On the single-leaf Frobenius Theorem and Its Applications

semi-Riemannian connections
Cartan–Ambrose–Hicks theorem
Affine immersions

Paolo Piccione

Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo

Recenti sviluppi della geometria complessa, differenziale, simplettica

Outline.

1. The single-leaf Frobenius theorem
   - Distributions and integral submanifolds
   - Horizontal distributions and horizontal liftings
   - The Levi form
   - The higher order Frobenius theorem

Outline.

1. The single-leaf Frobenius theorem
   - Distributions and integral submanifolds
   - Horizontal distributions and horizontal liftings
   - The Levi form
   - The higher order Frobenius theorem
2. The global single leaf Frobenius Theorem
   - Sprays on manifolds
   - The global result

Outline.

1. The single-leaf Frobenius theorem
   - Distributions and integral submanifolds
   - Horizontal distributions and horizontal liftings
   - The Levi form
   - The higher order Frobenius theorem
2. The global single leaf Frobenius Theorem
   - Sprays on manifolds
   - The global result
3. Levi–Civita connections
   - Levi form of the horizontal distribution of a connection
   - Connections arising from metric tensors
   - Left invariant connections in Lie groups
   - Constant connections in $\mathbb{R}^n$

Outline.

1. The single-leaf Frobenius theorem
   - Distributions and integral submanifolds
   - Horizontal distributions and horizontal liftings
   - The Levi form
   - The higher order Frobenius theorem
2. The global single leaf Frobenius Theorem
   - Sprays on manifolds
   - The global result
3. Levi–Civita connections
   - Levi form of the horizontal distribution of a connection
   - Connections arising from metric tensors
   - Left invariant connections in Lie groups
   - Constant connections in $\mathbb{R}^n$
4. Existence of affine maps
   - Affine manifolds and affine maps
   - The Cartan–Ambrose–Hicks Theorem
   - Higher order Cartan–Ambrose–Hicks theorem

Outline.

Outline

Smooth distributions and integral submanifolds

E **differentiable manifold**
$\mathcal{D} \subset TE$ a **smooth distribution** (constant rank)
$v \in \mathcal{D}$: v is ***horizontal***

Smooth distributions and integral submanifolds

E differentiable manifold
$\mathcal{D} \subset TE$ a smooth distribution (constant rank)
$v \in \mathcal{D}$: v is *horizontal*
Integral submanifold of $\mathcal{D}$: immersed submanifold $\Sigma \subset E$ with

$$T_e\Sigma = \mathcal{D}_e, \ \forall\, e \in \Sigma$$

Smooth distributions and integral submanifolds

E differentiable manifold
$\mathcal{D} \subset TE$ a smooth distribution (constant rank)
$v \in \mathcal{D}$: v is *horizontal*
Integral submanifold of $\mathcal{D}$: immersed submanifold $\Sigma \subset E$ with

$$T_e\Sigma = \mathcal{D}_e, \ \forall\, e \in \Sigma$$

$\mathcal{D}$ is *integrable* if $\exists$ an integral submanifold through every $e \in E$.

Smooth distributions and integral submanifolds

E differentiable manifold
$\mathcal{D} \subset TE$ a smooth distribution (constant rank)
$v \in \mathcal{D}$: v is *horizontal*
Integral submanifold of $\mathcal{D}$: immersed submanifold $\Sigma \subset E$ with

$$T_e\Sigma = \mathcal{D}_e, \ \forall\, e \in \Sigma$$

$\mathcal{D}$ is *integrable* if $\exists$ an integral submanifold through every $e \in E$.
$\mathcal{D}$ is *involutive* if $\boxed{X, Y \in \mathcal{D} \quad \Longrightarrow \quad [X, Y] \in \mathcal{D}}$

Smooth distributions and integral submanifolds

E differentiable manifold
$\mathcal{D} \subset TE$ a smooth distribution (constant rank)
$v \in \mathcal{D}$: v is *horizontal*
Integral submanifold of $\mathcal{D}$: immersed submanifold $\Sigma \subset E$ with

$$T_e\Sigma = \mathcal{D}_e, \ \forall\, e \in \Sigma$$

$\mathcal{D}$ is *integrable* if $\exists$ an integral submanifold through every $e \in E$.
$\mathcal{D}$ **is *involutive* if** $\boxed{X, Y \in \mathcal{D} \quad \Longrightarrow \quad [X, Y] \in \mathcal{D}}$

Theorem (Frobenius)

$\mathcal{D}$ is integrable $\Longleftrightarrow$ $\mathcal{D}$ is involutive.

Smooth distributions and integral submanifolds

E differentiable manifold
$\mathcal{D} \subset TE$ a smooth distribution (constant rank)
$v \in \mathcal{D}$: v is *horizontal*
Integral submanifold of $\mathcal{D}$: immersed submanifold $\Sigma \subset E$ with

$$T_e\Sigma = \mathcal{D}_e, \ \forall\, e \in \Sigma$$

$\mathcal{D}$ is *integrable* if $\exists$ an integral submanifold through every $e \in E$.
$\mathcal{D}$ is *involutive* if $\boxed{X, Y \in \mathcal{D} \quad \Longrightarrow \quad [X, Y] \in \mathcal{D}}$

Theorem (Frobenius)

$\mathcal{D}$ is integrable $\Longleftrightarrow$ $\mathcal{D}$ is involutive.

Involutivity is a *very strong* condition.

Total differential equations

In local coordinates: $U \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, $F : U \to \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{n-k})$
$\pi : U \to \mathbb{R}^k$ first projection

Total differential equations

In local coordinates: $U \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, $F : U \to \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{n-k})$
$\pi : U \to \mathbb{R}^k$ first projection

$\mathcal{D} = \mathrm{Gr}(F)$ is a π-horizontal distribution on U.

Total differential equations

In local coordinates: $U \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, $F : U \to \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{n-k})$
$\pi : U \to \mathbb{R}^k$ first projection

$\mathcal{D} = \mathrm{Gr}(F)$ is a π-horizontal distribution on U.

A horizontal section $s : V \subset \mathbb{R}^k \to \mathbb{R}^n$ is a map of the form $s(x) = \big(x, f(x)\big)$, where $f : V \to \mathbb{R}^{n-k}$ is a solution of the *total PDE*:

$$\mathrm{d}f(x) = F\big(x, f(x)\big).$$

Total differential equations

In local coordinates: $U \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, $F : U \to \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{n-k})$
$\pi : U \to \mathbb{R}^k$ first projection

$\mathcal{D} = \mathrm{Gr}(F)$ is a π-horizontal distribution on U.

A horizontal section $s : V \subset \mathbb{R}^k \to \mathbb{R}^n$ is a map of the form $s(x) = \big(x, f(x)\big)$, where $f : V \to \mathbb{R}^{n-k}$ is a solution of the *total PDE*:

$$\mathrm{d}f(x) = F\big(x, f(x)\big).$$

$f(x_0) = y_0$, given a curve $u : [0,1] \to V$ with $u(0) = x_0$ and $u(1) = x_1$, then $g = f \circ u : [0,1] \to \mathbb{R}^{n-k}$ is a solution of the IVP:

$$g'(t) = F\big(u(t), g(t)\big)u'(t), \quad u(0) = f(y_0).$$

Total differential equations

In local coordinates: $U \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, $F : U \to \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{n-k})$
$\pi : U \to \mathbb{R}^k$ first projection

$\mathcal{D} = \mathrm{Gr}(F)$ is a π-horizontal distribution on U.

