Orthogonal geodesic chords on Riemannian manifolds with concave boundary and applications

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symmetric bilinear form

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Lemma. $\overline{\Omega}$ strongly concave $\Longrightarrow$ $\overline{\Omega}$ concave.

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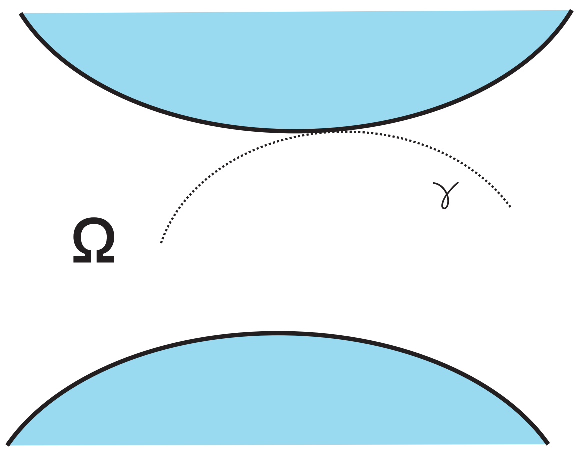

geodesics starting tangentially to $\partial\Omega$ move inside Ω

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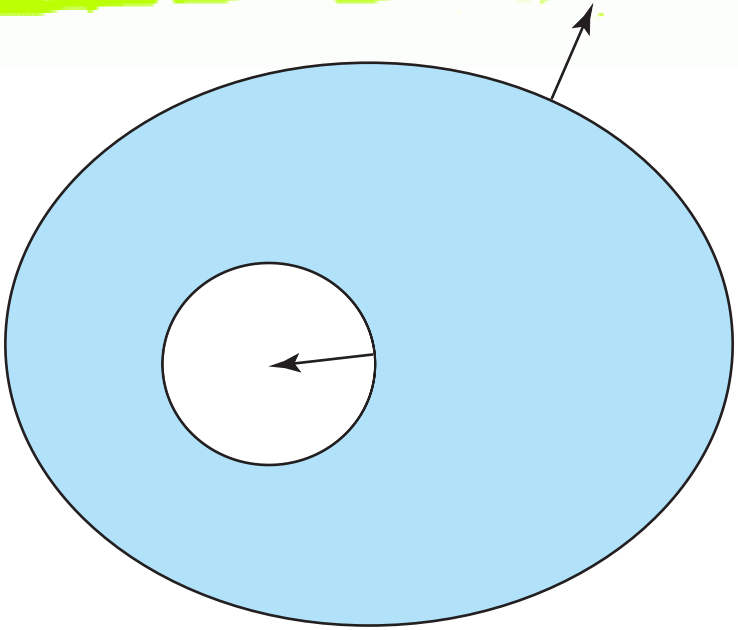

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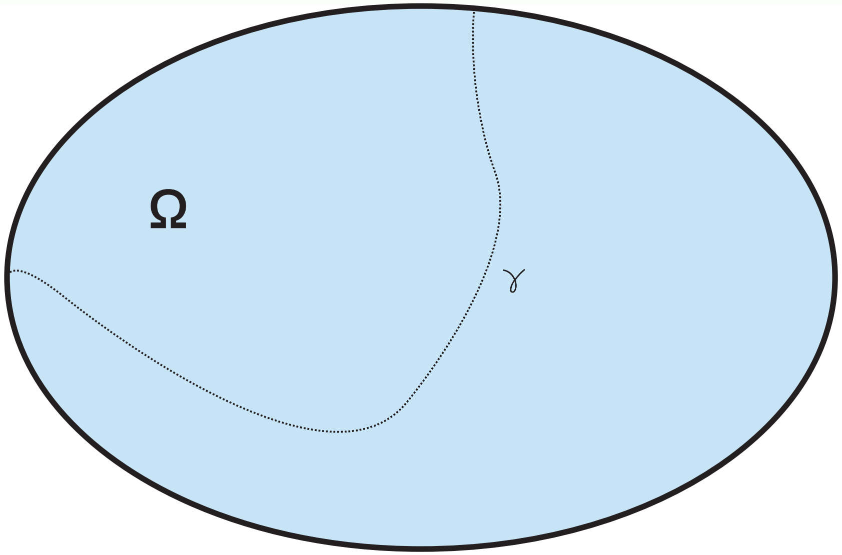
Back to proof

Orthogonal geodesic chords

Def.: An *orthogonal geodesic chord* (OGC) in $\overline{\Omega}$ is a non constant geodesic $\gamma : [a,b] \to \overline{\Omega}$ with $\gamma(a), \gamma(b) \in \partial\Omega$ and $\dot\gamma(a), \dot\gamma(b) \in T\big(\partial\Omega\big)^{\perp}$.



