Sexta quinzena

Problema 1.

$\Omega \subseteq \mathbb{R}^m$ aberto; $f, g : \Omega \to \mathbb{R}^n$.

i) Ω conexo;

$\exists\, C>0,\ \alpha>1$ t.q.: $\boxed{\|f(x)-f(y)\| \leq C\|x-y\|^\alpha}$ $(*)$

Afirmo: que f é diferenciável em Ω, e $df \equiv 0$.

Prova: Seja $a\in\Omega$, $v\in\mathbb{R}^m$ pequeno t.q. $a+h\in\Omega$.

Basta mostrar que:

$$\lim_{v\to 0} \frac{\|f(a+v)-f(a)-0.v\|}{\|v\|} = 0.$$

Tomo $x = a+v$, $y=a$ em $(*)$:

$$0 \leq \frac{\|f(a+v)-f(a)\|}{\|v\|} \leq C\|v\|^{\alpha-1} \to 0 \text{ qdo. } v\to 0,$$

pois $\alpha - 1 > 0$. //

Agora; fixo $a\in\Omega$ e seja $c = f(a)$.

$\boxed{f^{-1}(c) \text{ é fechado em } \Omega}$ $(\dagger)$

pois f é diferenciável $\Rightarrow$ contínua e $\{c\}\subseteq\mathbb{R}^n$ é fechado.

$x_0 \in f^{-1}(c)$

$\exists\, r>0$ t.q. $B_r(x_0) \subseteq \Omega$.

$B_r(x_0)$ convexa $\Rightarrow \forall\, x\in B_r(x_0)$, vale o DVM:

$$\|f(x)-f(x_0)\| \leq \underbrace{\left(\sup_{[x_0,x]} \|df(x)\|\right)}_{\equiv 0} \|x-x_0\| \Rightarrow f(x) = f(x_0) = c.$$

$\therefore\ B_r(x_0) \subseteq f^{-1}(c) \Rightarrow \boxed{f^{-1}(c) \text{ aberto em } \Omega}$ $(\dagger\dagger)$

$$\Omega \text{ conexo} \xRightarrow[(\dagger\dagger)]{(\dagger)} \Omega = f^{-1}(c) \Rightarrow \boxed{f \equiv c \text{ em } \Omega}$$ //

ii) f dif em $a\in\Omega$,

$(t_k)_{k\in\mathbb{N}}$ com $t_k \to 0$,

$(v_k)_{k\in\mathbb{N}}$ em $\mathbb{R}^m$ com $v_k \to v$

Vou assumir tacitamente que $t_k, v_k \neq 0\ \forall k\in\mathbb{N}$; e que $a + t_k v_k \in \Omega\ \forall k\in\mathbb{N}$ (ou p/ k suficientemente gde).

f diferenciável em a significa que posso escrever:

$$f(a+u) = f(a) + df(a)\cdot u + r(u) \text{ ; onde}$$

$$\lim_{u\to 0} \frac{r(u)}{|u|} = 0.$$

Logo:

$$\left\|\frac{f(a+t_k v_k)-f(a)}{t_k} - df(a)[v]\right\|$$

$$= \left\|\frac{df(a)[t_k v_k] + r(t_k v_k)}{t_k} - df(a)[v]\right\|$$

$$= \left\|\frac{t_k\, df(a)[v_k]}{t_k} + \frac{r(t_k v_k)}{t_k} - df(a)[v]\right\|$$

$$= \left\|df(a)[v_k - v] + \frac{r(t_k v_k)}{t_k}\right\|$$

$$\leq \|df(a)[v_k - v]\| + \frac{\|r(t_k v_k)\|}{|t_k|}$$

$$\leq \|df(a)\| \underbrace{\|v_k - v\|}_{\xrightarrow{k\to\infty} 0} + \underbrace{\|v_k\|}_{\text{limitado}} \underbrace{\left\|\frac{r(t_k v_k)}{\|t_k v_k\|}\right\|}_{\to 0 \text{ qdo. } k\to\infty,}$$

pois $t_k v_k \xrightarrow{k\to\infty} 0$.

