Problema 1. **Quarta quinzena**

$f: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$

homogênea de grau $k \in \mathbb{R}$: $\boxed{f(tx) = t^k f(x) \ \forall t > 0}$

i) $f \in C^1(\mathbb{R}^n \setminus \{0\})$

① $\forall j \in \{1, \dots, n\}$, $\frac{\partial f}{\partial x_j} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ homogênea do grau $k-1$.

Prova:

Sejam $x \in \mathbb{R}^n \setminus \{0\}$, $t > 0$ fixado

$$\frac{\partial f}{\partial x_j}(tx) = \lim_{h \to 0} \frac{f(tx + he_j) - f(tx)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(t\left[x + \frac{h}{t} e_j\right]\right) - f(tx)}{h}$$

$$= \lim_{h \to 0} \frac{t^k \left[f\left(x + \frac{h}{t} e_j\right) - f(x)\right]}{h}$$

$$= t^{k-1} \lim_{h \to 0} \frac{f\left(x + \frac{h}{t} e_j\right) - f(x)}{(h/t)}$$

$$= t^{k-1} \frac{\partial f}{\partial x_j}(x).$$

② $f \in C^\infty(\mathbb{R}^n \setminus \{0\}) \Rightarrow \forall \ell$, toda derivada de ordem ℓ é homogênea de grau $k - \ell$

Prova:

Caso base: $\ell = 1$ ok.

Suponha p/ ℓ.

Quero provar que se $i_1, \dots, i_{\ell+1} \in \{1, \dots, n\}$, então:

$$\frac{\partial^{\ell+1} f}{\partial x_{i_1} \cdots \partial x_{i_{\ell+1}}} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$$

é homogênea do grau $k - \ell$.

HI: $\frac{\partial^\ell f}{\partial x_{i_1} \cdots \partial x_{i_\ell}}$ é homogênea de grau $k - \ell$

$\overset{①}{\Longrightarrow} \frac{\partial^{\ell+1} f}{\partial x_{i_1} \cdots \partial x_{i_{\ell+1}}}$ é homogênea de grau $(k-\ell) - 1 = k - (\ell+1)$. //

ii) $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ homogênea de grau k. Queremos:

$$\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)\, x_j = k f(x)$$

Solução: fixe $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Seja $g: (0, +\infty) \to \mathbb{R}$, $g(t) = f(tx)$ $\overset{\text{hip.}}{\Longrightarrow}$ $g(t) = t^k f(x)$

Por um lado:

$$g'(t) = k\, t^{k-1} f(x)$$

Por outro:

$$g'(t) = (f \circ \gamma)'(t) \quad ; \quad \text{onde:}$$

$$\gamma(t) = tx$$

$$\overset{\text{Regra}}{\underset{\text{Cadeia}}{\Longrightarrow}} \quad g'(t) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle$$

$$= \langle \nabla f(tx), x \rangle$$

$$= \sum \frac{\partial f}{\partial x_j}(tx)\, x_j$$

Em particular qdo. $t = 1$: $\boxed{\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)\, x_j = k f(x)}$ //

iii) $f(x) = \langle x, x \rangle^{\frac{k}{2}} = \|x\|^k$ é homogênea do grau k

$$f(x) = \left(\sum_i x_i^2\right)^{\frac{k}{2}}$$

$$\Longrightarrow \frac{\partial f}{\partial x_j} = \frac{k}{\cancel{2}}\left(\sum x_j^2\right)^{\frac{k}{2}-1} (\cancel{2} x_j) = k\left(\sum_i x_i^2\right)^{\frac{k}{2}-1} x_j$$

$$\Longrightarrow \sum_j \frac{\partial f}{\partial x_j} x_j = \sum_j \left[k\left(\sum_i x_i^2\right)^{\frac{k}{2}-1} x_j\right] x_j$$

$$= k\left(\sum_i x_i^2\right)^{\frac{k}{2}} \cancel{\left(\sum x_i^2\right)^{-1}} \cancel{\left(\sum x_j^2\right)} = k f(x).$$ //