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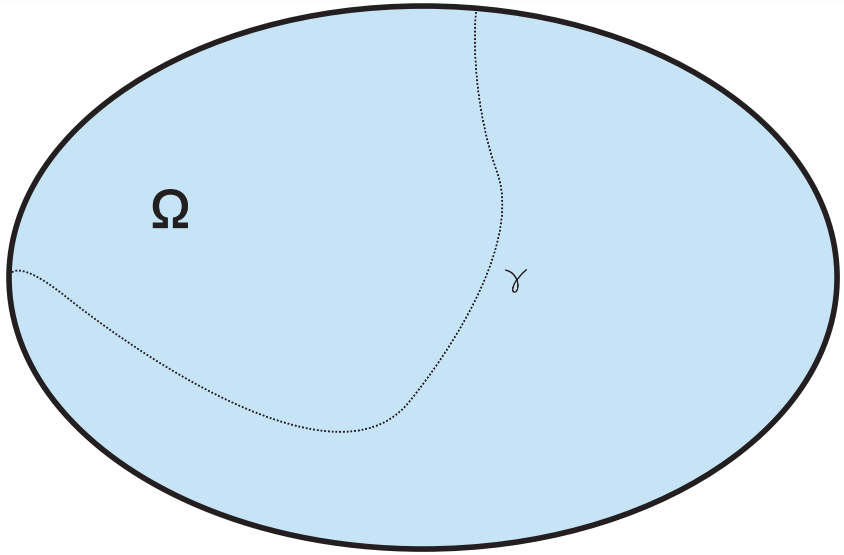
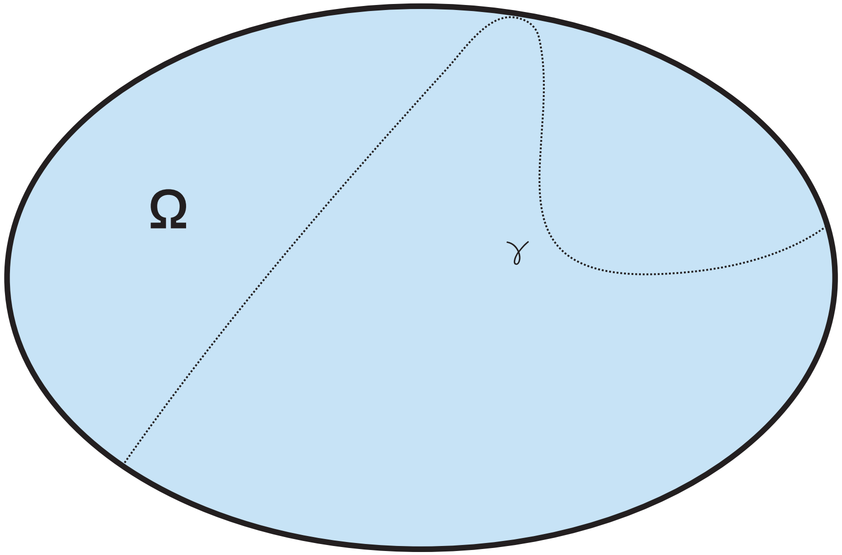
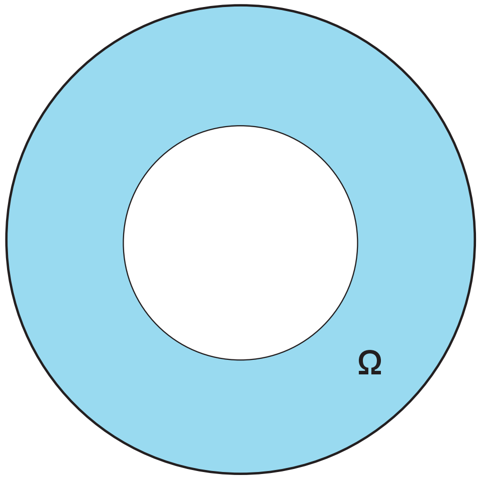




A weak orthogonal geodesic chord (WOCG).

WOGC's do *not exist* in the convex case.

Some examples – 1



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An OGC is *crossing* if its endpoints are in distinct connected components of $\partial\Omega$. It is easy to prove the existence of *one* crossing OGC whose length equals the distance between the two connected components of $\partial\Omega$.

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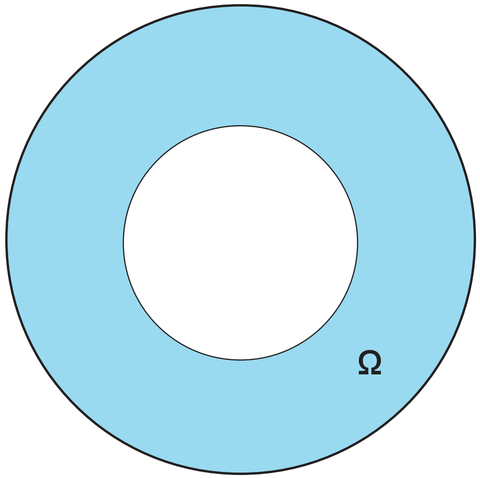

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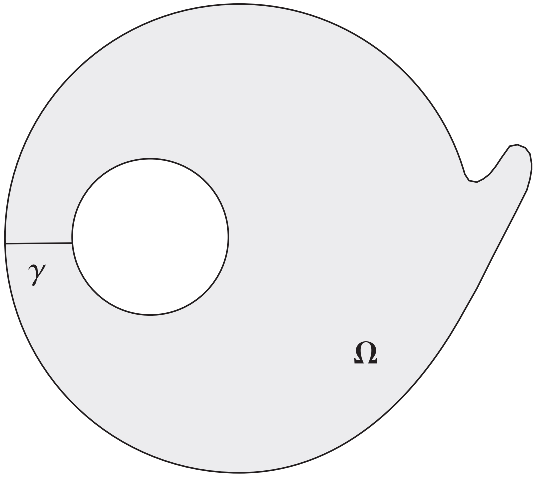
There may be *only one* OGC:



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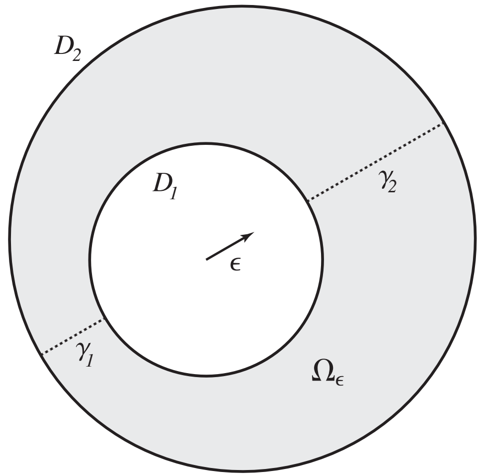
This is an optimal result

(in all dimensions):

Euclidean metric

$$g(x) = \psi(|x|) \cdot g_0(x), \quad \psi : \mathbb{R}^+ \to \mathbb{R}^+$$



convexity of the annulus

$$\frac{1}{2}\psi'(1) + \psi(1) \geq 0, \quad \psi'(2) + \psi(2) \leq 0.$$

$$\Omega_\varepsilon = \{x \in \mathbb{R}^m : 1 < |x|, \ |x - \varepsilon| < 2\}.$$

(Back to the central result)

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- if $\delta < \mathrm{foc}(\partial\Omega)$, every OGC in Ω' can be extended to an OGC in Ω
- if by absurd $\exists\delta_n \to 0$ and a sequence γ_n of WOGC's in $\phi^{-1}\big(\,]-\infty, -\delta_n[\,\big)$, then one would get infinitely many OGC's in Ω. QED

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Obs.: Again, the result is *optimal*. Recall example above (with opposite strict inequalities!)).

back to central symmetry

Central symmetry

Def.: (M, g) Riemannian man., $A \subset M$ is *centrally symmetric around* $x_0 \in M$ if exists an isometry $I : M \to M$, with $I^2 = I$, whose unique fixed pt is x_0, and such that $I(A) = A$.

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Theorem. Under the assumptions of the above theorem, if $\overline{\Omega}$ is centrally symmetric around some x_0, then there are at least $m = \dim(M)$ geometrically distinct OGC's $\gamma_1, \ldots, \gamma_m$ in $\overline{\Omega}$.