$$\therefore \left\|\frac{f(a+t_k v_k)-f(a)}{t_k} - df(a)[v]\right\| \xrightarrow{k\to\infty} 0,$$

que equivale a:

$$\boxed{\lim_{k\to\infty} \frac{f(a+t_k v_k)-f(a)}{t_k} = df(a)[v]}$$

iii) Obs. o enunciado contém um pequeno erro:

Hip: $\exists\, r>0$, $B_r(a) \subseteq \Omega$, e uma função:

$$A: \mathbb{R}^m \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n),$$

contínua em 0, t.q:

$$f(a+h) - f(a) = A(h)\cdot h$$

Quero provar: f é diferenciável em $a \in \Omega$.

Conjectura: $\boxed{df(a) = A(0)}$

Vou mostrar isso se mostrar que:

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - A(0)[h]\|}{\|h\|} = 0.$$

Por hip:

$$\frac{\|f(a+h) - f(a) - A(0)\cdot h\|}{\|h\|} = \frac{\|A(h)\cdot h - A(0)\cdot h\|}{\|h\|}$$

$$= \frac{\|(A(h) - A(0))\cdot h\|}{\|h\|}$$

$$\leq \|A(h) - A(0)\| \xrightarrow{h\to 0} 0,$$ pois

A é contínua em 0.

Logo, vale o desejado.

iv) $\begin{cases} f, g \text{ diferenciáveis em } a \\ f(a) = g(a) \end{cases}$

$\vdash$: $df(a) = dg(a) \iff \lim_{v \to 0} \dfrac{f(a+v) - g(a+v)}{\|v\|} = 0$.

$\Rightarrow$) Por hip:

$f(a+v) = f(a) + df(a)\cdot v + r_1(v)$, $\lim_{v\to 0} \dfrac{r_1(v)}{\|v\|}$;

$g(a+v) = g(a) + dg(a)\cdot v + r_2(v)$; $\lim_{v \to 0} \dfrac{r_2(v)}{\|v\|}$

$$\Rightarrow \frac{f(a+v) - g(a+v)}{\|v\|} = \frac{\cancel{f(a)} + \cancel{df(a)\cdot v} + r_1(v) - (\cancel{g(a)} + \cancel{dg(a)v} + r_2(v))}{\|v\|}$$

$$= \frac{r_1(v) - r_2(v)}{\|v\|} = \frac{r_1(v)}{\|v\|} - \frac{r_2(v)}{\|v\|} \xrightarrow{v \to 0} 0$$

$\Leftarrow$) Seja $v \in \mathbb{R}^m \setminus \{0\}$, fixado.

$v_k = \dfrac{v}{k} \Rightarrow v_k \to 0$ qdo. $k \to \infty$.

$\therefore$ $a + v_k \in \Omega$ qdo. k é gde. o suficiente.

Por hip, temos ...

$df(a)\cdot v_k - dg(a)\cdot v_k =$

$= f(a+v_k) - \cancel{f(a)} - r_1(v_k) - (g(a+v_k) - \cancel{g(a)} - r_2(v_k))$

$= f(a+v_k) - g(a+v_k) + r_2(v_k) - r_1(v_k)$

$$\frac{df(a)\cdot v_k - dg(a)\cdot v_k}{\|v_k\|} = \frac{f(a+v_k) - g(a+v_k)}{\|v_k\|} + \frac{r_2(v_k)}{\|v_k\|} - \frac{r_1(v_k)}{\|v_k\|}. \quad (*)$$

$$\frac{df(a)\cdot v_k}{\|v_k\|} = \frac{1}{\left\|\frac{v}{k}\right\|} df(a)\left[\frac{v}{k}\right] = \frac{\cancel{k}}{\|v\|}\frac{1}{\cancel{k}} df(a)[v] = \frac{df(a)[v]}{\|v\|}$$