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iv) $f \in C^\infty(\mathbb{R}^n)$ tq. $\boxed{f(tx) = t f(x) \ \forall t > 0 \ \forall x \in \mathbb{R}^n}$ ← hip

fato 1: $f(0) = 0$;

pois $x = 0 \overset{\text{hip}}{\Longrightarrow} \forall t > 0,\ f(0) = t f(0) \Rightarrow (1-t) f(0) = 0 \quad \forall t > 0$ |

fato 2: $\frac{\partial f}{\partial x_j}(0) = f(e_j) \quad \forall j$

pois $\frac{\partial f}{\partial x_j}(0) = \lim_{t\to 0} \frac{f(te_j) - f(0)}{t} = \lim_{t\to 0} \frac{t f(e_j)}{t} = f(e_j)$

fato 2 $\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = 0 \quad \forall x \neq 0$

Por i), em $\mathbb{R}^n \setminus \{0\}$ vale que $\frac{\partial^2 f}{\partial x_i \partial x_j}$ é homogênea de grau $\underbrace{k}_{=1} - 2 = -1$.

Assim:

$$x \neq 0 \Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(tx) = \frac{1}{t} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \quad \forall t > 0$$

Se $\frac{\partial^2 f}{\partial x_i \partial x_j} \neq 0 \Rightarrow$ qdo. $t \to 0^+$: $\begin{cases} \frac{\partial^2 f}{\partial x_i \partial x_j}(tx) \to \frac{\partial^2 f}{\partial x_i \partial x_j}(0) \\ \frac{1}{t} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \to \pm\infty \ \text{↯} \end{cases}$

$$\Longrightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = 0 \text{ em } \mathbb{R}^n \setminus \{0\}$$

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Taylor com resto de Lagrange:

$$f(x) = \underbrace{f(0)}_{=0} + \underbrace{df(0)\cdot x}_{=\sum_j \frac{\partial f}{\partial x_j}(0)\cdot x_j} + \underbrace{d^2 f(\theta x)\cdot x^2}_{\equiv 0}, \quad \theta \in (0,1)$$

$$\Rightarrow \boxed{f(x) = \sum_j a_j x_j} \; ; \text{ onde } \boxed{a_j = \frac{\partial f}{\partial x_j}(0) = f(e_j)}.$$

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Problema 2.

$\Omega \subseteq \mathbb{R}^n$ aberto

$f : \Omega \xrightarrow{C^2} \mathbb{R}$, $g = (g_1, \dots, g_n) : \tilde{\Omega} \to \mathbb{R}^n$ com

$g_j : \tilde{\Omega} \xrightarrow{C^2} \mathbb{R}$, $g(\tilde{\Omega}) \subseteq \Omega$

i) A regra da cadeia diz que se

$$\phi = (\phi_1, \dots, \phi_n) : \Omega \to \mathbb{R}^n$$
$$\phi(\Omega) \subseteq \tilde{\Omega}$$
$$\psi : \tilde{\Omega} \to \mathbb{R}$$

com: $\begin{cases} \phi \text{ diferenciável em } a \in \mathcal{U} \\ \psi \text{ diferenciável em } b = \phi(a) \in V \end{cases}$

$\Rightarrow \psi \circ \phi : \mathcal{U} \to \mathbb{R}$ é diferenciável em a com:

$$\boxed{\frac{\partial(\psi\circ\phi)}{\partial x_j}(a) = \sum_{k=1}^{n} \left(\frac{\partial \psi}{\partial x_k} \circ \phi\right) \frac{\partial \phi_k}{\partial x_j}(a)} \quad (*)$$

Queremos: $\dfrac{\partial^2 (f\circ g)}{\partial x_i \partial x_j}(a)$, onde $a \in \Omega$

1º Aplicamos $(*)$ com $\psi = f$ e $\phi = g$:

$$\frac{\partial (f\circ g)}{\partial x_j} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial g_k}{\partial x_j}$$

$$\frac{\partial^2 (f\circ g)}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left[\sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial g_k}{\partial x_j}\right]$$

$$= \sum_{k=1}^{n} \frac{\partial}{\partial x_i}\left[\left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial g_k}{\partial x_j}\right] \quad \text{Regra do Produto}$$

$$= \sum_{k=1}^{n} \left[\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial g_k}{\partial x_j} + \left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial^2 g_k}{\partial x_i \partial x_j}\right]$$

$$= \sum_{k=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial g_k}{\partial x_j} + \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial^2 g_k}{\partial x_i \partial x_j}$$

f, g C^2 $\Rightarrow$ $\dfrac{\partial f}{\partial x_k} \circ g$ dif. $\Rightarrow$ Aplicamos $(*)$ com $\psi = \dfrac{\partial f}{\partial x_k}$, $\phi = g$:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_k} \circ g\right) = \sum_{l=1}^{n} \left(\frac{\partial^2 f}{\partial x_l \partial x_k} \circ g\right) \frac{\partial g_l}{\partial x_i}$$