A short history of the problem

Two classical results:

- Ljusternik and Schnirelmann, 1932:

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- F. Giannoni, P. Majer, *On the effect of the domain about the multiplicity of the orthogonal geodesic chords*, Diff. Geom. Appl. 1997

The topology of the manifold

G & M's result:

- if the manifold is homeomorphic to an *annulus* and it is convex, then there are at least two OGC's;
- if the manifold has compact and convex boundary, and if the *LS-category* of the space of paths with endpoints on the boundary is infinite, then there are infinitely many OGC's.

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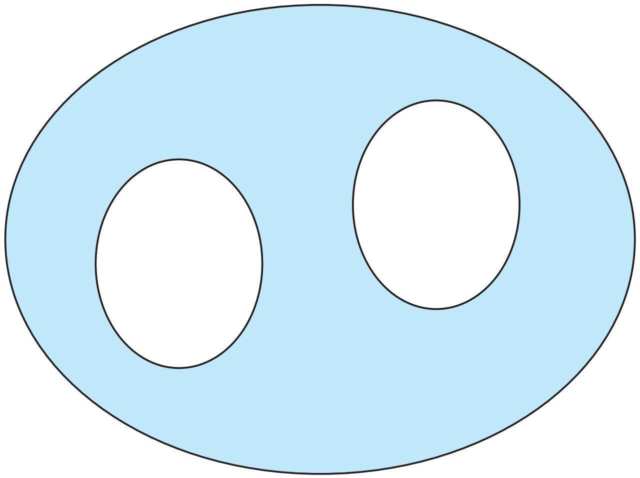

$\longleftarrow$ Example

Geodesics chords, homoclinics and brake orbits: a short bibliography

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These results will be reviewed later.

More bibliography

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Def.: $\mathcal{X}$ top. space, $\mathcal{Y} \subset \mathcal{X}$ is *contractible in* $\mathcal{X}$ if $\mathfrak{i} : \mathcal{Y} \to \mathcal{X}$ is homotopic to a constant.

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We prove two *deformation lemmas* for the sublevels of $\mathcal{F}$, and we prove a (PS) condition for $\mathcal{F}$, obtaining the existence of $\mathrm{cat}(\mathfrak{C}) = \mathrm{cat}(S^{m-1}) = 2$ distinct critical values of $\mathcal{F}$. For the *symmetric case*, a lower estimate is given by $\mathrm{cat}(\mathbb{R}P^{m-1}) = m$.

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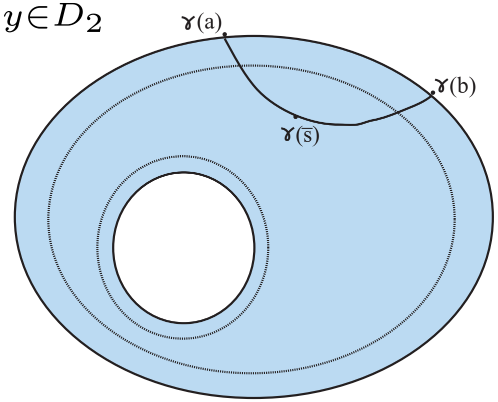

Prop.: If $\gamma : [a,b] \to \overline{\Omega}$ is a geo with $\gamma(a), \gamma(b) \in \partial\Omega$, then $\exists \bar{s} \in\ ]a,b[$ with $\phi(\gamma(\bar{s})) < -\delta_0$.

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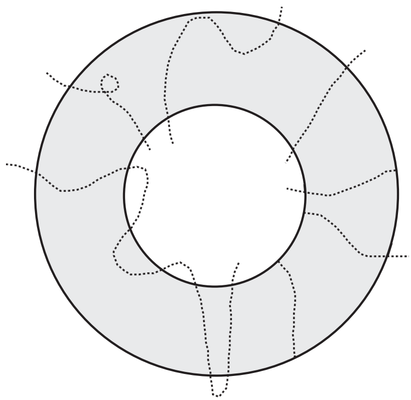

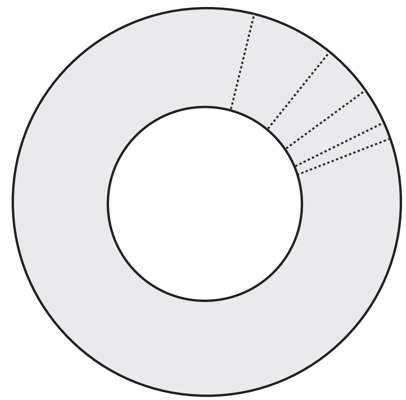

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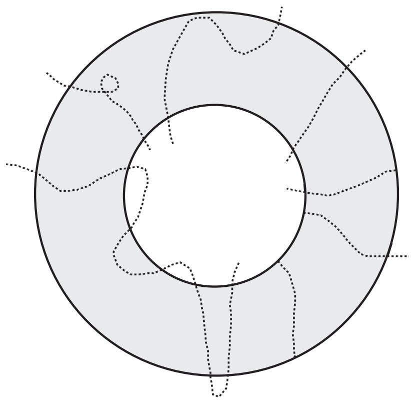

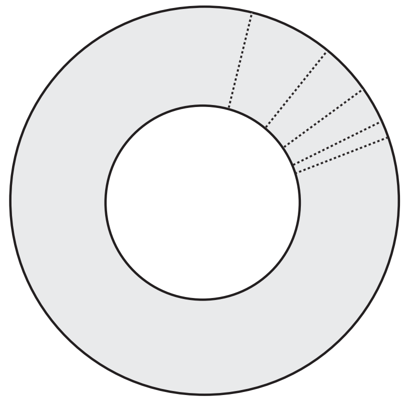

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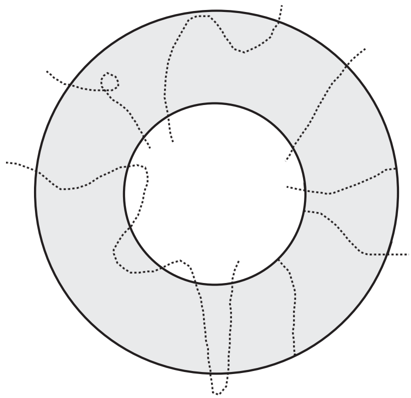