Analogamente p/ $dg(a)$. Concluo que:

$$\frac{df(a)\, v_k - dg(a)\cdot v_k}{\|v_k\|} = \frac{[df(a) - dg(a)][v]}{\|v\|} \text{ ñ}$$

depende de k. Fazendo $k \to \infty$ em $(*)$ concluo:

$$\forall v \in \mathbb{R}^n \setminus \{0\}, \ \frac{[df(a) - dg(a)][v]}{\|v\|} = 0$$

$$\Rightarrow \boxed{df(a)\cdot v = dg(a)\cdot v}$$

Como v era qualquer, coincidem. //

Problema 2

i) Pela regra da cadeia:

$$d(\varphi\circ f)(x) = d\varphi(f(x))\circ df(x)$$

Mas:

$\nabla\varphi(f(x))$ é a matriz de $d\varphi(f(x))\in(\mathbb{R}^m)^*$

$Jf(x)$ é a matriz de $df(x)\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)$

$\Rightarrow$ a matriz de $d\varphi(f(x))\circ df(x)$ é o produto das matrizes:

$$d\varphi(f(x))\circ df(x)[v] = \underbrace{\nabla\varphi(f(x)).\,Jf(x).v}_{=Tv}$$

ii) $\varphi\circ f(x) = c\ \forall x \Rightarrow d(\varphi\circ f)(x)=0\ \forall x$

$\Rightarrow \nabla\varphi(f(x))\,Jf(x)=0\ \forall x$

$\overset{\text{em}}{\Rightarrow}$ particular $\nabla\varphi(f(a))\,Jf(a)v=0\ \forall v\in\mathbb{R}^m$ (*)

Se $Jf(a)$ fosse invertível, seria isomorfismo, d'onde (*) implicaria:

$$\nabla\varphi(f(a))\,Jf(a)(\mathbb{R}^n) = \nabla f(a).\mathbb{R}^n=0$$

$\Rightarrow \nabla f(a)=0$, contradição!

iii) $x\mapsto\|f(x)\|$ cte. $\Rightarrow F:\Omega\to\mathbb{R}$ é cte.

$F(x)=\|f(x)\|^2=\langle f(x),f(x)\rangle$

$\varphi:\mathbb{R}^m\to\mathbb{R}$

$F(x)=\varphi\circ f(x)$, onde: $\varphi(x)=\langle x,x\rangle$, $\nabla\varphi(x)=2x$, e $\varphi\circ f\equiv c$

$\overset{i)}{\Rightarrow} dF(x)[v]=\nabla\varphi(f(x))\,Jf(x)[v] = 2\langle f(x), Jf(x)\cdot v\rangle\ \forall v\in\mathbb{R}^m$

Se $\|F\|\equiv0\Rightarrow f\equiv0\Rightarrow Jf=0$, óbvio

Se $\|f(x)\|\neq0\Rightarrow f(x)\neq0\ \forall x\in\mathbb{R}^m$.

$\Rightarrow \langle f(x), Jf(x)\cdot e_i\rangle=0,\ i=1,\dots,m$

$\Rightarrow \{Jf(x)e_1,\dots,Jf(x)e_m\}$ é um conjunto de m vetores no espaço $(m-1)$-dimensional $[f(x)]^\perp\Rightarrow$ LD $\Rightarrow$ Como compõem as colunas de $Jf(x)$, $\det Jf(x)=0$

Como x era qqer:

$$\boxed{Jf(x)\ \tilde{n}\text{ é invertível } \forall x}$$

iv) $\mu:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$, $\mu(x,y)=\langle x,y\rangle$ é bilinear

