$. Since $\varphi \notin \bigcap \boldsymbol{g}(\mathrm{MaxNon}(B,\alpha))$, there is a $Y \in \boldsymbol{g}(\mathrm{MaxNon}(B,\alpha))$ such that $\varphi \notin Y$. We have $Y \subseteq B$, $\alpha \notin \mathrm{Cn}(Y)$ and $\alpha \in \mathrm{Cn}(Y')$ for any $Y' \subseteq B$ such that $Y \subset Y'$. So, since $\boldsymbol{g}'$ is an extension of $\boldsymbol{g}$ to $\mathrm{Cn}^{**}(B,\alpha)$, there is an $X \in \boldsymbol{g}'(\mathrm{MaxNon}(\mathrm{Cn}^{**}(B,\alpha),\alpha))$ such that $Y \subseteq X$. And since $\varphi \notin Y$, $\varphi \in B$ and for any $Y' \subseteq B$ such that $Y \subset Y'$ we have $\alpha \in \mathrm{Cn}(Y')$, we conclude that $\varphi \notin X$ (otherwise, X would be a such Y' and α would be in $\mathrm{Cn}(X)$). □

Lemma 37. *Let* c *be a contraction operation for a set of sentences B. Let $\mathrm{c}^{\mathrm{Cn}^{**}}$ be a pseudo-contraction operation such that $\mathrm{c}^{\mathrm{Cn}^{**}}(B,\beta) \subseteq \mathrm{Cn}^{**}(B)$, where $(B \setminus \mathrm{c}(B,\beta)) \cap \mathrm{c}^{\mathrm{Cn}^{**}}(B,\beta) = \emptyset$ and $\mathrm{Cn}^{**}(B) := B \cup \mathrm{Cn'}(B \setminus \mathrm{c}(B,\beta))$ for all sentences β and the consequence relation Cn' is monotonic, subclassical and strictly weakening. If the ontology $\mathcal{O} := \langle \mathcal{O}_s, \mathcal{O}_r \rangle$ is such that $\mathcal{O}_s \subseteq \mathrm{c}(\mathcal{O},\beta) \cap \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\beta)$ for all sentences β and α is a sentence such that $\alpha \notin \mathrm{Cn}(\mathcal{O}_s)$, then the set $\mathcal{O}' := \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha) \setminus \mathcal{O}_s$ is a gentle repair of $\mathcal{O}$ with respect to α.*

Proof. We will start by proving that $\mathrm{Cn}(\mathcal{O}_s \cup \mathcal{O}') \subseteq \mathrm{Cn}(\mathcal{O}) \setminus \{\alpha\}$. By subclassicality of Cn', it follows that $\mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha) \subseteq \mathcal{O} \cup \mathrm{Cn}(\mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha))$, and by monotonicity, inclusion and idempotence of Cn we conclude that $\mathrm{Cn}(\mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)) \subseteq \mathrm{Cn}(\mathcal{O})$. Since $\mathcal{O}_s \subseteq \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)$, we have $\mathcal{O}_s \cup \mathcal{O}' = \mathcal{O}_s \cup (\mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)) \setminus \mathcal{O}_s) = \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)$, so $\mathrm{Cn}(\mathcal{O}_s \cup \mathcal{O}') \subseteq \mathrm{Cn}(\mathcal{O})$. To show that $\alpha \notin \mathrm{Cn}(\mathcal{O}_s \cup \mathcal{O}')$, since we have just shown that $\mathcal{O}_s \cup \mathcal{O}' = \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)$, from success of $\mathrm{c}^{\mathrm{Cn}^{**}}$ we have that $\alpha \notin \mathrm{Cn}(\mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha))$, and the result follows.

Now we have to prove that, for all $\varphi \in \mathcal{O}'$, either $\varphi \in \mathcal{O}_r$ or $\mathrm{Cn}(\{\varphi\}) \subset \mathrm{Cn}(\{\psi\})$ for some $\psi \in \mathcal{O}_r \setminus \mathcal{O}'$. Take some $\varphi \in \mathcal{O}'$. If $\varphi \in \mathcal{O}_r$, this part of the proof is done. Assume that $\varphi \notin \mathcal{O}_r$. As $\varphi \in \mathcal{O}'$, we have $\varphi \in \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)$, but