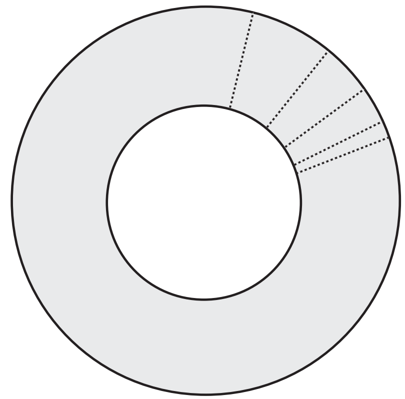

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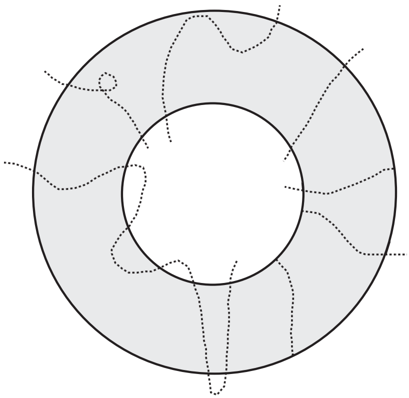
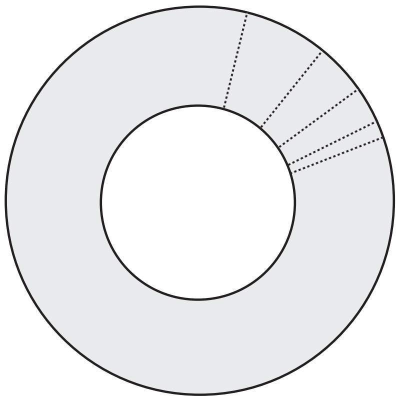
Define C_i = connected components of $\phi^{-1}\big([0,\delta_0]\big)$ containing D_i, $i=1,2$.

$$\boxed{\mathfrak{M} = \Big\{x \in H^1\big([0,1],\mathbb{R}^m\big) : \phi(x(s)) < \delta_0,\ x(0)\in C_1,\ x(1)\in C_2\Big\}}$$





$\boxed{\mathfrak{C} \cong \Big\{\text{orthogonal segments from } D_1 \text{ to } D_2\Big\}}$ $\quad M_0 = \sup\limits_{x\in\mathfrak{C}} \int_0^1 g(\dot x,\dot x)\,\mathrm{d}s.$

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Variationally critical portions of x are curves whose geodesic energy is not decreased by "infinitesimal variations" with curves *stretching outwards* from $\overline{\Omega}$.

first variation of the geodesic action functional

Classification of variationally critical portions

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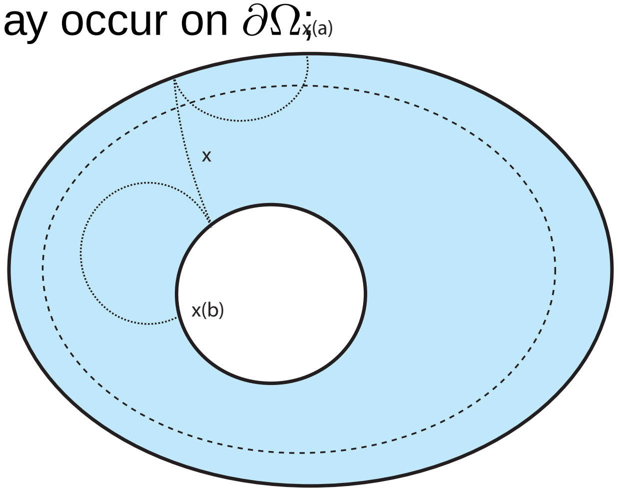

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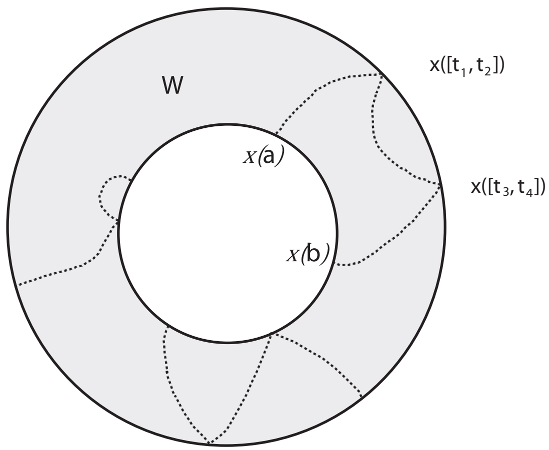
$[t_1,t_2]$ cusp interval of the irregular variationally critical portion of x



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Proof. Uses in a crucial way the fact that there is no WOGC.

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Prop.: If $(x_n) \subset \mathfrak{M}$, $[a_n, b_n] \in \mathcal{J}^0_{x_n}$ are M_0-intervals s.t. $x_n|_{[a_n,b_n]}$ is a **VCP** of x_n, then (up to subsequences) $a_n \to a$, $b_n \to b$, $\boxed{x_n|_{[a_n,b_n]} \to x_{[a,b]},}$ where $x|_{[a,b]}$ is a **VCP** of x.

Cor.: $\exists\, d_0 > 0$ such that $\boxed{\max \Theta_x(t_1, t_2) \ge d_0,}$ the max being taken over all $x \in \mathfrak{M}$, all M_0-intervals $[a, b] \in \mathcal{J}^0_x$ s.t. $x|_{[a,b]}$ is an irregular **VCP** of x, and all $[t_1, t_2] \subset [a, b]$ cusp interval.

Proof. Uses in a crucial way the fact that there is no WOGC.

The corollary tells us, in particular, that the **(VCP)**'s of first and of second type are *far from each other*.

The classical Palais–Smale condition

Let $\mathcal{X}$ be a smooth Banach manifold, and let $f : \mathcal{X} \to \mathbb{R}$ be a C^1-map.

f satisfies the (classical) Palais–Smale condition if every sequence $(x_n) \subset \mathcal{X}$ such that:

- $f(x_n)$ is bounded;
- $\mathrm{d}f(x_n) \to 0$ as $n \to \infty$,

admits a converging subsequence in $\mathcal{X}$.

The Palais–Smale condition

For $[a, b] \subset [0, 1]$, consider the set $\mathcal{Z}_{a,b}$ of curves in $\mathfrak{M}$ s.t. $x|_{[a,b]}$ is a **VCP**, not necessarily contained in $\overline{\Omega}$:

$$\mathcal{Z}_{a,b} = \Big\{ y : [a, b] \to \phi^{-1}\big(\,]-\infty, \delta_0[\,\big) : \int_a^b g\big(\dot{y}, \tfrac{\mathrm{D}}{\mathrm{d}t} V\big)\, \mathrm{d}t \geq 0 \; \forall\, V \in \mathcal{V}^+(y) \Big\}$$

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The following result plays the role of the classical Palais–Smale condition in our context:

Proposition (PS): For all $r > 0$, $\exists\, \theta(r), \mu(r) > 0$ with the following properties: for all $x \in \mathfrak{M}$ and all $[a,b] \in \mathcal{J}_x^0$ s.t.

