The Conley-Zehnder index and the saddle-center equilibrium

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We consider Hamiltonian systems with two degrees of freedom. We suppose the existence of a saddle-center equilibrium in a strictly convex component S of its energy level. Moser's normal form for such equilibriums and a theorem of Hofer, Wysocki and Zehnder are used to establish the existence of a periodic orbit in S with several topological properties. We also prove the explosion of the Conley-Zehnder index of any periodic orbit that passes close to the saddle-center equilibrium.

Key Words: Hamiltonian systems, Conley-Zehnder index, convexity, saddle-center, two-degrees of freedom.

1. INTRODUCTION

We consider Hamiltonian systems associated to a real-analytic function $H : \mathbb{R}^4 \to \mathbb{R}$. We assume the existence of an invariant component $S \subset H^{-1}(0)$ of the Hamiltonian vector field $\dot{x}= J_0 H_x(x)$, satisfying the following hypotheses:
(H1) S is ambient homeomorphic to S^3, i.e., there is a homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ such that $h(S) = S^3$;
(H2) $p_c \in S$ is a saddle-center equilibrium (See Section 2 for definitions);

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(H3) if $q \in S_0 \stackrel{def}{=} S\backslash\{p_c\}$, then q is a regular point of H and $\langle H_{xx}(q)v, v\rangle > 0$ for all $v \in T_qS$;

There are several Hamiltonians satisfying these properties. See [Sal] for examples of Hamiltonians of type "kinetic plus potential energy".

Let $N : S_0 \to S^3$ be the Gauss map defined by $N(p) \stackrel{def}{=} \frac{H_x(p)}{\|H_x(p)\|}$ and let $dN(p) : T_pS \to T_pS$ be its differential at $p \in S_0$. As $dN(p)$ is a linear self-adjoint operator, there exists an orthonormal basis $\{e_1(p), e_2(p), e_3(p)\}$ of T_pS such that $dN(p)e_i(p) = -k_i(p)e_i(p)$, $i = 1, 2, 3$, where k_i are the principal curvatures of S at p. We say that a hypersurface S has positive curvature at $p \in S$ if $k_i(p)k_j(p) > 0$ for all $1 \leq i, j \leq 3$.

It is imediate that if $v \in T_pS$ then

$$\langle dN(p)v, v\rangle = \frac{1}{\|H_x(p)\|} \langle H_{xx}(p)v, v\rangle$$

and, therefore, hypothesis (H3) implies that S_0 has positive curvature. In [Sal], it is proved that S is indeed a strictly convex hypersurface, i.e., it is the boundary of a convex set in $\mathbb{R}^4$.

When an energy level is regular and strictly convex, then a remarkable theorem of H. Hofer, E. Zehnder and K. Wysocki ([HWZ]) states that

Theorem 1. *Every strictly convex hypersurface $\widetilde{S} \subset H^{-1}(0)$, diffeomorphic to the sphere S^3, has an embedded disk $\sum$ with the following properties:*

(i) $\partial \sum$ is an unknotted periodic orbit P with Conley-Zehnder index 3 and self-linking number -1;

(ii) for all points in the interior of $\sum$, the Hamiltonian vector field X_H is transversal to $\sum$;

(iii) every orbit through $x \in \widetilde{S} \backslash P$ hits $\sum$ forward and backward in time;

The disk $\sum$ is called a global surface of section. In fact there is a S^1-family of such disks foliating $\widetilde{S} \backslash P$ called open book decomposition. The dynamics in $\widetilde{S}$ is, therefore, described by the diffeomorphism $\varphi : \sum \to \sum$ which is given by the first return map, providing an important simplification of the analysis of the flow in that energy level. An immediate consequence of this result and a theorem of J. Franks [Fr], is the existence of 2 or infinitely many periodic orbits in $\widetilde{S}$.

In this paper we give a first step in the direction of extending Theorem 1 for a non-regular hypersurface S satisfying hypothesis (H1)-(H3).

First, we use a normal form for a saddle-center equilibrium to estimate the Conley-Zehnder index $\mu_{CZ}(P)$ of a periodic orbit P near p_c. More precisely, we prove that the Conley-Zehnder index of P tends to $+\infty$ as the distance between P and p_c goes to zero. It is stated in the following

THEOREM 2. *Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a real-analytic function and $S \subset H^{-1}(0)$ be a hypersurface satisfying hypotheses (H1)-(H3). Then, given $K \in \mathbb{N}$, there exists a neighborhood W of p_c in S such that if a periodic orbit P of the Hamiltonian flow in S intersects W, then $\mu_{CZ}(P) > K$.*

Then we prove the existence of a periodic orbit in S with properties (i). Our method consists in regularizing S in a small neighborhood of p_c preserving convexity (See [Gho]). The hypothesis of real-analyticity comes only from the use of Moser's normal form, which is stated in the real-analytic case, but which may hold under less regularity. Then, as a corollary of theorems 1 and 2 we obtain the following

COROLLARY 3. *Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a real-analytic function and $S \subset H^{-1}(0)$ be a hypersurface satisfying hypotheses (H1)-(H3). Then there exists an unknotted periodic orbit P of the Hamiltonian flow in S with Conley-Zehnder index 3 and self-linking-number -1.*

Remark 4. We point out that the existence of a periodic orbit with properties (i) implies, in the regular case, the existence of the open book decomposition (See [HWZ]).

2. SADDLE-CENTER EQUILIBRIUMS

A point $p \in \mathbb{R}^4$ is an equilibrium point of the Hamiltonian flow of $H : \mathbb{R}^4 \to \mathbb{R}$ if $H_x(p) = 0$, i.e, if the solution starting at p is constant[1]. We say that p is a saddle-center equilibrium point of X_H if $J_0 H_{xx}(p)$ has two real eigenvalues $\pm\alpha$ ($\alpha > 0$) and two pure imaginary eigenvalues $\pm\omega i$ ($\omega > 0$).

A normal form for such equilibriums is given by J. Moser [Mo] (see also [Russ], [Hen] and [Rag]) and it states that it is possible to find a change of coordinates $\varphi : 0 \in V \subset \mathbb{R}^4 \to U$ where U is a neighborhood of p_c in $\mathbb{R}^4$, such that if $z = (q_1, q_2, p_1, p_2)$ are the coordinates in V, then the Hamiltonian flow in U is conjugate to the Hamiltonian flow in V (maybe reversing time parametrization) associated to the function $K : V \to \mathbb{R}$ given by

$$K(q_1, q_2, p_1, p_2) = -\alpha I_1 + \omega I_2 + O(I_1^2 + I_2^2) \tag{1}$$

[1]The equations of a Hamiltonian system associated to the function H are given by $\dot{x} = J_0 H_x(x)$, where $J_0 = \begin{pmatrix} 0 & I_{2\times 2} \\ -I_{2\times 2} & 0 \end{pmatrix}$ and $H_x(x)$ is the gradient of H at x.