$\mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha) \subseteq \mathrm{Cn}^{**}(\mathcal{O},\alpha) = \mathrm{Cn'}(\mathcal{O}\setminus\mathrm{c}(\mathcal{O},\alpha))\cup\mathcal{O}$. It was assumed that $\varphi \notin \mathcal{O}_r$, so $\varphi \notin \mathcal{O}$ (because $\varphi \in \mathcal{O}'$ implies that $\varphi \notin \mathcal{O}_s$), and thus φ must be in $\mathrm{Cn'}(\mathcal{O}\setminus\mathrm{c}(\mathcal{O},\alpha))$. Since Cn' is strictly weakening, either $\varphi \in \mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha)$ or there is a $\psi \in \mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha)$ such that $\mathrm{Cn}(\{\varphi\}) \subset \mathrm{Cn}(\{\psi\})$. The first case is not possible: since $\mathcal{O}_s \subseteq \mathrm{c}(\mathcal{O},\alpha)$, we have $\mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha) \subseteq \mathcal{O}_r$; hence, φ cannot be in $\mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha)$ because we assumed that $\varphi \notin \mathcal{O}_r$. Take some $\psi \in \mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha)$ such that $\mathrm{Cn}(\{\varphi\}) \subset \mathrm{Cn}(\{\psi\})$. Since $\mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha) \subseteq \mathcal{O}_r$, we know that $\psi \in \mathcal{O}_r$. Now it is left to show that $\psi \notin \mathcal{O}'$, i.e., $\psi \notin \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)$ or $\psi \in \mathcal{O}_s$. Since $\psi \in \mathcal{O}_r$ and $\mathcal{O}_s$ and $\mathcal{O}_r$ are assumed to be disjoint, $\psi \notin \mathcal{O}_s$. So we only have to show that $\psi \notin \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)$. We know that $\psi \in \mathcal{O}\setminus\mathrm{c}(\mathcal{O},\alpha)$, and the fact that $(\mathcal{O} \setminus \mathrm{c}(\mathcal{O},\alpha)) \cap \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha) = \emptyset$ implies that ψ is not in $\mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\alpha)$, finishing the proof. □

Theorem 38 (When a general partial meet pseudo-contraction is a gentle repair (Matos et al. 2019, adapted))**.** *Let* $\mathrm{gpmc}^{\mathrm{Cn}^{**}}_{\boldsymbol{g},\boldsymbol{g}'}$ *and* Cn^{**} *be as in Definition 31,* Cn^{**} *based on a consequence relation* Cn' *that satisfies subclassicality,* $\boldsymbol{g}$ *and* $\boldsymbol{g}'$ *satisfy* $\mathcal{O}_s$*-inclusion,* Cn' *be monotonic and strictly weakening, and* $\mathcal{O} = \langle\mathcal{O}_s,\mathcal{O}_r\rangle$*. If* $\alpha \notin \mathrm{Cn}(\mathcal{O}_s)$*, then* $\mathcal{O}' := \mathrm{gpmc}^{\mathrm{Cn}^{**}}_{\boldsymbol{g},\boldsymbol{g}'}(\mathcal{O},\alpha) \setminus \mathcal{O}_s$ *is a gentle repair of* $\mathcal{O}$ *w.r.t.* α*.*

Proof. The result follows from Lemma 37 by taking $\mathrm{pmc}_{\boldsymbol{g}}$ as c and $\mathrm{gpmc}^{\mathrm{Cn}^{**}}_{\boldsymbol{g},\boldsymbol{g}'}$ as $\mathrm{c}^{\mathrm{Cn}^{**}}$. Lemma 35 shows that the assumption $(B \setminus \mathrm{c}(B,\beta)) \cap \mathrm{c}^{\mathrm{Cn}^{**}}(B,\beta) = \emptyset$ holds, and the assumption that $\mathcal{O}_s \subseteq \mathrm{c}(\mathcal{O},\beta) \cap \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\beta)$ for all sentences β derives from $\mathcal{O}_s$-inclusion of $\boldsymbol{g}$ and $\boldsymbol{g}'$. □

Theorem 39 (When a general kernel pseudo-contraction is a gentle repair)**.** *Let* $\mathrm{gkc}^{\mathrm{Cn}^{**}}_{\boldsymbol{f},\boldsymbol{f}'}$ *and* Cn^{**} *be as in Definition 32,* Cn^{**} *based on a consequence relation* Cn' *that satisfies subclassicality,* $\boldsymbol{f}$ *and* $\boldsymbol{f}'$ *satisfy* $\mathcal{O}_s$*-exclusion,* Cn' *be monotonic and strictly weakening, and* $\mathcal{O} = \langle\mathcal{O}_s,\mathcal{O}_r\rangle$*. If* $\alpha \notin \mathrm{Cn}(\mathcal{O}_s)$*, then* $\mathcal{O}' := \mathrm{gkc}^{\mathrm{Cn}^{**}}_{\boldsymbol{f},\boldsymbol{f}'}(\mathcal{O},\alpha) \setminus \mathcal{O}_s$ *is a gentle repair of* $\mathcal{O}$ *w.r.t.* α*.*

Proof. The result follows from Lemma 37 by taking $\mathrm{kc}_{\boldsymbol{f}}$ as c and $\mathrm{gkc}^{\mathrm{Cn}^{**}}_{\boldsymbol{f},\boldsymbol{f}'}$ as $\mathrm{c}^{\mathrm{Cn}^{**}}$. The definition of extension of an incision function is enough to conclude that the assumption $(B \setminus \mathrm{c}(B,\beta)) \cap \mathrm{c}^{\mathrm{Cn}^{**}}(B,\beta) = \emptyset$ holds, and the assumption that $\mathcal{O}_s \subseteq \mathrm{c}(\mathcal{O},\beta) \cap \mathrm{c}^{\mathrm{Cn}^{**}}(\mathcal{O},\beta)$ for all sentences β derives from $\mathcal{O}_s$-exclusion of $\boldsymbol{f}$ and $\boldsymbol{f}'$. □