Conclusão:

$$\boxed{\frac{\partial^2 (f\circ g)}{\partial x_i \partial x_j} = \sum_{k=1}^{n} \sum_{l=1}^{n} \left(\frac{\partial^2 f}{\partial x_l \partial x_k} \circ g\right) \frac{\partial g_l}{\partial x_i} \frac{\partial g_k}{\partial x_j} + \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ g\right) \frac{\partial^2 g_k}{\partial x_i \partial x_j}}$$

ii) T ortogonal: $TT^* = I$, onde $T^*_{ij} = T_{ji}$

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \quad \text{Laplaciano}$$

Fato: $\sum_{j=1}^{n} T_{kj} T_{\ell j} = \delta_{k\ell}$

Prova: $TT^* = I \Rightarrow (TT^*)_{k\ell} = \delta_{k\ell}$

Mas: $(TT^*)_{k\ell} = \sum_{j=1}^{n} T_{kj} T^*_{j\ell}$

$= \sum_{j=1}^{n} T_{kj} T_{\ell j}$ //

Agora, aplicamos i) com $i=j$ e $g = T$:

$$\frac{\partial^2 (f\circ T)}{\partial x_j^2} = \sum_{\ell} \sum_{k} \left(\frac{\partial^2 f}{\partial x_\ell \partial x_k} \circ T\right) \frac{\partial T_\ell}{\partial x_j} \frac{\partial T_k}{\partial x_j} + \sum_{k} \left(\frac{\partial f}{\partial x_k} \circ T\right) \frac{\partial^2 T_k}{\partial x_j^2}$$

Precisamos de uma expressão p/ T_k. Como T é linear:

$$T(x) = T\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j Te_j$$

Lembrando:

$$\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix} = \begin{bmatrix} | & & | \\ Te_1 & \cdots & Te_n \\ | & & | \end{bmatrix} \Rightarrow Te_j = \sum_{k=1}^{n} T_{kj} e_k$$

$$\Rightarrow T(x) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} x_j T_{kj} e_k\right) \Rightarrow T_k = \sum_{j=1}^{n} x_j T_{kj} \Rightarrow \boxed{\begin{array}{l} \frac{\partial T_k}{\partial x_j} = T_{kj} \\ \frac{\partial^2 T_k}{\partial x_j^2} = 0 \end{array}}$$

$$\Rightarrow \frac{\partial^2 (f\circ T)}{\partial x_j^2} = \sum_{\ell=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial x_\ell \partial x_k} \circ T\right) T_{kj} T_{\ell j}$$

$$\Rightarrow \Delta(f\circ T) = \sum_{j=1}^{n}\sum_{\ell=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial x_\ell \partial x_k} \circ T\right) T_{kj} T_{\ell j}$$

$$= \sum_{\ell=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial x_\ell \partial x_k} \circ T\right)\left(\sum_{j=1}^{n} T_{kj} T_{\ell j}\right) \quad \text{Fato}$$

$$= \sum_{\ell=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial x_\ell \partial x_k} \circ T\right) \delta_{k\ell}$$

$$= \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2} \circ T = (\Delta f)\circ T.$$ //

iii) $q(v_1, v_2) = av_1^2 + 2cv_1v_2 + bv_2^2$. Queremos:

$A \neq 0$, $a+b=0 \Rightarrow \exists\, v, \tilde{v} \in \mathbb{R}^2$ com $q(v) > 0$ e $q(\tilde{v}) < 0$

$A \neq 0$ e $a+b=0 \Rightarrow$ ao menos 1 dentre $a, c \neq 0$.

$a \neq 0 \Rightarrow (1,0)$ e $(0,1)$ são t.q. $q(1,0)$ e $q(0,1)$ têm sinais opostos.