A horizontal section $s : V \subset \mathbb{R}^k \to \mathbb{R}^n$ is a map of the form $s(x) = \big(x, f(x)\big)$, where $f : V \to \mathbb{R}^{n-k}$ is a solution of the *total PDE*:

$$\mathrm{d}f(x) = F\big(x, f(x)\big).$$

$f(x_0) = y_0$, given a curve $u : [0,1] \to V$ with $u(0) = x_0$ and $u(1) = x_1$, then $g = f \circ u : [0,1] \to \mathbb{R}^{n-k}$ is a solution of the IVP:

$$g'(t) = F\big(u(t), g(t)\big)u'(t), \quad u(0) = f(y_0).$$

Involutivity of $\mathcal{D}$ is the integrability condition for such PDE.

Horizontal liftings (some terminology)

$\pi : E \to M$ submersion. $\mathcal{D} \subset TE$ is π*-horizontal* if $T_eE = \mathrm{Ker}(\pi_e) \oplus \mathcal{D}_e$ for all $e \in E$.

Horizontal liftings (some terminology)

$\pi : E \to M$ submersion. $\mathcal{D} \subset TE$ is π-*horizontal* if $T_eE = \mathrm{Ker}\big(\pi_e\big) \oplus \mathcal{D}_e$ for all $e \in E$.

$\widetilde{\gamma} : I \to E$ is *horizontal* if $\widetilde{\gamma}'(t) \in \mathcal{D}$ for all t. Given $\gamma : I \to M$ then a *horizontal lifting* of γ is a horizontal curve $\tilde{\gamma} : I \to E$ such that $\pi \circ \tilde{\gamma} = \gamma$.

Horizontal liftings (some terminology)

$\pi : E \to M$ submersion. $\mathcal{D} \subset TE$ is *π-horizontal* if $T_eE = \mathrm{Ker}\big(\pi_e\big) \oplus \mathcal{D}_e$ for all $e \in E$.

$\widetilde{\gamma} : I \to E$ is *horizontal* if $\widetilde{\gamma}'(t) \in \mathcal{D}$ for all t. Given $\gamma : I \to M$ then a *horizontal lifting* of γ is a horizontal curve $\tilde{\gamma} : I \to E$ such that $\pi \circ \tilde{\gamma} = \gamma$. By standard theory of ODE's, given $t_0 \in I$ and $x_0 \in \pi^{-1}\big(\gamma(t_0)\big)$ then $\exists!$ maximal horizontal lifting $\tilde{\gamma}$ of γ with $\tilde{\gamma}(t_0) = x_0$ defined in a subinterval of I around t_0.

Horizontal liftings (some terminology)

$\pi : E \to M$ submersion. $\mathcal{D} \subset TE$ is *π-horizontal* if $T_eE = \mathrm{Ker}\big(\pi_e\big) \oplus \mathcal{D}_e$ for all $e \in E$.

$\widetilde{\gamma} : I \to E$ is *horizontal* if $\widetilde{\gamma}'(t) \in \mathcal{D}$ for all t. Given $\gamma : I \to M$ then a *horizontal lifting* of γ is a horizontal curve $\tilde{\gamma} : I \to E$ such that $\pi \circ \tilde{\gamma} = \gamma$. By standard theory of ODE's, given $t_0 \in I$ and $x_0 \in \pi^{-1}\big(\gamma(t_0)\big)$ then $\exists!$ maximal horizontal lifting $\tilde{\gamma}$ of γ with $\tilde{\gamma}(t_0) = x_0$ defined in a subinterval of I around t_0.

A *local section* of a smooth submersion $\pi : E \to M$ is a locally defined smooth map $s : U \subset M \to E$ such that $\pi \circ s = \mathrm{Id}_U$. A local section s is called *horizontal* if the range of $\mathrm{d}s(m)$ is $\mathcal{D}_{s(m)}$, for all $m \in U$.

Λ-parametric family of curves

Λ-parametric family of curves ψ on M: $\psi : Z \subset \mathbb{R} \times \Lambda \to M$, Z open, such that: $I_\lambda = \{t \in \mathbb{R} : (t, \lambda) \in Z\} \subset \mathbb{R}$ is an interval containing the origin, for all $\lambda \in \Lambda$.

Λ-parametric family of curves

Λ-parametric family of curves ψ on M: $\psi : Z \subset \mathbb{R} \times \Lambda \to M$, Z open, such that: $I_\lambda = \{t \in \mathbb{R} : (t, \lambda) \in Z\} \subset \mathbb{R}$ is an interval containing the origin, for all $\lambda \in \Lambda$.

A *local right inverse* of ψ: a locally defined smooth map $\alpha : V \subset M \to Z$ such that $\psi(\alpha(m)) = m$, for all $m \in V$.

Λ-parametric family of curves

Λ-parametric family of curves ψ on M: $\psi : Z \subset \mathbb{R} \times \Lambda \to M$, Z open, such that: $I_\lambda = \big\{ t \in \mathbb{R} : (t, \lambda) \in Z \big\} \subset \mathbb{R}$ is an interval containing the origin, for all $\lambda \in \Lambda$.
A *local right inverse* of ψ: a locally defined smooth map $\alpha : V \subset M \to Z$ such that $\psi\big(\alpha(m)\big) = m$, for all $m \in V$.

Example

M manifold, ∇ connection on M. Given $x_0 \in M$, set $\Lambda = T_{x_0}M$.
Λ-parametric family of curves ψ on M: $\psi(t, \lambda) = \exp_{x_0}(t\lambda)$;

Λ-parametric family of curves

Λ*-parametric family of curves* ψ on M: $\psi : Z \subset \mathbb{R} \times \Lambda \to M$, Z open, such that: $I_\lambda = \big\{ t \in \mathbb{R} : (t, \lambda) \in Z \big\} \subset \mathbb{R}$ is an interval containing the origin, for all $\lambda \in \Lambda$.

A *local right inverse* of ψ: a locally defined smooth map $\alpha : V \subset M \to Z$ such that $\psi\big(\alpha(m)\big) = m$, for all $m \in V$.

Example

M manifold, ∇ connection on M. Given $x_0 \in M$, set $\Lambda = T_{x_0}M$. Λ-parametric family of curves ψ on M: $\psi(t, \lambda) = \exp_{x_0}(t\lambda)$; $Z = \Big\{ (t, \lambda) : t\lambda \in \mathrm{Dom}\big(\exp_{x_0}\big) \Big\}$.

Λ-parametric family of curves

Λ-parametric family of curves ψ on M: $\psi : Z \subset \mathbb{R} \times \Lambda \to M$, Z open, such that: $I_\lambda = \{t \in \mathbb{R} : (t, \lambda) \in Z\} \subset \mathbb{R}$ is an interval containing the origin, for all $\lambda \in \Lambda$.
A *local right inverse* of ψ: a locally defined smooth map $\alpha : V \subset M \to Z$ such that $\psi\big(\alpha(m)\big) = m$, for all $m \in V$.