where $I_1 = q_1p_1$ and $I_2 = \frac{q_2^2+p_2^2}{2}$ are first integrals of the flow. In V, the equations of motion are given by

$$\begin{aligned}
\dot{q}_1 &= \frac{\partial K(I_1,I_2)}{\partial I_1} q_1 \\
\dot{p}_1 &= -\frac{\partial K(I_1,I_2)}{\partial I_1} p_1 \\
\dot{q}_2 &= \frac{\partial K(I_1,I_2)}{\partial I_2} p_2 \\
\dot{p}_2 &= -\frac{\partial K(I_1,I_2)}{\partial I_2} q_2
\end{aligned}$$

The flow projected in the plane (q_1, p_1) has a hyperbolic saddle behavior and in the plane (q_2, p_2) it is similar to a center. Let $V_0 \overset{def}{=} V \cap \{K = 0\}$. Then, in local coordinates, the energy level of the saddle-center is given by two connected components C_1 and C_2, each one projecting onto the first and third quadrants of (q_1, p_1). These components have the origin as the unique point in common. We also have:

(i) $0 \in V_0$ is a saddle-center equilibrium of the Hamiltonian flow associated to K and $\varphi(0) = p_c$;

(ii) The sets $W^s(0) \overset{def}{=} \{z \in V_0 : p_1 = 0\}$ and $W^u(0) \overset{def}{=} \{z \in V_0 : q_1 = 0\}$ are, respectively, the local stable and unstable manifolds of 0 in V, which are one-dimensional;

2.1. Vector Bundle Trivialization of Hypersurfaces in $\mathbb{R}^4$

Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a C^∞ function and $c \in Im(H)$ a regular value of H. Let F_S be a C^∞ vector bundle over the hypersurface $S \overset{def}{=} H^{-1}(c)$, such that the fiber ξ_x at $x \in S$ is a n-dimensional subspace of T_xS ($n = 1$, 2 or 3). We say that F_S is trivial if there exists a homeomorphism $\beta_{F_S} : F_S \to S \times \mathbb{R}^n$ such that $pr1 \circ \beta_{F_S} = \pi_{F_S}$ where $pr1$ is the projection on the first component of $S \times \mathbb{R}^n$ and π_{F_S} is the projection of F_S onto the basis S. The homeomorphism β_{F_S} is called a trivialization of F_S. We will introduce a trivialization of TS presented in [CPR] and give a natural trivialization of two-dimensional vector bundles transversal to the Hamiltonian vector field X_H in S.

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $0' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be 2×2 matrices. Now define the 4×4 matrices by:

$$A_0 = \begin{pmatrix} I & 0' \\ 0' & I \end{pmatrix} A_1 = \begin{pmatrix} 0' & J \\ J & 0' \end{pmatrix} A_2 = \begin{pmatrix} J & 0' \\ 0' & -J \end{pmatrix} A_3 = \begin{pmatrix} 0' & I \\ -I & 0' \end{pmatrix} \quad (2)$$

The following properties of these orthogonal matrices will be useful:

$$A_1A_2 = A_3 \qquad A_2A_3 = A_1 \qquad A_3A_1 = A_2 \qquad A_i^2 = -I_{4\times 4} \tag{3}$$

For each $p \in S$, let

$$X_0(p) \stackrel{def}{=} \frac{H_x(p)}{\|H_x(p)\|} \tag{4}$$

where $H_x(p) \in T_p\mathbb{R}^4 \simeq \mathbb{R}^4$ is the gradient vector of H at x, which is normal to T_xS. Let

$$X_i(p) \stackrel{def}{=} A_iX_0(p), i = 1, 2, 3 \tag{5}$$

Then $\langle X_i(p), X_j(p)\rangle = \delta_{ij}$ for all $0 \le i, j \le 3$. It follows that the vectors $X_1(p)$, $X_2(p)$ and $X_3(p)$ give an orthonormal basis of T_pS. As $X_i(p) \neq 0$ for all $p \in S$, we can trivialize TS using the vector fields X_1, X_2 and X_3 by the following way: for $v \in T_pS$, we have $v = \sum_{i=1}^{3}\alpha^i X_i(p)$ and the trivialization $\beta_{TS} : TS \to S \times \mathbb{R}^3$ is given by

$$\beta_{TS}(p, v) = (p, \alpha^1, \alpha^2, \alpha^3) \tag{6}$$

The Hamiltonian vector field associated to the function H is given by $X_H(p) = A_3H_x(p)$ and, therefore, $X_H(p) = \|H_x(p)\| X_3(p)$. Considering the vector bundle F_2 over S with fibers generated by X_1 and X_2, its trivialization is given in the same way as β_{TS}, omitting the component α^3.

Let us consider now another two-dimensional vector bundle ξ over S, $\xi \subset TS$, such that the fiber ξ_p at $p \in S$ is transversal to $X_3(p)$. There exists a natural trivialization of ξ using the vector fields X_1 and X_2 as follows: as ξ_p is transversal to $X_3(p)$, then for each $v \in \xi_p$, $v \neq 0$, we have $v = \alpha^1X_1(p) + \alpha^2X_2(p) + \alpha^3X_3(p)$ where $(\alpha^1)^2 + (\alpha^2)^2 \neq 0$.

Let $\pi_{\xi_p} : \xi_p \to \{X_1, X_2\}$ be the canonical projection given by $\pi_{\xi_p}(v) = \alpha^1X_1(p) + \alpha^2X_2(p)$. Then π_{ξ_p} is an isomorphism and, therefore, we can define a basis for ξ_p given by $\{\widetilde{X}_1(p), \widetilde{X}_2(p)\}$ where

$$\widetilde{X}_i(p) = \pi_{\xi_p}^{-1}(X_i(p)), i = 1, 2 \tag{7}$$

Then we have

$$v = \alpha^1 \widetilde{X}_1(p) + \alpha^2 \widetilde{X}_2(p) \tag{8}$$

and the trivialization $\beta_\xi : \xi \to S \times \mathbb{R}^2$ of ξ, is given by

$$\beta_\xi(p, v) = (p, \alpha^1, \alpha^2) \tag{9}$$

Let ϕ_t be the Hamiltonian flow of H restricted to the hypersurface S, i.e., $\frac{d\phi_t(x)}{dt} = X_H(x)|_S$. Linearizing the equations along a solution $x(t) \subset S$, we get the flow $T\phi_t : TS \to TS$ which satisfies the following equation

$$\dot{y} = A_3 H_{xx}(x(t))y \tag{10}$$

where $y(t) \in T_{x(t)}S$. Denoting $y(t) = \sum_{i=1}^{3} \alpha^i(t) X_i(x(t))$ and $\alpha = \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix}$ then, by (10) and relations (3), we obtain

$$\dot{\alpha} = -J \tilde{S} \alpha \tag{11}$$

where $\tilde{S}$ is given by

$$\tilde{S} = \begin{pmatrix} \langle H_{xx}X_1, X_1 \rangle & \langle H_{xx}X_1, X_2 \rangle \\ \langle H_{xx}X_1, X_2 \rangle & \langle H_{xx}X_2, X_2 \rangle \end{pmatrix} + \langle H_{xx}X_3, X_3 \rangle I \tag{12}$$

Assume now that the Hessian of H is positive-definite when restricted to TS, i.e., S has positive curvature. Then we have the following