(a) $\frac{1}{2}\int_a^b g(\dot{x}, \dot{x})\,\mathrm{d}t \leq M_0$,

(b) $\|x|_{[a,b]} - y\|_{a,b} \geq r$ for all $y \in \mathcal{Z}_{a,b}$,

there exists a vector field $V_x : [a,b] \to \mathbb{R}^m$ such that:

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- $\int_a^b g\big(\dot x, \frac{\mathrm{D}}{\mathrm{d}t} V_x\big)\,\mathrm{d}t \leq -\mu(r)\|V_x\|_{a,b}$.

Preparation for the Deformation Lemmas 1

By the compactness of $\phi^{-1}(]-\infty,\delta_0])$, $\exists\,\ell_0, L_0 > 0$ s.t., denoting by $\|\cdot\|_E$ the Euclidean norm and by $\|\cdot\|$ the g-norm,

$$\ell_0\|v\|_E^2 \leq \|v\|^2 \leq L_0\|v\|_E^2, \quad \forall x \in \phi^{-1}(]-\infty,\delta_0]), \forall v \in \mathbb{R}^m.$$

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Moreover, $\exists\, G_0, L_1 = L_1(M_0) > 0$ s.t.

$$|g_x(v_1,v) - g_z(v_2,v)| \le G_0\left(\|v_1 - v_2\|_E\,\|v\|_E + \|x - z\|_E\,\|v_1\|_E\,\|v\|_E\right),$$

for all $x, z \in \phi^{-1}(]-\infty,\delta_0])$ and for any v_1, v_2, $v \in \mathbb{R}^m$, and

$$\left(\int_a^b \|\tfrac{\mathrm{D}}{\mathrm{d}s}V\|_E^2\,\mathrm{d}s\right)^{1/2} \le L_1\|V\|_{a,b}$$

for all $x \in \mathfrak{M}$ s.t. $\frac{1}{2}\int_a^b g(\dot{x},\dot{x})\,\mathrm{d}s \le M_0$, for all $V \in H^1([a,b],\mathbb{R}^N)$ along x, and for any $[a,b] \subset [0,1]$.

Lemmas 2

For $a, b \in [0,1]$, denote by $I_{a,b}$ the interval $[a,b]$ if $b \geq a$ and the interval $[b,a]$ if $b < a$; set:

$$\boxed{\mathcal{D}(x,\alpha,\beta,a,b) = \frac{1}{2}\int_{I_{a,\alpha}\cup I_{b,\beta}} g(\dot{x},\dot{x})\,\mathrm{d}t.}$$

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Lem.: Fix $K > 0$. For any $x, z \in \mathfrak{M}$, $[a,b] \subset [0,1]$, $[a_z, b_z] \subset [0,1]$, and $V \in H^1([0,1],\mathbb{R}^N)$, then if $\frac{1}{2}\int_a^b g(\dot{x},\dot{x})\,\mathrm{d}t \leq M_0$ and $\mathcal{D}(x,a_z,b_z,a,b) \leq K$, it is

$$\left|\int_a^b g_x\big(\dot{x}, \tfrac{\mathrm{D}}{\mathrm{d}t}V\big)\,\mathrm{d}t - \int_{a_z}^{b_z} g_z\big(\dot{z}, \tfrac{\mathrm{D}}{\mathrm{d}t}V\big)\,\mathrm{d}t\right| \leq$$

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Define: $\boxed{E(r) = \dfrac{\mu(r)^2}{32 L_1^2 L_0}}.$

Construction of local vector fields

Prop.: For $r > 0$, let $\theta(r), \mu(r) > 0$ be as in PS. For all $x \in \mathfrak{M}$ and for all $[a, b] \in \mathcal{J}_x^0$ for which (a) and (b) of PS hold, let V_x be the vector field in PS.
Extend V_x to $[0, 1]$ making it constant outside $[a, b]$.

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  (i) $g\big(\nabla\phi(z(s)), V_x(s)\big) \ge \frac{1}{2}\theta(r)\|V_x\|_{\alpha_x,\beta_x}$ for all $s \in [\alpha_x, \beta_x]$ with $0 \le \phi(z(s)) \le \frac{1}{2}\delta_0$;
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- for all $z \in \mathfrak{M}$, for all $[a_z, b_z] \in \mathcal{J}_z$ with $[a_z, b_z] \subset [\alpha_x, \beta_x]$, with $\|x - z\|_{a_z,b_z} < \rho(x)$ and with $\mathcal{D}(x, a_z, b_z, a, b) < E(r)$, then:
  $$\int_{a_z}^{b_z} g\big(\dot{z}, \tfrac{\mathrm{D}}{\mathrm{d}t}V_x\big)\,\mathrm{d}t \leq -\tfrac{1}{2}\mu(r)\|V_x\|_{\alpha_x,\beta_x}.$$

Interpretation of the constant $E(r)$

By the definition of $D(x, a_z, b_z, a, b)$, the number $E(r)$ gives a bound on the admissible difference between the energy of $x|_{[a,b]}$ and $x|_{[a_z,b_z]}$, to obtain a rate of decrease $\mu(r)/2$ for the quantity $\frac{1}{2}\int_{a_z}^{b_z} g(\dot{z}, \dot{z})\mathrm{d}s$, when $\|x - z\|_{a_z,b_z} < \rho(x)$.

“Genuine” crossing intervals

Def.: $\mathcal{D} \subset \mathfrak{C}$, $h : [0,1] \times \mathcal{D} \xrightarrow{C^0} \mathfrak{M}$, $\gamma \in \mathcal{D}$, $\tau \in [0,1]$. An interval $[a_\tau, b_\tau] \in \mathcal{J}_{h(\tau,\gamma)}$ is *h-genuine* if for all $\tau' \in [0,\tau]$ there exists $[a_{\tau'}, b_{\tau'}] \in \mathcal{J}_{h(\tau',\gamma)}$ such that $[a_\tau, b_\tau] \subset [a_{\tau'}, b_{\tau'}]$.