Example

M manifold, ∇ connection on M. Given $x_0 \in M$, set $\Lambda = T_{x_0}M$.
Λ-parametric family of curves ψ on M: $\psi(t, \lambda) = \exp_{x_0}(t\lambda)$;
$Z = \Big\{(t, \lambda) : t\lambda \in \mathrm{Dom}\big(\exp_{x_0}\big)\Big\}$.
A local right inverse of ψ: V_0 open neighborhood of $0 \in T_{x_0}M$ mapped diffeomorphically by $\exp_{x_0}$ onto an open neighborhood V of $x_0 \in M$.
Set:

$$\alpha(m) = \big(1, (\exp_{x_0}|_{V_0})^{-1}(m)\big), \qquad m \in V.$$

Λ-parametric family of curves

Λ-parametric family of curves ψ on M: $\psi : Z \subset \mathbb{R} \times \Lambda \to M$, Z open, such that: $I_\lambda = \big\{t \in \mathbb{R} : (t, \lambda) \in Z\big\} \subset \mathbb{R}$ is an interval containing the origin, for all $\lambda \in \Lambda$.
A *local right inverse* of ψ: a locally defined smooth map $\alpha : V \subset M \to Z$ such that $\psi\big(\alpha(m)\big) = m$, for all $m \in V$.

Example

M manifold, ∇ connection on M. Given $x_0 \in M$, set $\Lambda = T_{x_0}M$.
Λ-parametric family of curves ψ on M: $\psi(t, \lambda) = \exp_{x_0}(t\lambda)$;
$Z = \Big\{(t, \lambda) : t\lambda \in \mathrm{Dom}\big(\exp_{x_0}\big)\Big\}$.
A local right inverse of ψ: V_0 open neighborhood of $0 \in T_{x_0}M$ mapped diffeomorphically by $\exp_{x_0}$ onto an open neighborhood V of $x_0 \in M$.
Set:

$$\alpha(m) = \big(1, (\exp_{x_0}|_{V_0})^{-1}(m)\big), \qquad m \in V.$$

(same construction holds if one replaces the geodesic spray of a connection with an arbitrary spray).

The Levi form

If V, W are (local) horizontal fields, $[V, W]_p + \mathcal{D}_p \in T_pM/\mathcal{D}_p$ only depends on V_p, W_p.

The Levi form

If V, W are (local) horizontal fields, $[V, W]_p + \mathcal{D}_p \in T_pM/\mathcal{D}_p$ only depends on V_p, W_p.

Definition

$\mathfrak{L}_p^{\mathcal{D}} : \mathcal{D}_p \times \mathcal{D}_p \to T_pM/\mathcal{D}_p$ *Levi form* of $\mathcal{D}$ at p:
$\mathfrak{L}_p^{\mathcal{D}}(v, w) = [V, W]_p + \mathcal{D}_p$, where $v, w \in \mathcal{D}_p$ and V, W are local extensions of v and w to horizontal fields.

The Levi form

If V, W are (local) horizontal fields, $[V, W]_p + \mathcal{D}_p \in T_pM/\mathcal{D}_p$ only depends on V_p, W_p.

Definition

$\mathfrak{L}_p^{\mathcal{D}} : \mathcal{D}_p \times \mathcal{D}_p \to T_pM/\mathcal{D}_p$ *Levi form* of $\mathcal{D}$ at p:
$\mathfrak{L}_p^{\mathcal{D}}(v, w) = [V, W]_p + \mathcal{D}_p$, where $v, w \in \mathcal{D}_p$ and V, W are local extensions of v and w to horizontal fields.

$p \in M$ is an *involutive point* for $\mathcal{D}$ if $\mathfrak{L}_p^{\mathcal{D}} = 0$.

The Levi form

If V, W are (local) horizontal fields, $[V, W]_p + \mathcal{D}_p \in T_pM/\mathcal{D}_p$ only depends on V_p, W_p.

Definition

$\mathfrak{L}_p^{\mathcal{D}} : \mathcal{D}_p \times \mathcal{D}_p \to T_pM/\mathcal{D}_p$ *Levi form* of $\mathcal{D}$ at p:
$\mathfrak{L}_p^{\mathcal{D}}(v, w) = [V, W]_p + \mathcal{D}_p$, where $v, w \in \mathcal{D}_p$ and V, W are local extensions of v and w to horizontal fields.

$p \in M$ is an *involutive point* for $\mathcal{D}$ if $\mathfrak{L}_p^{\mathcal{D}} = 0$.
Obs.: If Σ is an integral submanifold, then every point of Σ is involutive.

The Levi form

If V, W are (local) horizontal fields, $[V, W]_p + \mathcal{D}_p \in T_pM/\mathcal{D}_p$ only depends on V_p, W_p.

Definition

$\mathfrak{L}_p^{\mathcal{D}} : \mathcal{D}_p \times \mathcal{D}_p \to T_pM/\mathcal{D}_p$ *Levi form* of $\mathcal{D}$ at p:
$\mathfrak{L}_p^{\mathcal{D}}(v, w) = [V, W]_p + \mathcal{D}_p$, where $v, w \in \mathcal{D}_p$ and V, W are local extensions of v and w to horizontal fields.

$p \in M$ is an *involutive point* for $\mathcal{D}$ if $\mathfrak{L}_p^{\mathcal{D}} = 0$.
Obs.: If Σ is an integral submanifold, then every point of Σ is involutive.
Conversely, if:

- Σ is *ruled* by curves tangent to $\mathcal{D}$
- every point of Σ is involutive

then Σ is an integral submanifold of $\mathcal{D}$.

The single leaf Frobenius theorem 1

Problem: We want to find conditions for the existence of *one* integral submanifold of $\mathcal{D} \subset TE$ through some given $e \in E$.

The single leaf Frobenius theorem 1

Problem: We want to find conditions for the existence of *one* integral submanifold of $\mathcal{D} \subset TE$ through some given $e \in E$.

Lemma

E manifold, $\mathcal{D} \subset TE$ distribution,
$\mathbb{R}^2 \supset U \ni (t, s) \longmapsto H(t, s) \in E$ smooth map.
$I \subset \mathbb{R}$ interval, $s_0 \in \mathbb{R}$ with $I \times \{s_0\} \subset U$.

The single leaf Frobenius theorem 1

Problem: We want to find conditions for the existence of *one* integral submanifold of $\mathcal{D} \subset TE$ through some given $e \in E$.

Lemma

E manifold, $\mathcal{D} \subset TE$ distribution,
$\mathbb{R}^2 \supset U \ni (t, s) \longmapsto H(t, s) \in E$ *smooth map.*
$I \subset \mathbb{R}$ *interval,* $s_0 \in \mathbb{R}$ *with* $I \times \{s_0\} \subset U$. *If:*

- $\frac{\partial H}{\partial t}(t, s) \in \mathcal{D}$ *for all* $(t, s) \in U$;

The single leaf Frobenius theorem 1

Problem: We want to find conditions for the existence of *one* integral submanifold of $\mathcal{D} \subset TE$ through some given $e \in E$.