$c \neq 0 \Rightarrow (1,1)$ e $(-1,1)$ são t.q. $q(1,1)$ e $q(1,-1)$ têm sinais opostos.

iv) $(x_0, y_0) \in \Omega$ pto crítico de $f \Rightarrow df(x_0,y_0) \equiv 0$.

$d^2 f(x_0,y_0) \neq 0$

$a := (x_0, y_0)$

Taylor infinitesimal:

$$f(a+v) = f(a) + df(a)\,v + \frac{1}{2} d^2 f(a)\cdot v^2 + r_2(v);$$

Onde $\lim\limits_{v \to 0} \frac{r_2(v)}{|v|^2} = 0$

Sejam q a forma quadrática $q(v) = \frac{1}{2} d^2 f(a)\cdot v^2$, temos:

$$\boxed{f(a+v) = f(a) + q(v) + r_2(v)}$$

$$f(a+v) = f(a) + |v|^2 \left(\frac{q(v)}{|v|^2} + \frac{r_2(v)}{|v|^2} \right)$$

$\exists\, v \in \mathbb{R}^2$ t.q. $q(v) > 0$, digamos: $q(v) = c > 0$.

$$f(a+tv) = f(a) + q(tv) + r_2(tv)$$
$$= f(a) + t^2 q(v) + r_2(tv)$$

$$\boxed{f(a+tv) = f(a) + t^2 |v|^2 \left[\frac{c}{|v|^2} + \frac{r_2(tv)}{t^2|v|^2} \right]}$$

$\lim\limits_{t \to 0} \frac{r(tv)}{t^2|v|^2} = 0 \Rightarrow \exists\, \delta > 0$ t.q. $0 < t < \delta$

$$\Rightarrow -\frac{c}{2|v|^2} < \frac{r(tv)}{t^2|v|^2} < \frac{c}{2|v|^2}$$

$\Rightarrow f(a+tv) > f(a) + t^2 c^2 > f(a) \quad \forall\, t$ peq.

Analogamente:

$f(a+t\tilde{v}) < f(a) - t^2 c^2 < f(a) \quad \forall\, t$ peq

$\therefore$ f ñ é pto. nem de máximo nem de mín. //

Problema 3.

i) $f: \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^2$, t.q.: $\forall\, (x,y) \in \Omega$, $(x^2+y^4) f(x,y) + f(x,y)^3 = 1$.

$\vdash$: $f \in C^\infty$.

Seja $h: \Omega \times \mathbb{R} \to \mathbb{R}$

$$h(x,y,z) = (x^2+y^4) z + z^3 \in C^\infty$$

$$\frac{\partial h}{\partial z}(x,y,z) = x^2 + y^4\ 3z^2$$

$\frac{\partial h}{\partial z} = 0 \iff x = y = z = 0$.

$\Rightarrow \frac{\partial h}{\partial z} \neq 0 \quad \forall$ pto. $(x,y,z) \in h^{-1}(1)$.

fixo $(x_0, y_0) \in \Omega$. Vou provar que f é C^∞ em Ω. Seja $z_0 = f(x_0,y_0) \Rightarrow$ $\boxed{h(x_0, y_0, z_0) = 1 \text{ e } \frac{\partial h}{\partial z}(x_0, y_0, z_0) \neq 0}$

$\Downarrow$ teorema das funções implícitas

$\exists$ bola $B \subseteq \mathbb{R}^2$, contendo (x_0,y_0), intervalo $J \subseteq \mathbb{R}$ contendo z_0 e $\xi : B \xrightarrow{C^\infty} J$ t.q. $h^{-1}(1) \cap (B \times J)$ é o gráfico de ξ.

f é contínua & $f(x_0,y_0) = z_0 \Rightarrow$ encolhendo B se necessário podemos supor $f(B) \subseteq J$. Daí:

$(x,y) \in B \Rightarrow f(x,y) \in J$ e $h(x, y, f(x,y)) = 1$

$\Rightarrow (x, y, f(x,y)) \in (B \times J) \cap h^{-1}(1) = \text{Graf}(\xi)$

$\Rightarrow f \equiv \xi$ em $B \Rightarrow f \in C^\infty$ em B.