THEOREM 5. *The vector* $\alpha(t) = (\alpha^1(t), \alpha^2(t)) \subset \mathbb{R}^2$, $\alpha(0) \neq (0,0)$*, turns around the origin always counter-clockwise.*

Proof. Consider the vector $\alpha(t) \wedge \dot{\alpha}(t) \in \mathbb{R}^3$ and $\vec{k}$ a unitary vector in $\mathbb{R}^3$ orthogonal to the plane of $\alpha(t)$. Then

$$\alpha(t) \wedge \dot{\alpha}(t) = \begin{pmatrix} \alpha^1 \alpha^2 \end{pmatrix} \tilde{S} \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} \vec{k}$$

As the Hessian of H restricted to TS is positive-definite, we obtain that $\tilde{S}$ is a positive-definite matrix. It follows that the component $\vec{k}$ of $\alpha(t) \wedge \dot{\alpha}(t)$ is always positive which proves that $\alpha(t)$ turns around the origin always counter-clockwise. ▮

3. THE GENERALIZED CONLEY-ZEHNDER INDEX OF PERIODIC ORBITS

The Conley-Zehnder index was first introduced in [CZ] and, roughly speaking, it measures how the orbits near a periodic orbit turn around it after choosing a referential. We will denote by P a periodic orbit, given by $x : [0,T] \to S$ where T is its minimum period and $x([0,T]) = P$.

We suppose that the hypersurface $S \subset H^{-1}(0)$ is diffeomorphic to S^3 and strictly convex. We assume that the origin is in its interior. The same thing can be done for a hypersurface satisfying hypothesis (H1)-(H3).

We consider the $1-$form $\lambda_0^S = \lambda_0 \mid_{TS}$ where $\lambda_0(x)(v) \stackrel{def}{=} \frac{1}{2} \sum_{j=1}^{2}[x_j(x)dy_j(v) - y_j(x)dx_j(v)]$ and (x_1, x_2, y_1, y_2) are the coordinates in $\mathbb{R}^4$.

It is easy to see that $d\lambda_0^S \wedge \lambda_0^S$ is non-degenerate in TS, i.e., it defines a volume form in S (λ_0 is called a contact form in S). Now we define the contact structure $\xi \subset TS$ by

$$\xi = \ker \lambda_0^S \tag{13}$$

The contact structure has the following properties:
(i) ξ is a two-dimensional vector bundle over S;
(ii) ξ_p is transversal to the Hamiltonian vector field $X_H(p)$ for all $p \in S$;
(iii) the $2-$form $d\lambda_0^S$ is non-degenerate in ξ;
(iv) it is always possible to find $H : \mathbb{R}^4 \to \mathbb{R}$ such that $S = H^{-1}(0)$ and the linearized flow preserves the 1-form λ_0^S. In this section we assume that H has this property. See also [HWZ].

The following explanation of the Conley-Zehnder index of a periodic orbit is based on [HK] and [HWZ], where the reader can find more details.

Let $v_D : D \to M$ be an embedding of the compact disk $D = \{z \in \mathrm{C} : |z| \leq 1\}$ in S such that $v_D(e^{2\pi it}) = x(tT)$. Let $\beta : v_D^*\xi \to D \times \mathbb{R}^2$ be a trivialization of $v_D^*\xi$. We define now the arc of 2×2 symplectic matrices, $\Phi : [0, T] \to Sp(1)$ along the periodic orbit $x(t) = \phi_t(x(0))$ by

$$\Phi(t) = \beta(e^{2\pi it/T}) \circ D\phi_t|_{\xi_{x(0)}} \circ \beta(1)^{-1}, 0 \leq t \leq T$$

This arc satisfies the following properties:
(i) $\Phi(0) = I$;
(ii) the periodic orbit P is non-degenerate if and only if the integer 1 is not an eigenvalue of $\Phi(T)$;
(iii) $\Phi(t + T) = \Phi(t)\Phi(T)$;

We give now an spectral definition of the generalized Conley-Zehnder index of the periodic orbit P.

Defining the symmetric matrix $A(t) \stackrel{def}{=} -J_0 \dot{\Phi}(t)\Phi^{-1}(t)$ it is easy to see that $A(t) = A(t + T)$.

Therefore, we can define the self-adjoint linear operator $L_A : H^{1,2}(\mathbb{R}/T\mathrm{Z}, \mathbb{R}^2) \to L^2(\mathbb{R}/T\mathrm{Z}, \mathbb{R}^2)$ given by $L_A \stackrel{def}{=} -J_0 \frac{d}{dt} - A(t)$. The spectrum $\sigma(L_A)$ of this operator has the following properties:
(i) $\sigma(L_A)$ is real and countable;
(ii) $\sigma(L_A)$ has no upper or lower bounds;
(iii) $\ker L_A = \{0\}$ if and only if 1 is not an eigenvalue of $\Phi(T)$.

Let $v \neq 0$ be an eigenvector in $L^2(\mathbb{R}/T\mathrm{Z}, \mathbb{R}^2)$ associated to $\lambda \in \sigma(L_A)$. Then

$$-J_0 \dot{v}(t) - A(t)v(t) = \lambda v(t),\ v(0) = v(T) \tag{14}$$

It follows that $v(t) \neq 0$ for $t \in [0, T]$ and, as v is periodic, we can associate to v a *winding number* $\omega(\lambda, v, A)$. It is possible to show that

(i) if v_1 and v_2 are two eigenvectors of L_A associated to the the same eigenvalue λ, then $\omega(\lambda, v_1, A) = \omega(\lambda, v_2, A)$. It follows that the *winding number* can be denoted by $\omega(\lambda, A)$;

(ii) for each $k \in$ Z, there are exactly two eigenvalues λ_1 and λ_2, counting its multiplicity such that $k = \omega(\lambda_1, A) = \omega(\lambda_2, A)$;

(iii) the map $\omega_A : \sigma(L_A) \to \mathbb{Z}$ given by $\omega_A(\lambda) = \omega(\lambda, A)$ is monotone increasing;

Let $\alpha(A)$ and $p(A)$ be the integers given by

$$\alpha(A) = \max\{\omega(\lambda, A) | \lambda \in \sigma(L_A) \cap (-\infty, 0)\}$$

$$p(A) = \begin{cases} 0, & \exists \lambda \in \sigma(L_A) \cap [0, \infty) \mid \omega(\lambda, A) = \alpha(A) \\ 1, & \text{otherwise} \end{cases}$$

Defining $\mu(A) = 2\alpha(A) + p(A)$, it is possible to show that it does not depend on the trivialization β. Moreover, as S is diffeomorphic to S^3, $\mu(A)$ also does not depend on the embedding v_D. Finally, the generalized Conley-Zehnder index of the periodic orbit P is defined by $\mu_{CZ}(P) \stackrel{def}{=} \mu(A)$.