Lemma

E manifold, $\mathcal{D} \subset TE$ distribution,
$\mathbb{R}^2 \supset U \ni (t,s) \longmapsto H(t,s) \in E$ smooth map.
$I \subset \mathbb{R}$ interval, $s_0 \in \mathbb{R}$ with $I \times \{s_0\} \subset U$. If:

- $\frac{\partial H}{\partial t}(t,s) \in \mathcal{D}$ *for all* $(t,s) \in U$;
- $\mathfrak{L}^{\mathcal{D}}_{H(t,s_0)} = 0$ *for all* $t \in I$;

The single leaf Frobenius theorem 1

Problem: We want to find conditions for the existence of *one* integral submanifold of $\mathcal{D} \subset TE$ through some given $e \in E$.

Lemma

E manifold, $\mathcal{D} \subset TE$ distribution,
$\mathbb{R}^2 \supset U \ni (t,s) \longmapsto H(t,s) \in E$ smooth map.
$I \subset \mathbb{R}$ interval, $s_0 \in \mathbb{R}$ with $I \times \{s_0\} \subset U$. If:

- $\frac{\partial H}{\partial t}(t,s) \in \mathcal{D}$ *for all* $(t,s) \in U$;
- $\mathfrak{L}^{\mathcal{D}}_{H(t,s_0)} = 0$ *for all* $t \in I$;
- $\frac{\partial H}{\partial s}(t_0,s_0) \in \mathcal{D}$ *for some* $t_0 \in I$

The single leaf Frobenius theorem 1

Problem: We want to find conditions for the existence of *one* integral submanifold of $\mathcal{D} \subset TE$ through some given $e \in E$.

Lemma

E manifold, $\mathcal{D} \subset TE$ distribution,
$\mathbb{R}^2 \supset U \ni (t,s) \longmapsto H(t,s) \in E$ smooth map.
$I \subset \mathbb{R}$ interval, $s_0 \in \mathbb{R}$ with $I \times \{s_0\} \subset U$. If:

- $\frac{\partial H}{\partial t}(t,s) \in \mathcal{D}$ *for all* $(t,s) \in U$;
- $\mathfrak{L}^{\mathcal{D}}_{H(t,s_0)} = 0$ *for all* $t \in I$;
- $\frac{\partial H}{\partial s}(t_0,s_0) \in \mathcal{D}$ *for some* $t_0 \in I$

then $\frac{\partial H}{\partial s}(t,s_0) \in \mathcal{D}$ *for all* $t \in I$.

The single leaf Frobenius theorem 2

Theorem (local single leaf Frobenius)

$\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution
$\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves with a local right inverse $\alpha : V \subset M \to Z$.
Let $\tilde{\psi} : Z \to E$ be a Λ-parametric family of curves on E such that $t \mapsto \tilde{\psi}(t, \lambda)$ is a horizontal lifting of $t \mapsto \psi(t, \lambda)$, for all $\lambda \in \Lambda$.

The single leaf Frobenius theorem 2

Theorem (local single leaf Frobenius)

$\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution
$\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves with a local right inverse $\alpha : V \subset M \to Z$.
Let $\tilde{\psi} : Z \to E$ be a Λ-parametric family of curves on E such that $t \mapsto \tilde{\psi}(t, \lambda)$ is a horizontal lifting of $t \mapsto \psi(t, \lambda)$, for all $\lambda \in \Lambda$. Assume that:

(a) *the Levi form of $\mathcal{D}$ vanishes on the range of $\tilde{\psi}$;*

The single leaf Frobenius theorem 2

Theorem (local single leaf Frobenius)

$\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution
$\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves with a local right inverse $\alpha : V \subset M \to Z$.
Let $\tilde{\psi} : Z \to E$ be a Λ-parametric family of curves on E such that $t \mapsto \tilde{\psi}(t, \lambda)$ is a horizontal lifting of $t \mapsto \psi(t, \lambda)$, for all $\lambda \in \Lambda$. Assume that:

(a) *the Levi form of $\mathcal{D}$ vanishes on the range of $\tilde{\psi}$;*

(b) *$\partial_\lambda \tilde{\psi}(0, \lambda) : T_\lambda \Lambda \to T_{\tilde{\psi}(0,\lambda)} E$ takes values in $\mathcal{D}$ for all $\lambda \in \Lambda$.*

The single leaf Frobenius theorem 2

Theorem (local single leaf Frobenius)

$\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution
$\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves with a local right inverse $\alpha : V \subset M \to Z$.
Let $\tilde{\psi} : Z \to E$ be a Λ-parametric family of curves on E such that $t \mapsto \tilde{\psi}(t, \lambda)$ is a horizontal lifting of $t \mapsto \psi(t, \lambda)$, for all $\lambda \in \Lambda$. Assume that:

(a) *the Levi form of $\mathcal{D}$ vanishes on the range of $\tilde{\psi}$;*

(b) *$\partial_\lambda \tilde{\psi}(0, \lambda) : T_\lambda \Lambda \to T_{\tilde{\psi}(0,\lambda)} E$ takes values in $\mathcal{D}$ for all $\lambda \in \Lambda$.*

Then $s = \tilde{\psi} \circ \alpha : V \to E$ is a local horizontal section of π.

The single leaf Frobenius theorem 2

Theorem (local single leaf Frobenius)

$\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution
$\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves with a local right inverse $\alpha : V \subset M \to Z$.
Let $\tilde{\psi} : Z \to E$ be a Λ-parametric family of curves on E such that $t \mapsto \tilde{\psi}(t, \lambda)$ is a horizontal lifting of $t \mapsto \psi(t, \lambda)$, for all $\lambda \in \Lambda$. Assume that:

(a) *the Levi form of $\mathcal{D}$ vanishes on the range of $\tilde{\psi}$;*

(b) *$\partial_\lambda \tilde{\psi}(0, \lambda) : T_\lambda \Lambda \to T_{\tilde{\psi}(0,\lambda)} E$ takes values in $\mathcal{D}$ for all $\lambda \in \Lambda$.*

Then $s = \tilde{\psi} \circ \alpha : V \to E$ is a local horizontal section of π.

Obs.: If $\lambda \mapsto \widetilde{\psi}(0, \lambda)$ is constant, then (b) is satisfied.

The higher order Frobenius theorem

$\mathcal{D} \subset TE$ smooth distribution
$\Gamma(TE)$ Lie algebra of vector fields on E
$\Gamma(\mathcal{D}) = \Gamma^0(\mathcal{D})$ space of horizontal vector fields

The higher order Frobenius theorem

$\mathcal{D} \subset TE$ smooth distribution
$\Gamma(TE)$ Lie algebra of vector fields on E
$\Gamma(\mathcal{D}) = \Gamma^0(\mathcal{D})$ space of horizontal vector fields

Define recursively $\Gamma^{r+1}(\mathcal{D}) \subset \Gamma(TE)$ as the space spanned by $\Gamma^r(\mathcal{D})$ and Lie brackets of the form $[X, Y]$, with $X \in \Gamma^r(\mathcal{D})$ and $Y \in \Gamma(\mathcal{D})$.
$\Gamma^\infty = \bigcup_{r=0}^{\infty} \Gamma^r(\mathcal{D})$: Lie subalgebra of TE spanned by $\Gamma(\mathcal{D})$.