For $(\mathcal{D}, h) \in \widetilde{\mathcal{H}}$ and $z \in h(1,\mathcal{D})$, set:

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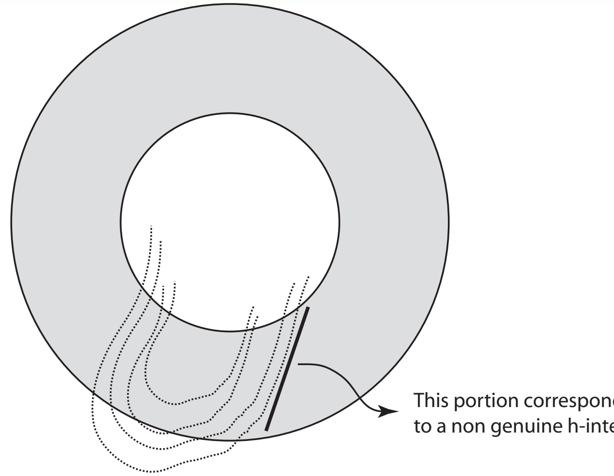

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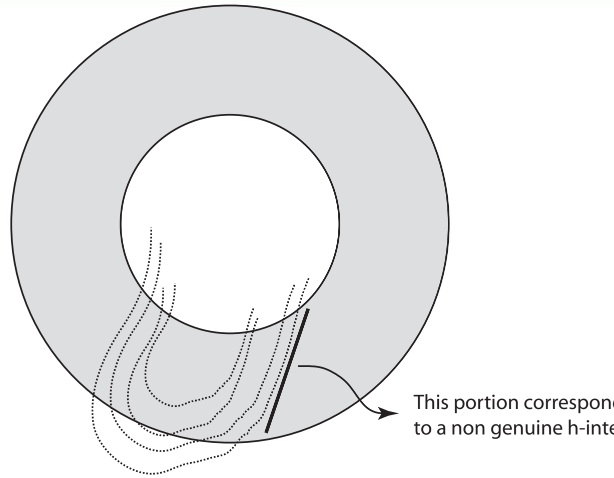

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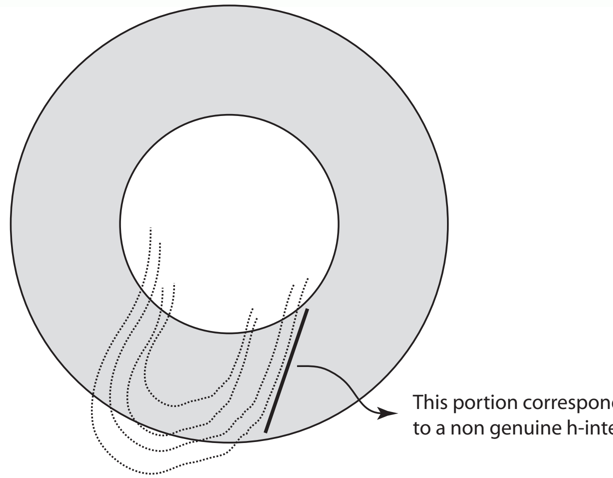

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Obs.: $\widehat{\mathcal{J}}_z^h(\mathcal{D})$ is always non empty. If $z \in h(1,\mathcal{D})$ and $[a,b] \in \mathcal{J}_z^h$, then $[a,b] \in \widehat{\mathcal{J}}_z^h(\mathcal{D})$.

Admissible homotopies

Def.: A set of *admissible homotopies* $\mathcal{H}$ of our variational problem (that will be used in a crucial preparatory deformation result) consists of all continuous maps $h : [0,1] \times \mathcal{D} \to \mathfrak{M}$, with $\mathcal{D}$ closed subset of $\mathfrak{C}$, such that:

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$$\mathcal{H} = \Big\{(\mathcal{D},h) : \mathcal{D} \text{ is a closed subset of } \mathfrak{C}, \text{ and } h : [0,1] \times \mathcal{D} \to \mathfrak{M} \text{ is a continuous homotopy satisfying (1), (2), (3) above.}\Big\}$$

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3. for all $x \in h(1,\mathcal{D})$, every $[a,b] \in \widehat{\mathcal{J}}^h_z(\mathcal{D})$ is an M_0-interval, i.e., $\frac{1}{2}\int_a^b g(\dot{x},\dot{x})\,\mathrm{d}t < M_0$.

$$\mathcal{H} = \Big\{(\mathcal{D},h) : \mathcal{D} \text{ is a closed subset of } \mathfrak{C}, \text{ and } h : [0,1]\times\mathcal{D} \to \mathfrak{M} \text{ is a continuous homotopy satisfying (1), (2), (3) above.}\Big\}$$

Obs. 1: Defining the *constant homotopy*: $h_0(\tau,x) \equiv x$ for all $x \in \mathfrak{C}$, $\mathcal{H}$ contains $(\mathfrak{C}, h_0)$.

Admissible homotopies

Def.: A set of *admissible homotopies* $\mathcal{H}$ of our variational problem (that will be used in a crucial preparatory deformation result) consists of all continuous maps $h : [0,1] \times \mathcal{D} \to \mathfrak{M}$, with $\mathcal{D}$ closed subset of $\mathfrak{C}$, such that:

1. $h(0,\cdot)$ is the inclusion of $\mathcal{D}$ into $\mathfrak{M}$;
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Obs. 2: There exists $N > 0$ (independent of x, $\mathcal{D}$ and h) such that $|\widehat{\mathcal{J}}_z^h(\mathcal{D})| \leq N$.

Concatenation of homotopies

$F_1, F_2 \subset \mathfrak{M}$ closed sets

$h_i : [0,1] \times F_i \xrightarrow{C^0} \mathfrak{M}$, $i = 1, 2$

If $h_1(1, F_1) \subset F_2$, then one defines the concatenation:
$h_1 \star h_2 : [0,1] \times F_1 \longrightarrow \mathfrak{M}$

$$h_1 \star h_2(t,x) = \begin{cases} h_1(2t, x), & \text{if } t \in [0, \frac{1}{2}]; \\ \\ h_2\big(2t-1, h_1(1,x)\big), & \text{if } t \in \left]\frac{1}{2}, 1\right]. \end{cases}$$

Consider the following functional $\mathcal{F} : \mathcal{H} \to \mathbb{R}^+$:

$$\boxed{\mathcal{F}(\mathcal{D}, h) = \sup \left\{ \frac{b-a}{2} \int_a^b g(\dot{x}, \dot{x})\, \mathrm{d}t : x \in h(1, \mathcal{D}),\ [a,b] \in \widehat{\mathcal{J}}_x^h(\mathcal{D}) \right\}}$$

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Obs. 2: for all $(\mathcal{D}, h) \in \mathcal{H}$, $\boxed{\frac{1}{2}\rho_0^2 \le \mathcal{F}(\mathcal{D}, h) \le \frac{1}{2}M_0}$.