$$\Rightarrow \boxed{d\mu(x,y)\cdot(v,w) = \mu(v,y)+\mu(x,w)}$$

$$h : \Omega \to \mathbb{R}$$
$$x \mapsto \langle f(x), g(x) \rangle$$

$$\Longrightarrow \quad h(x) = \mu(f(x), g(x))$$

$$\underset{\text{Cadeia}}{\overset{\text{Regra}}{\Longrightarrow}} \quad dh(x)[v] = d\mu\big(f(x), g(x)\big) \circ d(f, g)(x)[v]$$
$$= d\mu(f(x), g(x))\big(df(x)[v], dg(x)[v]\big)$$
$$= \mu\big(df(x)[v], g(x)\big) + \mu\big(f(x), dg(x)[v]\big)$$
$$= \langle df(x)[v], g(x) \rangle + \langle f(x), dg(x)[v] \rangle$$

$$\overset{\text{hip.}}{\Longrightarrow} \quad dh(x)[v] = 0 \ \forall v \in \mathbb{R}^m \Longrightarrow dh = 0 \underset{\text{conexo}}{\overset{\Omega}{\Longrightarrow}} \boxed{h = \langle f, g \rangle \ \text{cte.}}$$

Problema 3.

i) $f : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^n$.

Ω aberto convexo

$a, b \in \Omega \Rightarrow [a, b] \subseteq \Omega$. Dado $y \in \mathbb{R}^n$ seja:

$$\phi_y : [0,1] \longrightarrow \mathbb{R}$$

$$\phi_y(t) = \langle f(a + t(b-a)), y \rangle$$

$\Rightarrow$ ϕ_y é contínua em $[0,1]$ $\overset{\text{TVM}}{\Longrightarrow}$ $\exists\ \theta = \theta_y \in (0,1)$ tq
ϕ_y é derivável em $(0,1)$ na reta

$$\phi_y(1) - \phi_y(0) = \phi_y'(\theta)$$

$c_y \overset{\text{def}}{=} \lambda(\theta) \in [a, b]$ (ver abaixo)

$\phi_y' = \varphi \circ f \circ \lambda$, onde: $\varphi(x) = \langle x, y \rangle$, $\quad d\varphi(x)[v] = \langle v, y \rangle$
$\lambda(t) = a + t(b-a)$, $\quad \lambda' = (b-a)$

$$\Rightarrow \phi_y' = d\varphi(f \circ \lambda) \circ df(\lambda)[\lambda']$$

$$\Rightarrow \phi_y'(\theta) = d\varphi(f \circ \lambda(\theta)) \circ df(\lambda(\theta))[b-a]$$
$$= d\varphi(f(c_y)) \cdot \big[df(c_y)[b-a]\big]$$
$$= \langle df(c_y)[b-a], y \rangle$$

$$\phi_y(1) - \phi_y(0) = \langle f(\lambda(1)), y \rangle - \langle f(\lambda(0)), y \rangle = \langle f(b) - f(a), y \rangle$$

$$\Longrightarrow \quad \boxed{\langle f(b) - f(a), y \rangle = \langle df(c_y)[b-a], y \rangle}$$

ii) Sejam $x \neq y \in \Omega$ arbitrários

$$\lambda : [0,1] \to \Omega$$
$$\lambda(t) = x + t(y - x)$$

$$g : [0,1] \to \mathbb{R}^n$$
$$g(t) = \frac{f(\lambda(t))}{\|y - x\|}$$

$$g'(t) = \frac{1}{\|y-x\|}\, df(\lambda(t))[\lambda'(t)] = \frac{df(\lambda(t))[y-x]}{\|y-x\|}$$

$$\forall t\in(0,1),\ \|g'(t)\| = \frac{\|df(\lambda(t))[y-x]\|}{\|y-x\|} \leq \frac{\|df(\lambda(t))\|\,\|y-x\|}{\|y-x\|} = \|df(\lambda(t))\|$$

$$\underbrace{\|g(1)-g(0)\|}_{\frac{\|f(y)-f(x)\|}{\|y-x\|}} = \left\|\int_0^1 g'(t)\,dt\right\| \leq \int_0^1 \|g'(t)\|\,dt \leq \int_0^1 \|df(\lambda(t))\|\,dt \leq \int_0^1 \sup_\Omega \|df(z)\|\,dt$$

$$\Rightarrow \boxed{\sup_{x\neq y} \frac{\|f(x)-f(y)\|}{\|y-x\|} \leq \sup_{z\in\Omega} \|df(z)\|}$$

