Quinta quinzena

Problema 1.

$f, g : \mathbb{R}^2 \xrightarrow{C^2} \mathbb{R}$

i) Hip: $\exists\, a \in \mathbb{R}^2$ t.q: $\begin{cases} f(a) = g(a) = 0 \\ df(a) = df(a) = 0 \end{cases}$, $d^2f(x) = d^2g(x)\ \forall x \in \mathbb{R}^2$

T: $f = g$

Seja $h \overset{def}{=} f - g : \mathbb{R}^2 \xrightarrow{C^2} \mathbb{R} \overset{hip.}{\Rightarrow} h(a) = 0,\ dh(a) = 0,\ d^2h(x) = 0\ \forall x \in \mathbb{R}^2.$

Taylor com resto de Lagrange:

$$h(a+v) = h(a) + df(a)\cdot v + r_1(v) = r_1(v)$$

onde $r_1(v) = \frac{1}{2} d^2h(a+\theta v)\cdot v^{(2)} = 0$

$\Rightarrow \forall v \in \mathbb{R}^2,\ h(a+v) = 0 \Rightarrow \boxed{h = 0}$ //

ii) $f(x,y) = \int_x^y [3t^2 - 1]\, dt = \int_x^0 [\cdots]\, dt + \int_0^y [\cdots]\, dt \Rightarrow$

$$\boxed{f(x,y) = \int_0^y [3t^2-1]\, dt - \int_0^x [3t^2-1]\, dt}$$

$\frac{\partial f}{\partial x} = -3x^2 + 1$, $\frac{\partial f}{\partial y} = 3y^2 - 1$, (obs: $f \in C^\infty$)

Ptos. críticos $df(x_0, y_0) = 0$

$\Rightarrow \begin{cases} -3x_0^2 + 1 = 0 \\ 3y_0^2 - 1 = 0 \end{cases}$

$x_0^2 = \frac{1}{3}$, $y_0^2 = \frac{1}{3} \Rightarrow x_0 = \pm \frac{1}{\sqrt{3}}$, $y_0 = \pm \frac{1}{\sqrt{3}}$

$$\boxed{\text{Crit } f = \left\{ \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right), \left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right) \right\}}$$

Vamos calcular d^2f:

$\frac{\partial^2 f}{\partial x^2} = -6x$, $\frac{\partial^2 f}{\partial y^2} = 6y$,

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

$$\Rightarrow \boxed{d^2f(x,y) = \begin{bmatrix} -6x & 0 \\ 0 & 6y \end{bmatrix}}$$

Vamos lembrar do teorema:

$$\begin{cases} d^2f(a) \text{ positiva} \Rightarrow a \text{ mínimo local} \\ d^2f(a) \text{ negativa} \Rightarrow a \text{ máximo local} \\ d^2f(a) \text{ indefinida} \Rightarrow a \text{ pto. de sela} \end{cases}$$

Ptos. de interesse:

- $d^2f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = d^2f\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) = \begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} \end{bmatrix}$ indefinida

$d^2f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{bmatrix}$ negativa

$d^2f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} \end{bmatrix}$ positiva

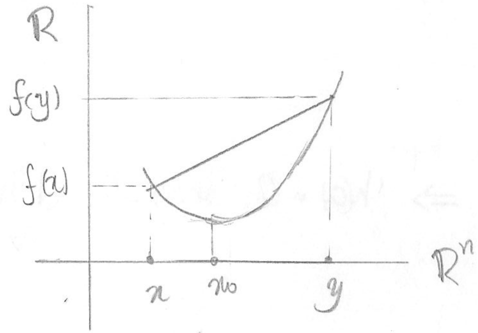
$d^2f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{bmatrix}$ indefinida

Conclusão:

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ e $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ são pto. de sela
$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ é máximo local
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ é mínimo local //

iii) f é côncava $\Rightarrow$ $-f$ é convexa

$$-f((1-t)x+ty) \leq (1-t)(-f(x)) + t(-f(y))$$
$$\Updownarrow$$
$$f((1-t)x+ty) \geq (1-t)f(x) + t f(y) \quad \forall t \in [0,1]$$



Quero provar:

x_0 pto. crit. $f \Rightarrow x_0$ máx. global de f

Outra caracterização:

$-f$ é convexa $\Leftrightarrow d^2(-f)(x)$ é uma forma quadrática ñ-negativa $\forall x \in \mathbb{R}^n$

$\Leftrightarrow d^2f(x)$ é uma forma quadrática ñ-positiva $\forall x \in \mathbb{R}^n$, i.e.

$$\forall v = (v_1, v_2), \quad \underbrace{\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) v_i v_j \leq 0}_{d^2f(x)\cdot v^{(2)} \leq 0}$$

Taylor com resto de Lagrange...