An “outward pushing” deformation

Lemma

$$\mathcal{Z}^1_{a,b} = \Big\{ y \in H^1\Big([a,b], \phi^{-1}\big(\,]-\infty, \delta_0[\,\big)\Big) : y|_{[a,b]} \text{ is an OGC,}$$
$$\text{or } y|_{[a,b]} \text{ is an irregular variational portion of first type} \Big\}$$

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Prop.: Let $r > 0$ and $0 < c_1 < c < c_2$ be fixed. Then there exists $\varepsilon_0 = \varepsilon_0(r, c) > 0$ such that, for all $(\mathcal{D}, h) \in \mathcal{H}$ satisfying:

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Interpretation: far from crossing OGC's and irregular VCP, the functional $\mathcal{F}$ decreases along homotopies of $\mathcal{H}$.

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- in a small neighborhood of portions of curves that are *far* from VCP, one uses integral curves of vector fields in $\mathcal{V}^+$, discussed above;
- in a small neighborhood of irregular VCP's of second type, one uses suitable *reparameterization flows*;
- one uses the methods of Degiovanni–Marzocchi (AMPA 1994) to build a *global flow* using local flows.

Flows far from VCP of first type

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This can be done thanks to the following crucial regularity result, due to Marino and Scolozzi (Boll. UMI 1982):

THM.: Let $y \in H^1\big([a,b],\overline{\Omega}\big)$ be such that

$$\int_a^b g\big(\dot y, \tfrac{\mathrm{D}}{\mathrm{d}t}V\big)\,\mathrm{d}t \geq 0, \quad \forall\, V \in \mathcal{V}^-(y) \text{ with } V(a) = V(b) = 0.$$

Then $y \in H^{2,\infty}\big([a,b],\overline{\Omega}\big)$, and in particular y is of class C^1.

On the class $\widetilde{\mathcal{H}}$

- Irregular VCP's of first type are not C^1, thus if a portion of curve is *close* to one of them it is *far* to VCP w.r.t. to $\mathcal{V}^-$.

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- Such homotopies h are constructed using vector fields in $\mathcal{V}^-$: they deform into curves far from irregular VCP's of first type, and the functional is not increasing by concatenation.

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4. for every $x \in h(1,\mathcal{D})$, and for every $[a,b] \in \widehat{\mathcal{J}}_x^h$, it is $\|H_0(1,x)|_{[a,b]} - y|_{[a,b]}\| \ge \bar{r}$ for any $y \in \mathfrak{M}$ such that $y|_{[a,b]}$ is an irregular VCP of first type.

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Combining the previous deformation Lemmas, one obtains:

1st Deformation Lemma: Let c be geometrically regular value. There exists $\varepsilon = \varepsilon(c) > 0$ such that, for all $(\mathcal{D}, h) \in \widetilde{\mathcal{H}}$ with $\boxed{\mathcal{F}(\mathcal{D}, h) \leq c + \varepsilon}$, there exists a continuous map $\eta : [0,1] \times h(1, \mathcal{D}) \to \mathfrak{M}$ such that $(\mathcal{D}, \eta \star h) \in \widetilde{\mathcal{H}}$ and $\boxed{\mathcal{F}(\mathcal{D}, \eta \star h) \leq c - \varepsilon}$.

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Corollary: Each c_i is a geometrically critical value.

Preparation for the 2nd Def. Lemma

Let $r_* > 0$ be fixed and $(\mathcal{D}, h) \in \widetilde{\mathcal{H}}$; consider the set:

$$\mathcal{W} = \mathcal{W}(\mathcal{D}, h, r_*) = \Big\{ x \in \mathfrak{M} : \exists\, [a,b] \in \widehat{\mathcal{J}}^h_x(\mathcal{D}) \text{ and a crossing OGC } \gamma : [a,b] \to \overline{\Omega} \\ \text{s.t. } \max_{s \in [a,b]} \operatorname{dist}\big(x(s), \gamma([a,b])\big) \le r_* \Big\}$$

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2. the set $\Big\{ A \in D_1 : \|A - \gamma(0)\| < 2r_* \text{ for some OGC } \gamma \text{ from } D_1 \text{ to } D_2 \Big\}$ is *contractible* in D_1.

(back to 2DL)

The 2nd Deformation Lemma

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Prop. 2: Assume that there are only a finite number of crossing OGC's from D_1 to D_2, and assume that $r_* > 0$ is so small so that properties (1) and (2) in the page above are satisfied. Then, for all $(\mathcal{D}, h) \in \widetilde{\mathcal{H}}$ there exists an open set $\mathcal{A}$ of $\mathfrak{C}$, with $h(1,\cdot)^{-1}(\mathcal{W}) \subset \mathcal{A}$, that is *contractible* in D_1.

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Corollary: Assume that there is only a finite number of crossing OGC's from D_1 to D_2. Then $c_1 < c_2$.

Some old and new results

We will now review some old and new results on periodic solutions of conservative dynamical systems.

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The existence of closed geodesics is clear on an *intuitive ground:* rest position of an elastic string whose initial position is a non null-homotopic closed curve.

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- apply the shortening method to a family of closed curves that cover simply a sphere;
- consider the **longest** curve of the family after each shortening process;
- a subsequence to this must converge to a closed geodesic, which is *not trivial*, because the sphere is not contractible.

Topological methods

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- take an *essential map* $f : S^k \to M$ and transfer to M a family of closed curve covering S^k;
- apply the curve shortening method to this family, and obtain a closed geodesic in M which is not trivial, due to the assumption that f represents a non zero element in $\pi_k(M)$.

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$H : \mathbb{R}^{2n} \to \mathbb{R}$, $\boxed{H(q,p) = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} p_i p_j + V(q),}$ $V : \mathbb{R}^n \to \mathbb{R}$, g^{ij} positive definite.

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Its image in the configuration space oscillates back and forth along a curve in D with endpoints in ∂D.

Obs.: By the conservation of energy, $p(0)=p(T)=0$. Since H is *even in* p, the solution can be continued to a $2T$-periodic solution according to the formulas: $q(-t)=q(t)$, $q(T+t)=q(T-t)$, $p(-t)=-p(t)$, $p(T-t)=-P(T-t)$ **brake orbit**

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Proof: apply the shortening method to a family of *diameters* of D. The main difficulty here is the fact that g_E vanishes on ∂D, and a *limit procedure* is employed to control the behaviour of geodesics near ∂D.

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$\mathbf{P} : \Sigma \xrightarrow{\cong} S^3$ radial projection **(picture)**,
$\vec{H} = \sum \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$,
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"Doubling trick": periodic solutions (x, y) of $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ with period $2T$ correspond to pairs (α, β) and (ξ, η) of solutions resp. of:

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Proof of THM 1: curve shortening method in Finsler geometry.