$z\in\Omega$, $h\neq 0$, $h_k = \frac{h}{k}$, k gde

$$f(z+h_k) = f(z) + df(z)[h_k] + \|h_k\|\rho(h_k)$$

$$\Rightarrow \frac{\|df(z)[h]\|}{\|h\|} \leq \underbrace{\frac{\|f(z+h_k)-f(z)\|}{\|h_k\|}}_{\leq \sup_{x\neq y}\frac{\|f(x)-f(y)\|}{\|x-y\|}} + \overbrace{\|\rho(h_k)\|}^{\to 0} \quad k\to\infty$$

$$\Rightarrow \forall h\neq 0 \quad \frac{\|df(z)h\|}{\|h\|} \leq \sup_{x\neq y}\frac{\|f(x)-f(y)\|}{\|y-x\|} \Rightarrow \|df(z)\| \leq \sup_{x\neq y}\frac{\|f(x)-f(y)\|}{\|y-x\|}$$

$$\Rightarrow \boxed{\sup_\Omega \|df(z)\| \leq \sup_{x\neq y}\frac{\|f(y)-f(x)\|}{\|y-x\|}}$$

iii)

$\Rightarrow)$ $x=y$ trivial

$x,y\in\Omega$, $x\neq y$

$$\overset{ii)}{\Rightarrow} \frac{\|f(x)-f(y)\|}{\|y-x\|} \leq \sup_\Omega \|df(z)\| \leq c$$

$$\Rightarrow \boxed{\|f(x)-f(y)\| \leq c\|y-x\|}$$

$\Leftarrow)$ $\forall x\neq y$, $\frac{\|f(y)-f(x)\|}{\|y-x\|} \leq c \Rightarrow \underbrace{\sup_{x\neq y}\frac{\|\cdot-\cdot\|}{\|\cdot\|}}_{\overset{ii)}{=}\sup_\Omega\|df(z)\|} \leq c$

$$\Rightarrow \boxed{\|df(z)\| \leq c\ \forall z\in\Omega}$$

iv) Objetivo:

probar que $\|g(x)\|\to 0$ qdo. $\|x\|\to+\infty$

I.e:

$\forall\varepsilon>0\ \exists R>0$:

$\|x\|>R \Rightarrow \|g(x)\|<\varepsilon$.

Hip: $\lim_{x\to\infty} df(x)[x] = 0$

$\Rightarrow \forall\varepsilon>0\ \exists R>0$: (*)

$\|x\|>0 \Rightarrow \|df(x)[x]\| < \frac{\varepsilon}{\log 2}$.

Dado x com $\|x\| > R$, seja:

$$\phi_x : [0,1] \longrightarrow \mathbb{R}^n$$

$$\phi_x(t) = f\left((1+t)\,x\right)$$

$$\Longrightarrow g(x) = \phi_x(1) - \phi_x(0).$$

$$\phi_x'(t) = df\left((1+t)\,x\right)[x]$$

$$g(x) = \int_0^1 \phi_x'(t)\,dt = \int_0^1 df\left((1+t)\,x\right)[x]\,dt$$

$$= \int_0^1 \frac{df\left((1+t)\,x\right)[(1+t)\,x]}{1+t}\,dt$$

$$y_t = (1+t)\,x \Longrightarrow \|y_t\| = (1+t)\,\|x\| \geq \|x\| > R$$

$$\Longrightarrow \|df(y_t)[y_t]\| < \frac{\varepsilon}{\log 2}$$

$$\overset{(*)}{\Longrightarrow} \|g(x)\| \leq \int_0^1 \frac{\|df(y_t)[y_t]\|}{1+t}\,dt$$

$$< \frac{\varepsilon}{\log 2} \int_0^1 \frac{dt}{1+t} = \varepsilon$$

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