The higher order Frobenius theorem

$\mathcal{D} \subset TE$ smooth distribution
$\Gamma(TE)$ Lie algebra of vector fields on E
$\Gamma(\mathcal{D}) = \Gamma^0(\mathcal{D})$ space of horizontal vector fields

Define recursively $\Gamma^{r+1}(\mathcal{D}) \subset \Gamma(TE)$ as the space spanned by $\Gamma^r(\mathcal{D})$ and Lie brackets of the form $[X, Y]$, with $X \in \Gamma^r(\mathcal{D})$ and $Y \in \Gamma(\mathcal{D})$.
$\Gamma^\infty = \bigcup_{r=0}^{\infty} \Gamma^r(\mathcal{D})$: Lie subalgebra of TE spanned by $\Gamma(\mathcal{D})$.

Theorem

If E is real analytic manifold and $\mathcal{D}$ is a real analytic distribution, then given $e_0 \in E$, there exists an integral submanifold of $\mathcal{D}$ through e_0 iff $X(e_0) \in \mathcal{D}_{e_0}$ for all $X \in \Gamma^\infty(\mathcal{D})$.

Outline

Sprays on manifolds

M manifold, $\pi : TM \to M$ tangent bundle, $\mathrm{d}\pi : T(TM) \to TM$,
$\bar{\pi} : T(TM) \to TM$
For $a \in \mathbb{R}$, $\mathfrak{m}_a : TM \to TM$ multiplication by a.

Definition

A *spray* on M is a vector field $\mathcal{S} : TM \to T(TM)$ such that:

- $\mathrm{d}\pi \circ \mathcal{S} = \bar{\pi} \circ \mathcal{S}$
- $a\, \mathrm{d}\mathfrak{m}_a \circ \mathcal{S} = S \circ \mathfrak{m}_a$ for all $a \in \mathbb{R}$.

Sprays on manifolds

M manifold, $\pi : TM \to M$ tangent bundle, $\mathrm{d}\pi : T(TM) \to TM$,
$\bar{\pi} : T(TM) \to TM$
For $a \in \mathbb{R}$, $\mathfrak{m}_a : TM \to TM$ multiplication by a.

Definition

A *spray* on M is a vector field $\mathcal{S} : TM \to T(TM)$ such that:

- $\mathrm{d}\pi \circ \mathcal{S} = \bar{\pi} \circ \mathcal{S}$
- $a\,\mathrm{d}\mathfrak{m}_a \circ \mathcal{S} = S \circ \mathfrak{m}_a$ for all $a \in \mathbb{R}$.

Integral curves $\lambda : I \to TM$ of $\mathcal{S}$ are of the form $\lambda = \gamma'$, $\gamma = \pi \circ \lambda$.
Given $\lambda = \gamma'$ integral curve, also $t \mapsto a \cdot \gamma'(at)$ is an integral curve of $\mathcal{S}$.

Example (Geodesic spray)

∇ connection on M, $\mathcal{S}(v)$ is the unique horizontal vector in $T_v(TM)$ with $\mathrm{d}\pi_v\big(\mathcal{S}(v)\big)$. Integral curves of $\mathcal{S}$ are γ', with γ geodesic.

Example (Geodesic spray)

∇ connection on M, $\mathcal{S}(v)$ is the unique horizontal vector in $T_v(TM)$ with $\mathrm{d}\pi_v\big(\mathcal{S}(v)\big)$. Integral curves of $\mathcal{S}$ are γ', with γ geodesic.

Example (one parameter subgroup spray)

G Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. $TG \cong G \times \mathfrak{g}$, hence:

$$T(TG) \cong T(G \times \mathfrak{g}) \cong (TG) \times (T\mathfrak{g}) \cong (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g})$$

$\mathcal{S}(g, X) = (g, X, X, 0)$, $g \in G$, $X \in \mathfrak{X}$ is a spray whose solutions are (translations of) one-parameter subgroups of G.

Example (Geodesic spray)

∇ connection on M, $\mathcal{S}(v)$ is the unique horizontal vector in $T_v(TM)$ with $\mathrm{d}\pi_v\big(\mathcal{S}(v)\big)$. Integral curves of $\mathcal{S}$ are γ', with γ geodesic.

Example (one parameter subgroup spray)

G Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. $TG \cong G \times \mathfrak{g}$, hence:

$$T(TG) \cong T(G \times \mathfrak{g}) \cong (TG) \times (T\mathfrak{g}) \cong (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g})$$

$\mathcal{S}(g, X) = (g, X, X, 0)$, $g \in G$, $X \in \mathfrak{X}$ is a spray whose solutions are (translations of) one-parameter subgroups of G.

Local theory of solutions of sprays totally analogous to geodesics. There exist *normal neighborhoods* of every point.

Theorem (Global single-leaf Frobenius theorem)

E, M manifolds, $\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution, $\mathcal{S}$ spray on M. Fix $x_0 \in M$ and $e_0 \in \pi^{-1}(x_0) \in E$.

Theorem (Global single-leaf Frobenius theorem)

E, M manifolds, $\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution, $\mathcal{S}$ spray on M. Fix $x_0 \in M$ and $e_0 \in \pi^{-1}(x_0) \in E$. Assume:

- *every piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$ admits a horizontal lifting $\widetilde{\gamma} : [a, b] \to E$ with $\widetilde{\gamma}(a) = e_0$;*

Theorem (Global single-leaf Frobenius theorem)

E, M manifolds, $\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution, $\mathcal{S}$ spray on M. Fix $x_0 \in M$ and $e_0 \in \pi^{-1}(x_0) \in E$. Assume:

- *every piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$ admits a horizontal lifting $\widetilde{\gamma} : [a, b] \to E$ with $\widetilde{\gamma}(a) = e_0$;*
- *if $\widetilde{\gamma} : [a, b] \to E$ is as above, then $\mathfrak{L}^{\mathcal{D}}_{\widetilde{\gamma}(b)} = 0$;*

Theorem (Global single-leaf Frobenius theorem)

E, M manifolds, $\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution, $\mathcal{S}$ spray on M. Fix $x_0 \in M$ and $e_0 \in \pi^{-1}(x_0) \in E$. Assume:

- *every piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$ admits a horizontal lifting $\widetilde{\gamma} : [a, b] \to E$ with $\widetilde{\gamma}(a) = e_0$;*
- *if $\widetilde{\gamma} : [a, b] \to E$ is as above, then $\mathfrak{L}^{\mathcal{D}}_{\widetilde{\gamma}(b)} = 0$;*
- *M is (connected and) simply connected.*

Theorem (Global single-leaf Frobenius theorem)

E, M manifolds, $\pi : E \to M$ submersion, $\mathcal{D} \subset TE$ horizontal distribution, $\mathcal{S}$ spray on M. Fix $x_0 \in M$ and $e_0 \in \pi^{-1}(x_0) \in E$. Assume:

- *every piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = x_0$ admits a horizontal lifting $\widetilde{\gamma} : [a, b] \to E$ with $\widetilde{\gamma}(a) = e_0$;*
- *if $\widetilde{\gamma} : [a, b] \to E$ is as above, then $\mathfrak{L}^{\mathcal{D}}_{\widetilde{\gamma}(b)} = 0$;*
- *M is (connected and) simply connected.*

Then, there exists a unique global horizontal section s of E with $s(x_0) = e_0$.