Como (x_0,y_0) era qqer., $f \in C^\infty$ em geral. //

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ii) Dada $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$ t.q. $g(x) = f(x) + f(x)^5$ $\forall x \in \mathbb{R}^n$, com $g \in C^r(\mathbb{R}^n)$.

$\vdash$: $f \in C^r(\mathbb{R}^n)$

Seja $h : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$

$$h(x,y) = g(x) - (y + y^5) \in C^r$$

$$\frac{\partial h}{\partial y}(x,y) = -(1 + 5y^4) \neq 0 \quad \forall (x,y) \in \mathbb{R}^{n+1}$$

$\Rightarrow \forall x_0 \in \mathbb{R}^n$ $\exists$ bola $B \subseteq \mathbb{R}^n$ contendo x_0, intervalo $J \subseteq \mathbb{R}$ contendo 0 e $\xi : B \xrightarrow{C^r} J$ t.q. $h^{-1}(0) \cap (B \times J)$ é o gráfico de ξ.

Pelo mesmo argumento usado antes, ξ e f coincidem em $B \Rightarrow f \in C^r(B)$. Como x_0 era qqer., $f \in C^r(\mathbb{R}^n)$. //

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iii) $f : \Omega \to \mathbb{R}$ dada

defina $g(x) = \int_0^{f(x)} (1 + t^2)\, dt$

$\vdash$: Se g for de classe $C^\infty \Rightarrow f$ será de classe C^∞.

Seja $\phi : \mathbb{R} \to \mathbb{R}$, dada por:

$$\phi(y) = \int_0^y (1+t^2)\, dt$$

$\Rightarrow \phi'(y) = 1 + y^2 > 0$ $\forall y \in \mathbb{R} \Rightarrow \phi$ é um difeo C^∞ da reta $\Rightarrow \exists\, \phi^{-1} : \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$.

temos: $g = \phi \circ f \Rightarrow f = \phi^{-1} \circ g$ é contínua.

Seja $h : \Omega \times \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$

$$h(x,y) = g(x) - \phi(y)$$

$\Rightarrow \dfrac{\partial h}{\partial y} = \phi'(y) \neq 0$ em $\Omega \times \mathbb{R}$

$\Rightarrow \forall (x_0, y_0) \in \Omega \times \mathbb{R}\ \exists$ bola $B \subseteq \Omega$ contendo x_0, intervalo $J \subseteq \mathbb{R}$ contendo y_0 e $\xi : B \xrightarrow{C^\infty} J$ t.q. $h^{-1}(0) \cap (B \times J)$ é o gráfico de ξ.

Pelo mesmo argumento usado antes, ξ e f coincidem em $B \Rightarrow f \in C^\infty(B)$. Como x_0 era qqer, $f \in C^\infty(\Omega)$.

iv) $\xi : I \to \mathbb{R}$ contínua t.q. $f(x, \xi(x)) = 0\ \forall x \in I$.
$f : \mathbb{R}^2 \xrightarrow{C^1} \mathbb{R}$, $\dfrac{\partial f}{\partial y} \neq 0$ em todo pto.

$\boxed{\text{Af: } \xi \in C^1}$. De fato:

Sejam $x_0 \in I$, $y_0 = \xi(x_0) \Rightarrow \dfrac{\partial f}{\partial y}(x_0, y_0) \neq 0$ & $f(x_0, y_0) = 0$.

Pelo teorema das funções implícitas, $\exists$ aberto $\mathcal{U} \subseteq I$ contendo x_0, intervalo J contendo y_0 e $\eta : \mathcal{U} \xrightarrow{C^1} J$ t.q. $f^{-1}(0) \cap (\mathcal{U} \times J) = \text{Graf}\, \eta$.

Como ξ é contínua, restringindo $\mathcal{U}$ se necessário podemos supor $\xi(\mathcal{U}) \subseteq J$. Logo:

$x \in \mathcal{U} \Rightarrow \xi(x) \in J$ e $f(x, \xi(x)) = 0$
$\Rightarrow (x, \xi(x)) \in (\mathcal{U} \times J)$ e $(x, \xi(x)) \in f^{-1}(0)$
$\Rightarrow (x, \xi(x)) \in (\mathcal{U} \times J) \cap f^{-1}(0) = \text{Graf}\, \eta \Rightarrow \xi \equiv \eta$ em $\mathcal{U} \Rightarrow \xi \in C^1$. //