$f(x_0+v) = f(x_0) + \underbrace{df(x_0)\cdot v}_{=0} + \underbrace{\frac{1}{2}\, d^2 f(x_0+\theta v)\cdot v^{(2)}}_{\leq 0}$, $\theta \in (0,1)$

$\Rightarrow \quad f(x_0+v) - f(x_0) \leq 0 \quad \forall\, v \in \mathbb{R}^n$

$\Rightarrow \quad \boxed{f(x_0) \geq f(x_0+v) \quad \forall\, v\in\mathbb{R}^n} \Rightarrow x_0$ máx. global. //

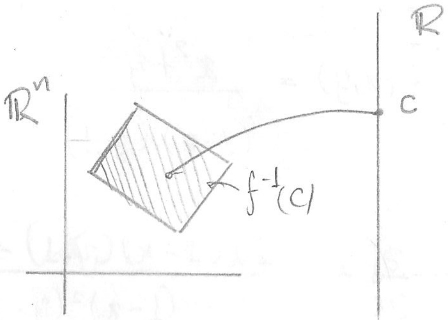
Problema 2

$f : \mathbb{R}^n \xrightarrow{C^1} \mathbb{R}$, $c \in \mathbb{R}$ t.q. $f^{-1}(c) \neq \emptyset$, cpto.

i) $n \geq 2$

$F := \{x\in\mathbb{R}^n : f(x) \geq c\}$

$G := \{x\in\mathbb{R}^n : f(x) \leq c\}$



†: Ao menos um dentre F ou G é cpto.

Fato 1. F, G são ambos fechados.

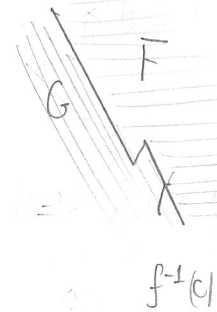
Prova. $F = \underbrace{f^{-1}}_{\text{contínua}} \underbrace{(-\infty, c]}_{\text{fechado}}$, e análogamente p/ G. #

não[F cpto ou G cpto] $\Leftrightarrow \underbrace{F \text{ ñ cpto e } G \text{ ñ cpto}}_{⊛}$

Suponho ⊛ por absurdo.

⊛ $\Leftrightarrow$ F ilimitado & G ilimitado

$$\boxed{\begin{array}{l} \mathbb{R}^n = F\cup G \\ F\cap G = f^{-1}(c) \end{array}}$$



$f^{-1}(c)$ cpto. $\Rightarrow \exists\, M>0$ t.q.

$f^{-1}(c) \subseteq \underbrace{[-M,M]^n}_{:=Q}$

F & G ilimitados $\Rightarrow \exists\, x\in F$, $y \in G$ t.q.:

$\|x\|_\infty > M$, $\|y\|_\infty > M$ e $f(x) < c$, $f(y) > c$.

$\therefore \quad x, y \in \mathbb{R}^n \setminus Q$.

pb. $n \geq 2$

$\mathbb{R}^n \setminus Q$ é conexo por caminhos $\Rightarrow \exists$ caminho $\gamma : [0,1] \xrightarrow{C^0} \mathbb{R}^n\setminus Q$ ligando x a y.

Considere $\begin{cases} \phi : [0,1] \longrightarrow \mathbb{R} \\ \phi = f\circ\gamma \end{cases}$

ϕ é contínua e $\begin{cases} \phi(0) = f(x) < c \\ \phi(1) = f(y) > c \end{cases}$

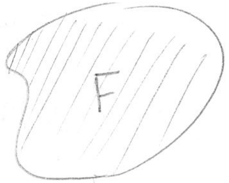
$\therefore \exists\, \theta \in (0,1)$ t.q. $\phi(\theta) = c \Rightarrow f(\gamma(\theta)) = c$

$\therefore\ \gamma(\theta) \in f^{-1}(c)$, mas $\gamma(\theta) \notin \Omega$, contradição!

Logo, ao menos um dentre F ou G é limitado e portanto cpto. #

ii) Já sabemos que ao menos um dentre F e G é cpto. Suponha F. Então $x_0 \in F$ t.q.

$$f(x_0) = \min_F f$$



Afirmo que x_0 é min de f em $\mathbb{R}^n$.