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They also obtain a *multiplicity result* in the case that the E-sublevel of the potential is homeomorphic to a disk, under a certain nonresonance assumption: the maximum diameter of the disk should have g_E-length smaller than twice the length of the shortest g_E-geodesic chord.

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We want to extend the MJ-principle to *brake orbits*.

Maupertuis–Jacobi principle for brake orbits

Thm.: E regular value of V, $x :]a,b[\longrightarrow \Omega_E$ s.t.: $\boxed{C^0 \cap H^1_{\mathrm{loc}}}$

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- (p,q) can be extended to a $2T$-periodic brake orbit of energy E.

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We need a Maupertuis–Jacobi principle for homoclinics.

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If E reg. value of V, $\overline{\Omega_E}$ compact, set $d_E : \Omega \to [0, +\infty[$:

$$d_E(Q) = \inf \left\{ \int_0^1 (E - V) g(\dot{x}, \dot{x})^{\frac{1}{2}} \, \mathrm{d}t : x \in H^1\big([0,1], \overline{\Omega_E},\ x(0) = Q,\ x(1) \in \partial\Omega \right\}.$$

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Lem 4: Set $\psi = \frac{1}{2} d_E^2 : \Omega_E \to \mathbb{R}^+$; for y near $\partial\Omega_E$:

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- if $\overline{\Omega}$ is *centrally symmetric*, also $\overline{\Omega_*}$ is cent. symmetric.

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Lem 2: $\lambda_E : \Omega_E \to [0,+\infty[$ is *continuous*.

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Lem 3: $\exists \widehat{\rho} > 0$ s.t., setting $\psi(y) = \frac{1}{2}\lambda_Q(y)^2$, for $\mathrm{dist}(y, x_0) \le \widehat{\rho}$:

$$\mathrm{Hess}(\psi)_y[v,v] > 0, \quad \text{for } v \neq 0 \text{ with } \mathrm{d}\psi_y[v] = 0.$$

M–J principle for homoclinics

THM: x_0 nondegenerate max of V, $V(x_0) = E$, E regular value of V, $V^{-1}\big(\,]-\infty, E[\,\big) \bigcup \{x_0\}$ homeomorhic to an open ball in $\mathbb{R}^m$.

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- if $V^{-1}\big(\,]-\infty, E[\,\big) \bigcup \{x_0\}$ and V are centrally symmetric around x_0, then so is $\overline{\Omega_*}$.

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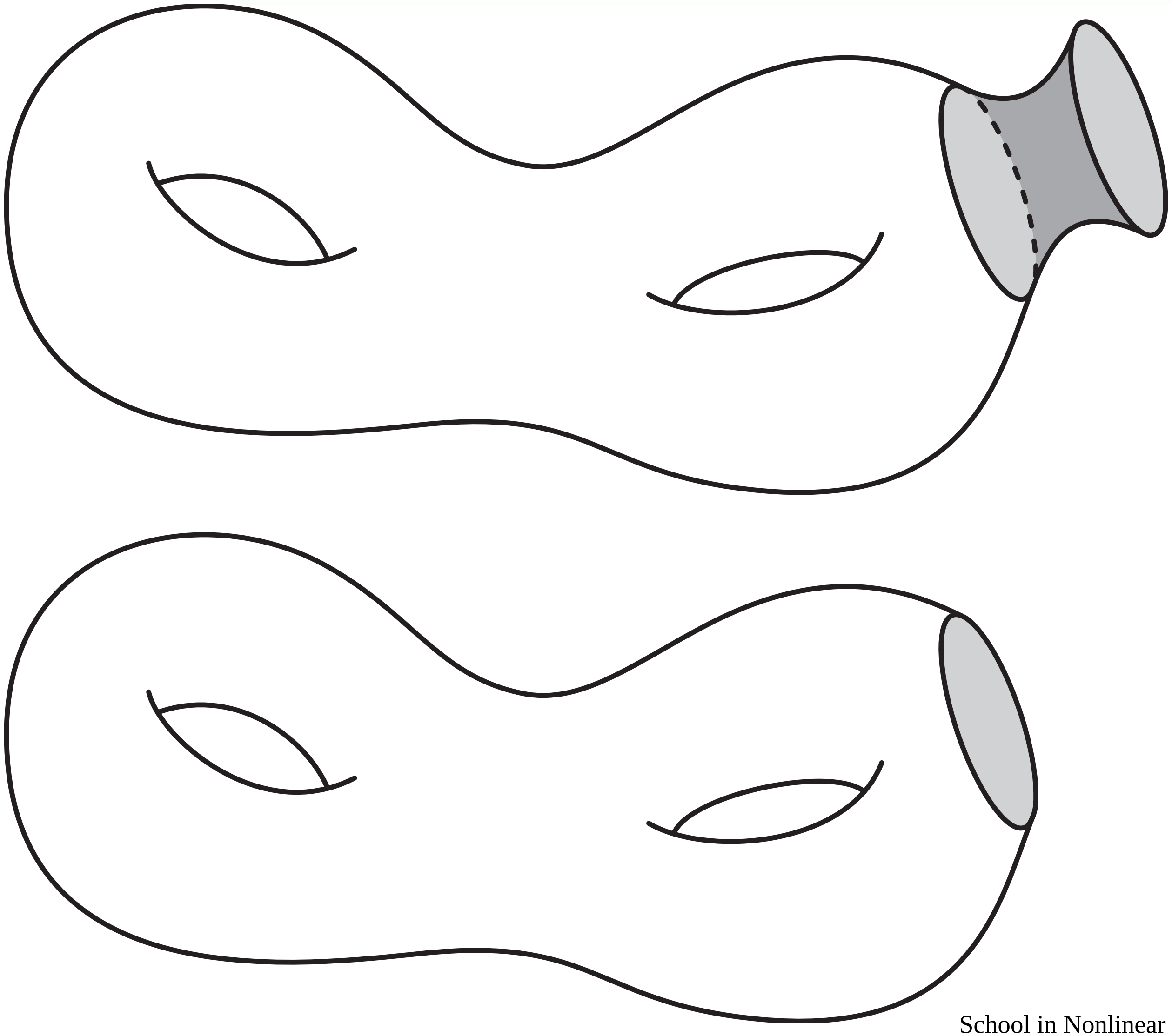
Theorem 4: Under the assumptions of THM 3, if (M, g) and V are *centrally symmetric* around x_0, then there are at least m geometrically distinct homoclinics of **(LP)** emanating from x_0.

Gluing a convex collar to a manifold with boundary



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