Now we give the geometric definition of the Conley-Zehnder index, which only works for non-degenerate periodic orbits. Consider the arc of symplectic matrixes $\Phi : [0, T] \to Sp(1)$ given as above. Let $z \in$ C$\backslash\{0\}$ and $\rho(t)$ be a continuous argument of the solution $z(t) = \Phi(t)z$, i.e., $\rho(t)$ is a continuous real function in $[0, T]$ such that $e^{2\pi i \rho(t)} = \frac{z(t)}{|z(t)|}$. Let $\Delta(z) \stackrel{def}{=} \rho(T) - \rho(0)$ and $I(\Phi) \stackrel{def}{=} \{\Delta(z) : z \in$ C$\backslash\{0\}\}$. The set $I(\Phi)$ is an interval satisfying $|I(\Phi)| < 1/2$ and, therefore, we can define

$$\widetilde{\mu}\,(\Phi) = \begin{cases} 2k + 1 & I(\Phi) \subset (k, k+1) \\ 2k & k \in I(\Phi) \end{cases}$$

By the same reasons as before $\widetilde{\mu}\,(\Phi)$ does not depend on the trivialization β and on the embedding v_D (see [HWZ] and [HK]). We can, therefore, define a new index $\widetilde{\mu}_2\,(P)$ of the non-degenerate periodic orbit P by $\widetilde{\mu}_2$ $(P) \stackrel{def}{=} \widetilde{\mu}\,(\Phi)$. From [HK] we have the following

Proposition 6. *If P is a non-degenerate periodic orbit then $\mu_{CZ}(P) = \widetilde{\mu}_2$ (P).*

We can calculate $\widetilde{\mu}_2\,(P)$ also for degenerate periodic orbits giving a good estimate of the Conley-Zehnder index $\mu_{CZ}(P)$ as the following proposition shows

Proposition 7. *If P is a degenerate periodic orbit then either $\widetilde{\mu}_{CZ}$ $(P) = \widetilde{\mu}_2\ (P)$ or $\widetilde{\mu}_{CZ}\ (P) = \widetilde{\mu}_2\ (P) - 1$.*

Proof. We know that there is an eigenvector $v \neq 0$ in $\ker L_A$. By (14) v satisfies $\dot{v}\ (t) = \dot{\Phi}\ (t)\Phi^{-1}(t)v(t)$ and, therefore, by unicity of solutions of O.D.E., we have $v(t) = \Phi(t)v(0)$. As v is periodic, we have $\widetilde{\mu}_2\ (P) = 2\omega(0, A)$. By the properties of $\omega(\lambda, A)$ and the definition of $\alpha(A)$ we conclude that either $\alpha(A) = \omega(0, A)$ (in this case $p(A) = 0$) or $\alpha(A) = \omega(0, A) - 1$ ($p(A) = 1$). It follows that either $\widetilde{\mu}_{CZ}\ (P) = \widetilde{\mu}\ (A) = 2\omega(0, A) = \widetilde{\mu}_2\ (P)$ or $\widetilde{\mu}_{CZ}\ (P) = \widetilde{\mu}\ (A) = 2\omega(0, A) - 1 = \widetilde{\mu}_2\ (P) - 1$. ∎

4. EQUIVALENCE OF THE HAMILTONIAN FLOW

In this section, we show that some of the properties of the Hamiltonian flow in S do not depend on the choice of the Hamiltonian function.

Let S be a $C^{k \geq 2}$ connected and orientable hypersurface in $\mathbb{R}^4$. Let $H, G : \mathbb{R}^4 \to \mathbb{R}$ be two $C^{k \geq 2}$ functions such that $S \subset H^{-1}(c_1)$ and $S \subset G^{-1}(c_2)$ and, for all $p \in S$, p is a regular point of H and G. Let $x : 0 \in I \subset \mathbb{R} \to S$ be the solution of

$$\frac{dx(t)}{dt} = J_0 H_x(x(t)), x(0) = x_0 \in S \tag{15}$$

and let $\widetilde{x}$: $0 \in I_0 \subset \mathbb{R} \to S$ be the solution of

$$\frac{d\,\widetilde{x}\ (t)}{dt} = J_0 G_x(\widetilde{x}\ (t)) \tag{16}$$

with the same initial conditions of x, i.e., $\widetilde{x}\ (0) = x_0$.

Proposition 8. *There exists a time reparametrization $k : 0 \in \widetilde{I} \subset I_0 \to I$, $k(0) = 0$ such that $\widetilde{x}\ (t) = x(k(t))$ for all $t \in \widetilde{I}$.*

Proof. As $G_x(p)$ and $H_x(p)$ are non-zero and normal to S, there exists a non-zero C^1 function $f : S \to \mathbb{R}$ such that $G_x(p) = f(p)H_x(p)$. Let $k : 0 \in \widetilde{I} \subset \mathbb{R} \to I$ be the solution of

$$\dot{k} = f(x(k(t))), k(0) = 0 \tag{17}$$

Let $\widetilde{r}\ (t) \overset{def}{=} x(k(t))$. Then $\widetilde{r}\ (0) = x_0$ and

$$\frac{d\,\widetilde{r}\ (t)}{dt} = \frac{dx(k(t))}{dt}\,\dot{k} = f(x(k(t)))J_0 H_x(x(k(t))) = J_0 G_x(\widetilde{r}\ (t))$$

By unicity of solutions, we have $\widetilde{x}(t) = \widetilde{r}(t)$ for all $t \in \widetilde{I}$. ▮

The linearized flow of X_H restricted to TS, along a solution $x(t)$, is given by

$$\dot{y} = J_0 H_{xx}(x(t))y \tag{18}$$

where $y(t) \in T_{x(t)}S$ for all $t \in I$. By the same meanings, the linearized flow of X_G restricted to TS along a solution $\widetilde{x}(t)$ is given by

$$\dot{\widetilde{y}} = J_0 G_{xx}(\widetilde{x}(t))\,\widetilde{y} \tag{19}$$

where $\widetilde{y}(t) \in T_{\widetilde{x}(t)}S$. Using the trivialization given by (6), we have the following proposition

Proposition 9. *Considering the same hypotheses and notations of Proposition 8, if $y(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \in T_{x(t)}S$ is a solution of (18) and $\widetilde{y}(t) = (\widetilde{\alpha}_1(t), \widetilde{\alpha}_2(t), \widetilde{\alpha}_3(t)) \in T_{\widetilde{x}(t)}S$ is a solution of (19) with $\widetilde{y}(0) = y(0)$, then $\widetilde{\alpha}_1(t) = \alpha_1(k(t))$ and $\widetilde{\alpha}_2(t) = \alpha_2(k(t))$ for all $t \in \widetilde{I}$, where k is defined by (17).*

Proof. Let

$$M(x) \stackrel{def}{=} -J \begin{pmatrix} \langle H_{xx}X_1, X_1\rangle + \langle H_{xx}X_3, X_3\rangle & \langle H_{xx}X_1, X_2\rangle \\ \langle H_{xx}X_1, X_2\rangle & \langle H_{xx}X_2, X_2\rangle + \langle H_{xx}X_3, X_3\rangle \end{pmatrix}(x) \tag{20}$$

and

$$\widetilde{M}(x) \stackrel{def}{=} -J \begin{pmatrix} \langle G_{xx}X_1, X_1\rangle + \langle G_{xx}X_3, X_3\rangle & \langle G_{xx}X_1, X_2\rangle \\ \langle G_{xx}X_1, X_2\rangle & \langle G_{xx}X_2, X_2\rangle + \langle G_{xx}X_3, X_3\rangle \end{pmatrix}(x) \tag{21}$$