De fato, dado $x \in \mathbb{R}^n$:

se $f(x) \geq c \Rightarrow f(x) \geq f(x_0)$, pois $x_0 \in F$, donde $f(x_0) \leq c$;

se $f(x) \leq c \Rightarrow x \in F \Rightarrow f(x) \geq f(x_0)$, pois $f(x_0) = \min_F f$.

$$\therefore\ \boxed{\forall x \in \mathbb{R}^n,\ f(x) \geq f(x_0)}$$

O caso G cpto é análogo. #

iii) Não podemos concluir a mesma coisa.

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x \Rightarrow \forall\ c \in \mathbb{R},\ f^{-1}(c) = \{c\}$ é cpto. e ñ-vazio, mas f ñ assume máximo nem mínimo e

$$F = (-\infty, c],\quad G = [c, +\infty)$$

ñ são cptos.

iv) $\Omega = (-1, 1)^2$

$$f(x,y) = \frac{x^2+y^2}{(1-x)(y-1)}$$

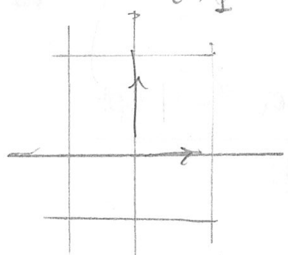
$$\frac{\partial f}{\partial x} = \frac{2x(1-x)\cancel{(y-1)} - x^2(-1)\cancel{(y-1)}}{(1-x)^2(y-1)^2}$$

$$\frac{\partial f}{\partial x} = -\frac{x^2-2x}{(1-x)^2(y-1)}\ ;\quad \frac{\partial f}{\partial y} = \frac{y^2-2y}{(y-1)^2(1-x)}$$

?

checar

$$\lim_{t \to 1^-} f(t,0) = \lim_{t \to 1^-} \frac{t^2}{1-t} = +\infty$$



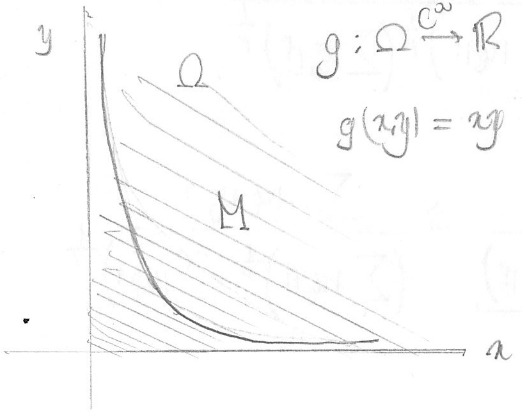
$$\lim_{t \to 1^-} f(0,t) = \frac{t^2}{t-1} = -\infty$$

#

$M = \{(x,y)\in\mathbb{R}^2 : x>0,\ y>0,\ xy=1\}$

Problema 3.

i) $\Omega = \{(x,y)\in\mathbb{R}^2 : x>0,\ y>0\}$



$g : \Omega \xrightarrow{C^\infty} \mathbb{R}$

$g(x,y) = xy \Rightarrow \frac{\partial g}{\partial x} = y,\ \frac{\partial g}{\partial y} = x$

$\boxed{\therefore\ \nabla g(x,y) \neq (0,0)\ \forall (x,y)\in\Omega}$

$\Rightarrow$ 1 valor regular de g

$\Rightarrow \boxed{M = g^{-1}(0)}$ hiperfície C^∞ de $\mathbb{R}^2$. //

ii) $f(x,y) = \frac{x^p}{p} + \frac{y^q}{q}$, $p,q>0$, $\frac{1}{p}+\frac{1}{q}=1$

Em M, $y=\frac{1}{x} \Rightarrow f\left(x,\frac{1}{x}\right) = \frac{x^p}{p} + \frac{1}{q\,x^q} > 0$ em M,

e: $f\left(x,\frac{1}{x}\right) \longrightarrow +\infty$ qdo. $x \to 0^+$ e qdo. $x \to +\infty$

$\therefore\ \exists$ ao (menos um) pto. de mínimo p/ f em M.

Vamos procurar os ptos. críticos de $f|_M$:

(Multiplicadores de Lagrange) $\nabla f = \lambda \nabla g$, $g=1$

$\frac{\partial f}{\partial x} = \frac{p\,x^{p-1}}{p} = x^{p-1}$

$\frac{\partial f}{\partial y} = \frac{q\,y^{q-1}}{q} = y^{q-1}$

Obs. $\underbrace{\frac{1}{p}}_{>0} + \underbrace{\frac{1}{q}}_{>0} = 1 \Rightarrow \frac{1}{p} < 1 \Rightarrow p>1 \Rightarrow \boxed{p-1>0}$