Global higher order Frobenius theorem

Theorem

E, M real analytic manifolds, $\pi : E \longrightarrow M$ real analytic submersion, $\mathcal{D} \subset TE$ real analytic horizontal distribution.

Global higher order Frobenius theorem

Theorem

E, M real analytic manifolds, $\pi : E \to M$ real analytic submersion, $\mathcal{D} \subset TE$ real analytic horizontal distribution.
Assume:

- *M is connected and simply connected;*
- *if $\gamma : I \to M$ is real analytic, $t_0 \in I$, $e_0 \in \pi^{-1}\big(\gamma(t_0)\big)$, then there exists a horizontal lifting $\widetilde{\gamma} : I \to E$ with $\widetilde{\gamma}(t_0) = e_0$.*

Global higher order Frobenius theorem

Theorem

E, M real analytic manifolds, $\pi : E \to M$ real analytic submersion, $\mathcal{D} \subset TE$ real analytic horizontal distribution.
Assume:

- *M is connected and simply connected;*
- *if $\gamma : I \to M$ is real analytic, $t_0 \in I$, $e_0 \in \pi^{-1}\big(\gamma(t_0)\big)$, then there exists a horizontal lifting $\widetilde{\gamma} : I \to E$ with $\widetilde{\gamma}(t_0) = e_0$.*

Then, every local horizontal section of π defined on a nonempty connected open subset of M extends to a global horizontal section of π.

Global higher order Frobenius theorem

Theorem

E, M real analytic manifolds, $\pi : E \to M$ real analytic submersion, $\mathcal{D} \subset TE$ real analytic horizontal distribution.
Assume:

- *M is connected and simply connected;*
- *if $\gamma : I \to M$ is real analytic, $t_0 \in I$, $e_0 \in \pi^{-1}\big(\gamma(t_0)\big)$, then there exists a horizontal lifting $\widetilde{\gamma} : I \to E$ with $\widetilde{\gamma}(t_0) = e_0$.*

Then, every local horizontal section of π defined on a nonempty connected open subset of M extends to a global horizontal section of π.
In particular, if $\mathcal{D}$ satisfies the assumptions of the Higher Order Frobenius theorem at some point $e_0 \in E$, then π admits a global horizontal section.

Outline

Levi form of the horizontal distribution of a connection

$\pi : E \to M$ vector bundle, $E_m = \pi^{-1}(m)$ fiber,
∇ connection on E.
$R_m : T_mM \times T_mM \times E_m \to E_m$ *curvature* of ∇:

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$$

Levi form of the horizontal distribution of a connection

$\pi : E \to M$ vector bundle, $E_m = \pi^{-1}(m)$ fiber,
∇ connection on E.
$R_m : T_m M \times T_m M \times E_m \to E_m$ *curvature* of ∇:

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$$

$\mathcal{D} \subset TE$ *horizontal distribution* of ∇: given $\gamma : I \to M$, a horizontal lifting $\widetilde{\gamma} : I \to E$ is a ∇-parallel section of E along γ.

Levi form of the horizontal distribution of a connection

$\pi : E \to M$ vector bundle, $E_m = \pi^{-1}(m)$ fiber,
∇ connection on E.
$R_m : T_mM \times T_mM \times E_m \to E_m$ *curvature* of ∇:

$$R(X, Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi$$

$\mathcal{D} \subset TE$ *horizontal distribution* of ∇: given $\gamma : I \to M$, a horizontal lifting $\widetilde{\gamma} : I \to E$ is a ∇-parallel section of E along γ.

- For $\xi \in E_m$, $T_\xi E/\mathcal{D}_\xi \cong T_\xi(E_m) = \mathrm{Ker}(\mathrm{d}\pi_\xi)$.
- $T_\xi(E_m) \cong E_m$
- $\mathrm{d}\pi_\xi : \mathcal{D}_\xi \xrightarrow{\cong} T_mM$.
- $\mathfrak{L}_\xi^{\mathcal{D}} : T_mM \times T_mM \longrightarrow E_m$

Levi form of the horizontal distribution of a connection

$\pi : E \to M$ vector bundle, $E_m = \pi^{-1}(m)$ fiber,
∇ connection on E.
$R_m : T_mM \times T_mM \times E_m \to E_m$ *curvature* of ∇:

$$R(X, Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi$$

$\mathcal{D} \subset TE$ *horizontal distribution* of ∇: given $\gamma : I \to M$, a horizontal lifting $\widetilde{\gamma} : I \to E$ is a ∇-parallel section of E along γ.

- For $\xi \in E_m$, $T_\xi E/\mathcal{D}_\xi \cong T_\xi(E_m) = \mathrm{Ker}(\mathrm{d}\pi_\xi)$.
- $T_\xi(E_m) \cong E_m$
- $\mathrm{d}\pi_\xi : \mathcal{D}_\xi \xrightarrow{\cong} T_mM$.
- $\mathfrak{L}^{\mathcal{D}}_\xi : T_mM \times T_mM \longrightarrow E_m$

Lemma

$\mathfrak{L}^{\mathcal{D}}_\xi(v, w) = -R_m(v, w)\xi$, $m \in M$, $\xi \in E_m$.

Corollary

$\pi : E \to M$ *vector bundle,* ∇ *connection on* E.
$\psi : Z \subset \mathbb{R} \times \Lambda$ *a* Λ *parametric family of curves in* M
$\alpha : V \subset M \to Z$ *local right inverse of* ψ
$\widetilde{\psi} : Z \to E$ *a section of* E *along* ψ.

Corollary

$\pi : E \to M$ *vector bundle,* ∇ *connection on* E*.*
$\psi : Z \subset \mathbb{R} \times \Lambda$ *a* Λ *parametric family of curves in* M
$\alpha : V \subset M \to Z$ *local right inverse of* ψ
$\widetilde{\psi} : Z \to E$ *a section of* E *along* ψ*. If:*

- $t \mapsto \widetilde{\psi}(t, \lambda)$ *is parallel for all* $\lambda \in \Lambda$*;*
- $\lambda \mapsto \widetilde{\psi}(0, \lambda)$ *is parallel;*
- $R_{\psi(t,\lambda)}(v, w)\widetilde{\psi}(t, \lambda) = 0$ *for all* $v, w \in T_{\psi(t,\lambda)}M$ *and all* $(t, \lambda) \in Z$

then $\widetilde{\psi} \circ \alpha$ *is a (local) parallel section of* E*.*

Corollary

$\pi : E \longrightarrow M$ vector bundle with connection ∇.
$\mathcal{S}$ spray on M, $x_0 \in M$, $e_0 \in \pi^{-1}(x_0)$.