As $G_x(x) = f(x)H_x(x)$, we have $G_{xx}(x)v = f(x)H_{xx}(x)v$ for all $v \in T_xS$. It follows that $\widetilde{M}(x) = f(x)M(x)$. By (11), we obtain

$$\begin{pmatrix} \dot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \end{pmatrix} = M(x(t)) \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}$$

and if $\beta_1(t) \stackrel{def}{=} \alpha_1(k(t))$ and $\beta_2(t) \stackrel{def}{=} \alpha_2(k(t))$ then $\beta_1(0) = \alpha_1(0)$, $\beta_2(0) = \alpha_2(0)$ and

$$\begin{aligned}\begin{pmatrix} \dot{\beta}_1(t) \\ \dot{\beta}_2(t) \end{pmatrix} &= \dot{k} \begin{pmatrix} \dot{\alpha}_1(k(t)) \\ \dot{\alpha}_2(k(t)) \end{pmatrix} = \dot{k}\, M(x(k(t))) \begin{pmatrix} \alpha_1(k(t)) \\ \alpha_2(k(t)) \end{pmatrix} \\ &= f(\tilde{x}(t)) M(\tilde{x}(t)) \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} = \begin{pmatrix} \dot{\tilde{\alpha}}_1(t) \\ \dot{\tilde{\alpha}}_2(t) \end{pmatrix}\end{aligned}$$

By unicity of solutions, $\tilde{\alpha}_i(t) = \beta_i(t)$, $i = 1, 2$. ∎

The expression "Hamiltonian flow in S" will be used whenever it is not necessary to mention the Hamiltonian function which defines the hypersurface S.

4.1. Geometric estimate of the Conley-Zehnder index

The geometric method to calculate the Conley-Zehnder index of a periodic orbit depends on the linearized flow restricted to the contact structure ξ. The trivialization of ξ, given by (7), implies that the rotation angle $\rho(t)$ of the geometric definition of the Conley-Zehnder index can be calculated using equation (11). Proposition 9 shows that this angle is unchanged by the choice of the Hamiltonian function.

For instance, consider the irrational ellipsoid $E = H^{-1}(1)$ where $H = x_1^2 + p_1^2 + \frac{x_2^2+p_2^2}{r^2}$ and r^2 is an irrational number greater than 1. The Hamiltonian vector field X_H in E has exactly 2 periodic orbits given by $P_1 = \{x_1^2 + p_1^2 = 1, x_2 = p_2 = 0\}$ and $P_2 = \{x_2^2 + p_2^2 = r^2, x_1 = p_1 = 0\}$. Both of them are non-degenerate and, therefore, we can calculate $\mu_{CZ}(P_1)$ and $\mu_{CZ}(P_2)$ using the geometric method described above. The projection of the linearized flow on the plane generated by X_1 and X_2 along the periodic orbits P_1 and P_2 is given by equation (11), i.e.,

$$\begin{pmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} = \begin{pmatrix} 0 & -(2 + \frac{2}{r^2}) \\ 2 + \frac{2}{r^2} & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

The minimum period of P_1 is π . The change of the argument of a solution in this period is $(1 + \frac{1}{r^2})2\pi$. As $1 < (1 + \frac{1}{r^2}) < 2$, we have $\mu_{CZ}(P_1) = 3$. The minimum period of P_2 is πr^2 and the change of argument of a solution in this period is $(1 + r^2)2\pi$. We conclude that $\mu_{CZ}(P_2) = 2k + 1$, where k is the integer that satisfies $k < 1 + r^2 < k + 1$.

When $r = 1$ we get the sphere S^3. All of its orbits are periodic and degenerate. By symmetry, the Conley-Zehnder index $\mu_{CZ}(P)$ does not depend on the choice of the periodic orbit P and, as they are degenerate, we cannot use the geometric method to calculate it. However, we can estimate it using Proposition 7. In this case, we get $\widetilde{\mu}_2\ (P) = 4$ and, therefore, $\mu_{CZ}(P) = 3$ or $\mu_{CZ}(P) = 4$. But a strictly convex hypersurface, like the ellipsoid, and in particular the sphere, always has a periodic orbit with Conley-Zehnder index equal to 3 (See [HWZ]). So $\mu_{CZ}(P) = 3$ for all periodic orbits of S^3.

5. INVARIANT SETS OF HYPERSURFACES IN $\mathbb{R}^4$

In this section we present a sufficient condition for the non-existence of periodic orbits in some subsets of a hypersurface in $\mathbb{R}^4$.

Let $S \subset H^{-1}(0)$ be a $C^{k\geq 1}$ connected and orientable hypersurface where $H : \mathbb{R}^4 \to \mathbb{R}$ is a smooth function. We assume that S is invariant by the Hamiltonian flow associated to H.

THEOREM 10. *Let $K \subset S$ be a proper compact subset of S. Suppose that there exists a vector $N \in \mathbb{R}^4$, such that for all $x \in K$, we have $\langle X_0(x), N\rangle > 0$, where $X_0(x)$ is given by (4). Then every solution of the Hamiltonian flow in S through a point $p \in K$ has points in $S\backslash K$ both forward and backward in time. In particular, there is no periodic orbits totally inside K.*

Proof. Let $X_i = A_i N$, $i = 1, 2, 3$ where A_i are defined by (2). The set $X \overset{def}{=} \{X_1, X_2, X_3, N\}$ defines an orthonormal basis for $\mathbb{R}^4$. If $x \in S$, then $x = x_1X_1 + x_2X_2 + x_3X_3 + x_4N$. Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \in S$ be a solution of the Hamiltonian flow in this basis such that $x(0) = p \in K$. We have

$$\dot{x}\ (t) = \dot{x}_1\ (t)X_1 + \dot{x}_2\ (t)X_2 + \dot{x}_3\ (t)X_3 + \dot{x}_4\ (t)N$$

$$\dot{x}_3\ (t) = \left\langle \dot{x}\ (t), X_3 \right\rangle$$

By hypothesis, we have that $\langle X_0(x), N\rangle > 0$ for all $x \in U_K$, where U_K is a neighborhood of K in S. Using $A_3^t = -A_3$, $A_3A_3 = -I_{4\times 4}$ and that K is compact, we obtain the existence of a constant $\varepsilon > 0$ such that

$$\begin{aligned} \dot{x}_3\ (t) &= \left\langle \dot{x}\ (t), X_3 \right\rangle = \langle A_3H_x(x(t)), X_3\rangle = \left\langle H_x(x(t)), A_3^tA_3N \right\rangle \\ &= \|H_x(x(t))\| \left\langle \frac{H_x(x(t))}{\|H_x(x(t))\|}, N \right\rangle = \|H_x(x(t))\| \left\langle X_0(x), N\right\rangle > \varepsilon \end{aligned}$$

It follows that the boundedness of K implies that $x(t)$ cannot stay inside K for $|t|$ arbitrarily large and the theorem is proved. ∎

Now let $E \subset \mathbb{R}^4$ be a hyperplane tangent to S and consider an orthonormal coordinate system (x_1, x_2, x_3, x_4) of $\mathbb{R}^4$ such that $E = \{x_4 = 0\}$. Let $W \subset S$ be the graph of a $C^{k\geq 1}$ function $f : U \subset E \to \mathbb{R}$, i.e., $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | (x_1, x_2, x_3) \in U, x_4 = f(x_1, x_2, x_3)\}$ where U is an open connected subset of E.