Analogamente: $\boxed{q-1>0}$

$\nabla f = (x^{p-1}, y^{q-1})$, $\nabla g = (y, x)$.

$$(*)\ \begin{cases} x^{p-1} = \lambda y \\ y^{q-1} = \lambda x \\ xy = 1 \end{cases}$$

$$y = \frac{1}{x} \Rightarrow \begin{cases} x^{p-1} = \frac{\lambda}{x} \\ \frac{1}{x^{q-1}} = \lambda x \end{cases} \Leftrightarrow \begin{cases} \boxed{x^p = \lambda} \Rightarrow x = \lambda^{\frac{1}{p}} > 0 \\ 1 = \lambda x^q \Rightarrow 1 = \lambda\left(\lambda^{\frac{q}{p}}\right) \end{cases}$$

$\Rightarrow \lambda^{1+\frac{q}{p}} = 1 \Rightarrow \lambda^{\frac{p+q}{p}} = 1 \overset{(*)}{\Rightarrow} \lambda^q = 1 \Rightarrow \lambda = 1^{\frac{1}{q}} \Rightarrow \boxed{\lambda = 1}$

$\boxed{(*)\ \frac{1}{p}+\frac{1}{q} = \frac{p+q}{pq} = 1}$

$\Rightarrow x=1,\ y=1 \Rightarrow$ único pto. crítico: $\boxed{\mathrm{Crit}\, f|_M = \{(1,1)\}}$

$$\Rightarrow \forall (x,y) \in M, \quad f(x,y) \geq f(1,1) = 1$$

$$\Rightarrow \boxed{\forall\, x,y \text{ t.q. } xy = 1, \quad \frac{x^p}{p} + \frac{y^q}{q} \geq 1}$$

iii)

Dados $x>0$, $y>0$, sejam

$$\tilde{x} = \frac{x}{(xy)^{\frac{1}{p}}} > 0, \quad \tilde{y} = \frac{y}{(xy)^{\frac{1}{q}}} > 0$$

$$\Rightarrow \tilde{x}\tilde{y} = \frac{x}{(xy)^{\frac{1}{p}}} \frac{y}{(xy)^{\frac{1}{q}}} = \frac{xy}{(xy)^{\frac{1}{p}+\frac{1}{q}}} = 1$$

ii)

$$\Rightarrow \frac{\tilde{x}^p}{p} + \frac{\tilde{y}^q}{q} \geq 1 \Rightarrow \frac{x^p}{p(xy)} + \frac{y^q}{q(xy)} \geq 1$$

$$\therefore \boxed{\frac{x^p}{p} + \frac{y^q}{q} \geq xy}$$

iv) $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n$

P/ cada $j \in \{1, \dots, n\}$, tome:

$$x_j = \frac{|u_j|}{\left(\sum |u_j|^p\right)^{\frac{1}{p}}}, \quad y_j = \frac{|v_j|}{\left(\sum |v_j|^q\right)^{\frac{1}{q}}}$$

iii)

$$\Rightarrow \frac{x_j^p}{p} + \frac{y_j^q}{q} \geq x_j y_j$$

$$\Rightarrow \frac{|u_j|^p}{p\left(\sum |u_j|^p\right)} + \frac{|v_j|^q}{q\left(\sum |v_j|^q\right)} \geq \frac{|u_j|\,|v_j|}{\left(\sum |u_j|^p\right)^{\frac{1}{p}}\left(\sum |v_j|^p\right)^{\frac{1}{p}}}$$

somando:

$$\Rightarrow \frac{1}{p} \underbrace{\sum_j \frac{|u_j|^p}{\left(\sum |u_j|^p\right)}}_{=1} + \frac{1}{q} \underbrace{\sum_j \frac{|v_j|^p}{\left(\sum |v_j|^p\right)}}_{=1} \geq \frac{\sum |u_j v_j|}{\left(\sum |u_j|^p\right)^{\frac{1}{p}} \left(\sum |v_j|^q\right)^{\frac{1}{q}}}$$

$$\Rightarrow \frac{\sum |u_j v_j|}{\left(\sum |u_j|^p\right)^{\frac{1}{p}} \left(\sum |v_j|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \sum |u_j v_j| \leq \left(\sum |u_j|^p\right)^{\frac{1}{p}} \left(\sum |v_j|^q\right)^{\frac{1}{q}}$$

Mas: $\left|\sum u_j v_j\right| \leq \sum |u_j v_j|$. Logo:

$$\boxed{\left|\sum_{j=1}^n u_j v_j\right| \leq \left(\sum_{j=1}^n |u_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |v_j|^q\right)^{\frac{1}{q}}}$$

(Desigualdade de Hölder). //