Corollary

$\pi : E \to M$ vector bundle with connection ∇.
$\mathcal{S}$ spray on M, $x_0 \in M$, $e_0 \in \pi^{-1}(x_0)$. Assume:

- *if $\gamma : [a, b] \to M$ is a piecewise solution of $\mathcal{S}$ with $\gamma(a) = x_0$, and $\widetilde{\gamma} : [a, b] \to E$ is a section of E along γ with $\widetilde{\gamma}(a) = e_0$, then $R_{\gamma(b)}(v, w)\widetilde{\gamma}(b) = 0$ for all $v, w \in T_{\gamma(b)}M$;*
- *M is (connected and) simply connected.*

Corollary

$\pi : E \to M$ vector bundle with connection ∇.
$\mathcal{S}$ spray on M, $x_0 \in M$, $e_0 \in \pi^{-1}(x_0)$. Assume:

- *if $\gamma : [a, b] \to M$ is a piecewise solution of $\mathcal{S}$ with $\gamma(a) = x_0$, and $\widetilde{\gamma} : [a, b] \to E$ is a section of E along γ with $\widetilde{\gamma}(a) = e_0$, then $R_{\gamma(b)}(v, w)\widetilde{\gamma}(b) = 0$ for all $v, w \in T_{\gamma(b)}M$;*
- *M is (connected and) simply connected.*

Then there exists a unique global parallel section s of E with $s(x_0) = e_0$.

Lemma

$\pi : E \to M$ real analytic vector bundle with real analytic connection ∇.

Lemma

$\pi : E \longrightarrow M$ real analytic vector bundle with real analytic connection ∇. If M is simply connected, then any local parallel section s of E, defined on a non empty connected open subset $U \subset M$, extends to a (unique) global parallel section.

Lemma

$\pi : E \to M$ real analytic vector bundle with real analytic connection ∇. If M is simply connected, then any local parallel section s of E, defined on a non empty connected open subset $U \subset M$, extends to a (unique) global parallel section.

Corollary

$\pi : E \to M$ real analytic vector bundle with real analytic connection ∇.

Lemma

$\pi : E \to M$ real analytic vector bundle with real analytic connection ∇. If M is simply connected, then any local parallel section s of E, defined on a non empty connected open subset $U \subset M$, extends to a (unique) global parallel section.

Corollary

$\pi : E \to M$ real analytic vector bundle with real analytic connection ∇. Given $x \in M$ and $e \in \pi^{-1}(x)$, assume $\nabla^k R(v_1, \ldots, v_{k+2})e = 0$ for all $v_1, \ldots, v_{k+2} \in T_xM$ and all $k \geq 0$.

Lemma

$\pi : E \to M$ real analytic vector bundle with real analytic connection ∇. If M is simply connected, then any local parallel section s of E, defined on a non empty connected open subset $U \subset M$, extends to a (unique) global parallel section.

Corollary

$\pi : E \to M$ real analytic vector bundle with real analytic connection ∇. Given $x \in M$ and $e \in \pi^{-1}(x)$, assume $\nabla^k R(v_1, \ldots, v_{k+2})e = 0$ for all $v_1, \ldots, v_{k+2} \in T_x M$ and all $k \geq 0$.
Then there exists a local parallel section s of E, defined around x, with $s(x) = e$. If M is simply connected, then there exists a global parallel section s of E with $s(x) = e$.

Levi–Civita connections

$\pi : E \longrightarrow M$ vector bundle with connection ∇.
∇ induces connections on all vector bundle obtained by functorial constructions on E.

Levi–Civita connections

$\pi : E \longrightarrow M$ vector bundle with connection ∇.
∇ induces connections on all vector bundle obtained by functorial constructions on E.

Example

$E^* \otimes E^*$ vector bundle over M with fiber at m the space of all bilinear forms on E_m. ∇ induces a connection ∇^{bil} on $E^* \otimes E^*$:

$$(\nabla^{\mathrm{bil}}_X g)(\xi,\eta) = X\big(g(\xi,\eta)\big) - g\big(\nabla_X \xi,\eta\big) - g\big(\xi,\nabla_X \eta\big)$$

The curvature tensor R^{bil} of ∇^{bil} is:

$$\big(R^{\mathrm{bil}}(X,Y)g\big)(\xi,\eta) = -g\big(R(X,Y)\xi,\eta\big) - g\big(\xi,R(X,Y)\eta\big)$$

Levi–Civita connections

$\pi : E \to M$ vector bundle with connection ∇.
∇ induces connections on all vector bundle obtained by functorial constructions on E.

Example

$E^* \otimes E^*$ vector bundle over M with fiber at m the space of all bilinear forms on E_m. ∇ induces a connection ∇^{bil} on $E^* \otimes E^*$:

$$(\nabla^{\mathrm{bil}}_X g)(\xi,\eta) = X\big(g(\xi,\eta)\big) - g\big(\nabla_X \xi, \eta\big) - g\big(\xi, \nabla_X \eta\big)$$

The curvature tensor R^{bil} of ∇^{bil} is:

$$\big(R^{\mathrm{bil}}(X,Y)g\big)(\xi,\eta) = -g\big(R(X,Y)\xi,\eta\big) - g\big(\xi, R(X,Y)\eta\big)$$

Definition

∇ *symmetric* connection on TM, g semi-Riemannian metric tensor on M. ∇ is the *Levi–Civita* connection of g if $\nabla^{\mathrm{bil}} g = 0$.

Characterization of Levi–Civita connections

Problem: given a *symmetric* ∇, when does there exist g semi-Riemannian metric with $\nabla^{\mathrm{bil}} g = 0$?

Characterization of Levi–Civita connections

Problem: given a *symmetric* ∇, when does there exist g semi-Riemannian metric with $\nabla^{\mathrm{bil}} g = 0$?
Equivalently, does there exist a ∇^{bil}-parallel section g of $T^*M \otimes T^*M$ (symmetric and nondegenerate)?

Characterization of Levi–Civita connections

Problem: given a *symmetric* ∇, when does there exist g semi-Riemannian metric with $\nabla^{\mathrm{bil}} g = 0$?
Equivalently, does there exist a ∇^{bil}-parallel section g of $T^*M \otimes T^*M$ (symmetric and nondegenerate)?

Idea: Given $m_0 \in M$ and a nondegenerate symmetric bilinear form g_0, one can *spread* g_0 by *parallel transport* along the curves of a Λ-parametric family, or along solutions of a spray.

Characterization of Levi–Civita connections

Problem: given a *symmetric* ∇, when does there exist g semi-Riemannian metric with $\nabla^{\mathrm{bil}} g = 0$?
Equivalently, does there exist a ∇^{bil}-parallel section g of $T^*M \otimes T^*M$ (symmetric and nondegenerate)?

Idea: Given $m_0 \in M$ and a nondegenerate symmetric bilinear form g_0, one can *spread* g_0 by *parallel transport* along the curves of a Λ-parametric family, or along solutions of a spray.

Frobenius theorem gives us that the metric g obtained in this way is a solution of the problem if and only if $R^{\mathrm{bil}}(\cdot,\cdot)g = 0$. Recalling the form of R^{bil}, this is equivalent to the g-antisymmetry of R. More precisely:

Theorem

M manifold, ∇ symmetric connection on TM, $m_0 \in M$,
$g_0 : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$, $\mathcal{S}$ spray on M.