Corollary 11. *Let $K \subset W$ be a compact subset of W. Then the thesis of Theorem 10 holds for K.*

Proof. Consider the coordinates (x_1, x_2, x_3, x_4) as before. Then we have

$$X_0(x) = \pm \frac{1}{\|(-f_{x_1}(x), -f_{x_2}(x), -f_{x_3}(x), 1)\|}(-f_{x_1}(x), -f_{x_2}(x), -f_{x_3}(x), 1)$$

It follows that $\langle X_0(x), N\rangle > 0$ for all $x \in S$ where $N = \pm(0, 0, 0, 1)$. Here the symbol $\pm$ means the appropriate choice of $+$ or $-$. The compact K satisfies the hypothesis of Theorem 10 finishing the proof of the corollary. ▮

Consider now $\phi(t, x)$ the Hamiltonian flow associated to H in S. Let $K \subset S$ be a compact subset with the following properties:

(i) K is diffeomorphic to B^3, the 3-dimensional unit ball;

(ii) $\partial K = A^+ \cup A^- \cup \gamma$, where A^+ and A^- are diffeomorphic to D^2 and γ is diffeomorphic to S^1 $(\partial A^+ = \partial A^- = \gamma)$. The vector field $X_H(x)$ is transversal to ∂K for all $x \in A^+ \cup A^-$. If $x \in A^+$ then $\phi(t, x) \in K$ for $t > 0$ small and $\phi(t, x) \notin K$ for $t < 0$ small. If $x \in A^-$ then $\phi(t, x) \notin K$ for $t > 0$ small and $\phi(t, x) \in K$ for $t < 0$ small. The vector field is tangent to K in γ and $\phi(t, x) \notin K$ if $x \in \gamma$ and $t \neq 0$ is small;

(iii) There exists a vector $N \in \mathbb{R}^4$ such that $\langle X_0(x), N\rangle > 0$ for all $x \in K$;

Proposition 12. *Under hypotheses (i), (ii) and (iii), there exists a diffeomorphism $\phi_K : A^+ \to A^-$ which describes the flow $\phi(t, x)$ in K, i.e., if $x \in A^+$, then there exists $t_x > 0$ such that $\phi_K(x) = \phi(t_x, x) \in A^-$ and $\phi(t, x) \in \overset{\circ}{K}$ for all $0 < t < t_x$.*

Proof. By Theorem 10, we know that every solution through a point in K exits K both forward and backward in time. It follows that a solution $x(t)$ satisfying $x(0) \in A^+$ must hit A^-, i.e., there exists $t_x > 0$ such that $x(t_x) \in A^-$ and $x(t) \in \overset{\circ}{K}$ for all $0 < t < t_x$. Let $\phi_K : A^+ \to A^-$ be defined by $\phi_K(x) \overset{def}{=} x(t_x) = \phi(t_x, x)$. The function ϕ_K is well-defined and, considering the vector field $-X_H$, we see that ϕ_K is bijective. The transversality of the vector field in $A^+ \cup A^-$ and the regularity of ∂K implies that ϕ_K is a local diffeomorphism and, therefore, ϕ_K is a diffeomorphism. ▮

6. ESTIMATES OF THE CONLEY-ZEHNDER INDEX OF PERIODIC ORBITS NEAR THE SADDLE-CENTER EQUILIBRIUM

Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a real-analytic function. Let $S \subset H^{-1}(0)$ be a hypersurface satisfying hypotheses (H1)-(H3).

As mentioned before, by Moser's normal form for a saddle-center equilibrium, we can find coordinates in a neighborhood of p_c such that the flow can be represented in a very simple form. We will use these coordinates to estimate the Conley-Zehnder index of periodic orbits in S with points near p_c.

We know that $H_x(p_c) = 0$ and, therefore, in a neighborhood of p_c, the function H is given by

$$H(x) = \frac{1}{2}\langle B(x-p_c),(x-p_c)\rangle + R_0(x-p_c) \tag{22}$$

where $x = (x_1, x_2, x_3, x_4)$ and $\|R_0(x)\| \leq r_0 \|x\|^3$. The matrix B is the Hessian of H at p_c and J_0B has a pair of real eigenvalues $\pm \bar{\alpha}$ ($\bar{\alpha} > 0$), and a pair of pure imaginary eigenvalues $\pm \bar{\omega}\, i$ ($\bar{\omega} > 0$).

Let U be a neighborhood of p_c in $\mathbb{R}^4$ where Moser 's normal form is valid. Let $\phi : V \to U$ be the change of coordinates which conjugates the flow generated by H in U (maybe after changing the sign of H, see [Rag]) with the flow generated by the Hamiltonian $K : V \to \mathbb{R}$ given by

$$K(I_1, I_2) = -\bar{\alpha}\, I_1 + \bar{\omega}\, I_2 + \mathcal{O}(I_1^2 + I_2^2) \tag{23}$$

where $I_1 = q_1p_1$, $I_2 = \frac{q_2^2+p_2^2}{2}$ and the coordinates in V are $y = (q_1, q_2, p_1, p_2)$.

The set $S_0 \stackrel{def}{=} S\backslash\{p_c\}$ is a regular hypersurface in $\mathbb{R}^4$, invariant by the flow of X_H. The vector fields X_i, $i = 1, 2, 3$, given by (5) provide a trivialization of TS_0 as defined in (6). We will start estimating X_i over the stable manifold of p_c.

We know that in V, the local stable manifold of the saddle-center is given by $\tilde{W}^s_V = \{q_1 \in \mathbb{R} | (q_1, 0, 0, 0) \in V\}$, i.e., it is a line segment r in V generated by $v_1 = (1, 0, 0, 0)$. We can assume, without losing generality, that the stable manifold of p_c in S_0 corresponds in the local coordinates to the points in $\tilde{W}^s_V$ which satisfy $q_1 > 0$, i.e., S_0 corresponds to a component in V which projects in the first quadrant of the (q_1, p_1) plane. The case $q_1 < 0$ is identical. Then, by the diffeomorphism ϕ, the local stable manifold of p_c in S_0 can be approximated by the line segment s through p_c in the direction of the vector $u_1 \stackrel{def}{=} D\phi(0)v_1$. We know that $\phi(y) = p_c + D\phi(0)y + L_0(y)$, where $\|L_0(y)\| \leq l_0 \|y\|^2$, $l_0 > 0$, and in S_0, the stable manifold of p_c is locally given by $W^s_{S_0} \stackrel{def}{=} \phi(W^s_V) = \{p_c + q_1 u_1 + Z_0(q_1), 0 < q_1 \leq \delta\}$ where $\|Z_0(x)\| \leq z_0 x^2$ with $z_0 > 0$. Let $X_0(x) = \frac{H_x(x)}{\|H_x(x)\|}$ be the normal vector to S_0 at $x \in S_0$ and $X_i(x)$, $i = 1, 2, 3$, be the vectors given by (5).

LEMMA 13. *If $x \in W^s_{S_0}$, then $X_0(x) \stackrel{x \to p_c}{\longrightarrow} \frac{Bu_1}{\|Bu_1\|}$ and, therefore, $X_i(x) \stackrel{x \to p_c}{\longrightarrow} \frac{A_i B u_1}{\|Bu_1\|}$, $i = 1, 2, 3$.*

Proof. By (22) we have $H_x(x) = B(x - p_c) + R_1(x - p_c)$ where $\|R_1(x)\| \leq r_1 \|x\|^2$. Let $x(q_1) \stackrel{def}{=} \phi(q_1, 0, 0, 0) \in W^s_{S_0}$, we have

$$H_x(x(q_1)) = q_1 B u_1 + B Z_0(q_1) + R_1(q_1 u_1 + Z_0(q_1)) \tag{24}$$

Moreover

$$\begin{aligned} \|BZ_0(q_1)\| &\leq \|B\| \, \|Z_0(q_1)\| \leq \tilde{z}_0 \, q_1^2 \\ \|R_1(q_1 u_1 + Z_0(q_1))\| &\leq r_1 \, \|q_1 u_1 + Z_0(q_1)\|^2 \\ \|q_1 u_1 + Z_0(q_1)\| &\leq ||u_1|| q_1 + z_0 q_1^2 \leq \hat{r}_0 \, q_1 \end{aligned} \tag{25}$$

Using (24) and (25) we have $H_x(x(q_1)) = q_1 B u_1 + R_2(q_1)$ where $\|R_2(q_1)\| \leq r_2 q_1^2$ and $r_2 = r_1 \, \hat{r}_0^2 + \tilde{z}_0 > 0$. It follows that

$$X_0(x(q_1)) = \frac{Bu_1 + R_2(q_1)/q_1}{\|Bu_1 + R_2(q_1)/q_1\|} \stackrel{q_1 \to 0}{\longrightarrow} \frac{Bu_1}{\|Bu_1\|} \tag{26}$$

In $W^s_{S_0}$, $q_1 \to 0$ if and only if $x \to p_c$. ∎

Let $m(t, x)$ be the Hamiltonian flow in U generated by the function (22). Let $n(t, y)$ be the Hamiltonian flow in V associated to the function (23) and let $m_t(x) \stackrel{def}{=} m(t, x)$ and $n_t(y) \stackrel{def}{=} n(t, y)$. These local flows are conjugated by the diffeomorphism $\phi : V \to U$, i.e.,

$$\phi \circ n_t = m_t \circ \phi \tag{27}$$

which implies

$$D_y \phi D_y n_t = D_x m_t D_y \phi \tag{28}$$

Let $x : [0, \infty) \to U$ be a solution of

$$\dot{x} = J_0 H_x(x)$$

which satisfies $\lim_{t\to\infty} x(t) = p_c$, $x(0) \in S_0$. Let $X_i : [0,\infty) \to TS_0$, $i = 1,2,3$, be the orthonormal vectors defined in (5) such that $\{X_i(t)\}_{i=1,2,3}$ generate $T_{x(t)}S_0$. We know that the solution $y : [0,\infty) \to V$ given by $y(t) = n_t(\phi^{-1}(x(0)))$ is conjugated to $x(t)$ by ϕ and corresponds to a local branch of the stable manifold of 0 in V. The solution $y(t)$ satisfies

$$\dot{y} = J_0 K_y(y)$$

where $K_y(y)$ is the gradient vector of K at y. Then $y(t) = (q_{10}e^{-\bar{\alpha} t}, 0, 0, 0)$.

Let $V_0 \overset{def}{=} V\backslash\{0\}$ and consider the orthonormal vectors $\{Y_i(t)\}_{i=1,2,3}$ given by (5) which generate $T_{y(t)}V_0$. In the coordinates $y = (q_1, q_2, p_1, p_2)$ in V, we have

$$\begin{aligned} Y_1(t) &= Y_1 = (0,1,0,0) \\ Y_2(t) &= Y_2 = (0,0,0,-1) \\ Y_3(t) &= Y_3 = (-1,0,0,0) \end{aligned} \tag{29}$$

for all $t \in [0,\infty)$.

Let $v : [0,\infty) \to TS_0$ be a solution of the linearized flow over $x(t)$

$$\dot{v} = J_0 H_{xx}(x(t))v$$

such that $\alpha_1(t)^2 + \alpha_2(t)^2 \neq 0$ where $v(t) = \alpha_1(t)X_1(t) + \alpha_2(t)X_2(t) + \alpha_3(t)X_3(t)$.

We want to estimate the number of turns around the origin of the projection of $v(t)$ into the plane generated by $X_1(t)$ and $X_2(t)$, i.e., the number of laps of the vector $(\alpha_1(t), \alpha_2(t)) \in \mathbb{R}^2$ around the origin. Let $\widetilde{X}_i\,(t) \overset{def}{=} D_x\phi^{-1}(x(t))X_i(t), i = 1,2,3$ and $\widetilde{T}\,(t)$ be the plane generated by the vectors $\widetilde{X}_1\,(t)$ and $\widetilde{X}_2\,(t)$.

As $X_H(x(t))$ is transversal to the plane generated by $X_1(t)$ and $X_2(t)$, we have that $\widetilde{T}\,(t)$ is transversal to $X_K(y(t))$, i.e., $\widetilde{T}\,(t)$ is transversal to $Y_3(t)$. We can, therefore, consider the isomorphism $\pi_t : \widetilde{T}\,(t) \to span\{Y_1(t), Y_2(t)\}$ given by the projection along $Y_3(t)$. Then we define another basis for $\widetilde{T}\,(t)$, $\widetilde{Y} \overset{def}{=} \{\widetilde{Y}_1\,(t), \widetilde{Y}_2\,(t)\}$ given by

$$\widetilde{Y}_i\,(t) = \pi_t^{-1}(Y_i), i = 1,2$$

Defining $w(t) \overset{def}{=} D_x\phi^{-1}(x(t))v(t)$, we have

$$w(t) \;=\; \beta^1(t)Y_1(t) + \beta^2(t)Y_2(t) + \beta^3(t)Y_3(t) \tag{30}$$

$$
\begin{aligned}
&= \alpha^1(t)\, \widetilde{X}_1\,(t) + \alpha^2(t)\, \widetilde{X}_2\,(t) + \alpha^3(t)\, \widetilde{X}_3\,(t) \\
&= \beta^1(t)\, \widetilde{Y}_1\,(t) + \beta^2(t)\, \widetilde{Y}_2\,(t) + \alpha^3(t)\, \widetilde{X}_3\,(t)
\end{aligned}
$$

which implies

$$
\alpha^1(t)\, \widetilde{X}_1\,(t) + \alpha^2(t)\, \widetilde{X}_2\,(t) = \beta^1(t)\, \widetilde{Y}_1\,(t) + \beta^2(t)\, \widetilde{Y}_2\,(t) \tag{31}
$$

By (28), we know that $w(t)$ is solution of

$$
\dot{w}\,(t) = J_0 K_{yy}(y(t))w(t)
$$

where $K_{yy}(y(t))$ is the Hessian Matrix of K in $y(t)$. It is easy to see that

$$
K_{yy}(y(t)) = \begin{pmatrix} 0 & 0 & -\bar{\alpha} & 0 \\ 0 & \bar{\omega} & 0 & 0 \\ -\bar{\alpha} & 0 & c_1 e^{-2\bar{\alpha}t} & 0 \\ 0 & 0 & 0 & \bar{\omega} \end{pmatrix} \tag{32}
$$

We claim that

$$
\begin{pmatrix} \beta^1(t) \\ \beta^2(t) \end{pmatrix} = \begin{pmatrix} \cos\bar{\omega}\,t & -\sin\bar{\omega}\,t \\ \sin\bar{\omega}\,t & \cos\bar{\omega}\,t \end{pmatrix} \begin{pmatrix} \beta^1(0) \\ \beta^2(0) \end{pmatrix}
$$

for all $t \geq 0$.

To see this, let $\beta(t) = \begin{pmatrix} \beta^1(t) \\ \beta^2(t) \end{pmatrix}$. We know from (11) that

$$
\dot{\beta} = -JS\beta
$$

where, by (12),

$$
S = \begin{pmatrix} \langle K_{yy}(y(t))Y_1(t), Y_1(t)\rangle & \langle K_{yy}(y(t))Y_1(t), Y_2(t)\rangle \\ \langle K_{yy}(y(t))Y_1(t), Y_2(t)\rangle & \langle K_{yy}(y(t))Y_2(t), Y_2(t)\rangle \end{pmatrix} + \langle K_{yy}Y_3(t), Y_3(t)\rangle\, I
$$

Then, by (29) and (32), we have

$$
\begin{aligned}
\langle K_{yy}(y(t))Y_1(t), Y_1(t)\rangle &= \bar{\omega} \\
\langle K_{yy}(y(t))Y_1(t), Y_2(t)\rangle &= 0 \\
\langle K_{yy}(y(t))Y_2(t), Y_2(t)\rangle &= \bar{\omega} \\
\langle K_{yy}Y_3(t), Y_3(t)\rangle &= 0
\end{aligned}
$$

We conclude that

$$S(t) = \begin{pmatrix} \bar{\omega} & 0 \\ 0 & \bar{\omega} \end{pmatrix}$$

and, therefore,

$$\dot{\beta} = \begin{pmatrix} 0 & -\bar{\omega} \\ \bar{\omega} & 0 \end{pmatrix} \beta$$

proving the claim.

The solution $(\beta^1(t), \beta^2(t))$ corresponds, therefore, to a circular orbit around the origin with constant angular velocity. It follows that the projection of $w(t)$ in $\widetilde{T}(t)$ turns around the origin counter-clockwise infinitely many times in the basis $\widetilde{Y}$.

By Lemma 13, we know that the vectors $X_i(t)$, $i = 1, 2$ and 3, converge as $t \to \infty$. It follows that $\widetilde{X}_i(t)$ and $\widetilde{Y}_i(t)$ also converge. The vector $(\beta^1(t), \beta^2(t))$ turns around the origin infinitely many times and, therefore, by (31), the vector $\beta^1(t)\ \widetilde{Y}_1(t) + \beta^2(t)\ \widetilde{Y}_2(t)$ also turns around the origin infinitely many times in $\widetilde{X}$. We conclude, finally, that the vector $(\alpha^1(t), \alpha^2(t))$ also turns around the origin counter-clockwise infinitely many times.

This means that the linearized flow over a branch of the stable manifold of the saddle-center has an oscillatory behavior when projected in the plane generated by the vectors X_1 and X_2. Now we are able to prove Theorem 2.

7. PROOF OF THEOREM 2

We know that the equations used to estimate the Conley-Zehnder index of a periodic orbit are

$$\dot{x} = J_0 H_x(x) \tag{33}$$

$$\begin{pmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} = M(x(t)) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{34}$$

where $M(x)$ is defined in (20).

Let $x_s(t)$ be a solution in S which converges to p_c as $t \to \infty$. From the results in the previous section we know that a solution $(\alpha^1(t), \alpha^2(t))$ of the linearized equation (34) over $x_s(t)$ turns counter-clockwise around the origin infinitely many times as $t \to \infty$. We can choose $T_0 > 0$ such that $(\alpha^1(t), \alpha^2(t))$ turns $K+1$ times around the origin between $t = 0$ and $t = T_0$. We can also find a neighborhood W_1 of $x_s(0)$ such that any periodic orbit which crosses W_1 has period greater than T_0.

By continuous dependence of the solutions of (33) and (34) with respect to the initial conditions, there exists a neighborhood $W_0 \subset W_1$ of $x_s(0)$ such that the solution $\alpha(t)$ over any solution $x(t)$ which starts in W_0 turns around the origin at least K many times in the time interval $[0, T_0]$.

By Theorem 5, we know that the solution $(\alpha^1(t), \alpha^2(t))$ always rotate counter-clockwise and, therefore, after completing the minimum period of a periodic orbit which intersects W_0, the solution $(\alpha^1(t), \alpha^2(t))$ will have turned around the origin at least K many times. By Propositions 6 and 7, the Conley-Zehnder index of a periodic orbit can be estimated by the number of total laps of $(\alpha^1(t), \alpha^2(t))$ using the geometric method and, therefore, $\mu_{CZ}(P) > K$ for all periodic orbits crossing W_0. By Moser´s normal form, it is easy to see that it is possible to find a neighborhood W of p_c such that if a periodic orbit P intersects W then P intersects W_0. It follows that $\mu_{CZ}(P) > K$ for all periodic orbits P intersecting W. ▮

8. PROOF OF COROLLARY 3

By Theorem 2 we can find a small neighborhood W of p_c in S such that Moser's normal form is valid and all periodic orbits intersecting W have Conley-Zehnder index greater than 3. Let $S_W \overset{def}{=} S_0 \backslash W$. By a Theorem of M. Ghomi [Gho], it is possible to extend S_W to a strictly convex hypersurface $\widetilde{S}$ which is diffeomorphic to S^3. Applying Theorem 1 for $\widetilde{S}$, we have a periodic orbit $P \subset \widetilde{S}$ with the desired properties. Now we show that $P \subset S_W \subset S_0$. Using normal form of p_c, it is easy to see that we can choose W such that the regularized part $R_W \overset{def}{=} \widetilde{S} \backslash S_W$ satisfies the hypotheses of Proposition 12. It follows that P cannot be totally inside R_W. Also P cannot intersect R_W because $\mu_{CZ}(P) = 3$ and, as in the proof of Theorem 2, all periodic orbits intersecting R_W must have Conley-Zehnder index greater than 3. This implies that $P \subset S_0$, finishing the proof of Corollary 3. ▮

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