Theorem

M manifold, ∇ symmetric connection on TM, $m_0 \in M$, $g_0 : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$, $\mathcal{S}$ spray on M. Assume:

- *given a piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = m_0$, $P_\gamma^{-1} R_{\gamma(b)} P_\gamma : T_{m_0}M \to T_{m_0}M$ is g_0-antisymmetric;*
- *M is (connected and) simply connected.*

Theorem

M manifold, ∇ symmetric connection on TM, $m_0 \in M$, $g_0 : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$, $\mathcal{S}$ spray on M. Assume:

- *given a piecewise solution $\gamma : [a, b] \to M$ of $\mathcal{S}$ with $\gamma(a) = m_0$, $P_\gamma^{-1} R_{\gamma(b)} P_\gamma : T_{m_0}M \to T_{m_0}M$ is g_0-antisymmetric;*
- *M is (connected and) simply connected.*

Then, g_0 extends to a semi-Riemannian metric on M whose Levi–Civita connection is ∇.

Theorem

If M is a simply connected real analytic manifold with real analytic symmetric connection ∇. If g is a semi–Riemannian metric defined on a non empty open connected subset of M whose Levi–Civita connection is ∇, then g extends to a globally defined semi-Riemannian metric tensor on M whose Levi–Civita connection is ∇.

Theorem

M real analytic, ∇ real analytic symmetric connection on TM. Given $x_0 \in M$ and a nondegenerate symmetric bilinear form g_0 on $T_{x_0}M$ if:

$$(\nabla^k R)(v_1, \ldots, v_{k+2}) : T_{x_0}M \longrightarrow T_{x_0}M$$

is g_0-antisymmetric for all $v_1, \ldots, v_{k+2} \in T_{x_0}M$ and all $k \geq 0$, then g_0 extends to a locally defined semi-Riemannian metric tensor g whose Levi–Civita connection is ∇.

Theorem

M real analytic, ∇ real analytic symmetric connection on TM. Given $x_0 \in M$ and a nondegenerate symmetric bilinear form g_0 on $T_{x_0}M$ if:

$$(\nabla^k R)(v_1, \ldots, v_{k+2}) : T_{x_0}M \longrightarrow T_{x_0}M$$

is g_0-antisymmetric for all $v_1, \ldots, v_{k+2} \in T_{x_0}M$ and all $k \geq 0$, then g_0 extends to a locally defined semi-Riemannian metric tensor g whose Levi–Civita connection is ∇.
If M is simply connected, then g_0 extends to a globally defined semi-Riemannian metric g with Levi–Civita connection ∇.

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)
∇ determined by a linear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\Gamma(X, Y) = \nabla_X Y, \quad X, Y \text{ left invariant vector fields}$$

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)
∇ determined by a linear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\Gamma(X, Y) = \nabla_X Y, \quad X, Y \text{ left invariant vector fields}$$

Identify Γ with the map $\mathfrak{g} \ni X \mapsto \Gamma(X, \cdot) \in \mathrm{Lin}(\mathfrak{g})$.

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)
∇ determined by a linear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\Gamma(X, Y) = \nabla_X Y, \quad X, Y \text{ left invariant vector fields}$$

Identify Γ with the map $\mathfrak{g} \ni X \mapsto \Gamma(X, \cdot) \in \mathrm{Lin}(\mathfrak{g})$.
Torsion of ∇: $\boxed{T(X, Y) = \Gamma(X, Y) - \Gamma(Y, X) - [X, Y]}$
(∇ symmetric iff $\Gamma : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g})$ is a Lie algebra homomorphism)

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)
∇ determined by a linear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\Gamma(X, Y) = \nabla_X Y, \quad X, Y \text{ left invariant vector fields}$$

Identify Γ with the map $\mathfrak{g} \ni X \mapsto \Gamma(X, \cdot) \in \mathrm{Lin}(\mathfrak{g})$.
Torsion of ∇: $\boxed{T(X, Y) = \Gamma(X, Y) - \Gamma(Y, X) - [X, Y]}$
(∇ symmetric iff $\Gamma : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g})$ is a Lie algebra homomorphism)
Curvature of ∇: $\boxed{R(X, Y) = \big[\Gamma(X), \Gamma(Y)\big] - \Gamma\big([X, Y]\big)}$

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)
∇ determined by a linear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\Gamma(X, Y) = \nabla_X Y, \quad X, Y \text{ left invariant vector fields}$$

Identify Γ with the map $\mathfrak{g} \ni X \mapsto \Gamma(X, \cdot) \in \mathrm{Lin}(\mathfrak{g})$.
Torsion of ∇: $\boxed{T(X, Y) = \Gamma(X, Y) - \Gamma(Y, X) - [X, Y]}$
(∇ symmetric iff $\Gamma : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g})$ is a Lie algebra homomorphism)
Curvature of ∇: $\boxed{R(X, Y) = \big[\Gamma(X), \Gamma(Y)\big] - \Gamma\big([X, Y]\big)}$
Parallel transport of Y along $t \mapsto \exp(tX)$: $\boxed{t \mapsto e^{-t\Gamma(X)} Y}$

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)
∇ determined by a linear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\Gamma(X, Y) = \nabla_X Y, \quad X, Y \text{ left invariant vector fields}$$

Identify Γ with the map $\mathfrak{g} \ni X \mapsto \Gamma(X, \cdot) \in \mathrm{Lin}(\mathfrak{g})$.
Torsion of ∇: $\boxed{T(X, Y) = \Gamma(X, Y) - \Gamma(Y, X) - [X, Y]}$
(∇ symmetric iff $\Gamma : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g})$ is a Lie algebra homomorphism)
Curvature of ∇: $\boxed{R(X, Y) = \big[\Gamma(X), \Gamma(Y)\big] - \Gamma\big([X, Y]\big)}$
Parallel transport of Y along $t \mapsto \exp(tX)$: $\boxed{t \mapsto e^{-t\Gamma(X)} Y}$

Theorem

∇ symmetric left-invariant connection on G, $h : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ nondegenerate symmetric bilinear form.

G Lie group, ∇ left invariant connection on G
(i.e., zero Christoffel symbols in a left invariant referential)
∇ determined by a linear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\Gamma(X, Y) = \nabla_X Y, \quad X, Y \text{ left invariant vector fields}$$

Identify Γ with the map $\mathfrak{g} \ni X \mapsto \Gamma(X, \cdot) \in \mathrm{Lin}(\mathfrak{g})$.
Torsion of ∇: $\boxed{T(X, Y) = \Gamma(X, Y) - \Gamma(Y, X) - [X, Y]}$
(∇ symmetric iff $\Gamma : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g})$ is a Lie algebra homomorphism)
Curvature of ∇: $\boxed{R(X, Y) = \big[\Gamma(X), \Gamma(Y)\big] - \Gamma\big([X, Y]\big)}$
Parallel transport of Y along $t \mapsto \exp(tX)$: $\boxed{t \mapsto e^{-t\Gamma(X)} Y}$

Theorem

∇ symmetric left-invariant connection on G, $h : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ nondegenerate symmetric bilinear form.
Then, h extends to a local semi-Riemannian metric on G whose Levi–Civita connection is ∇ iff:

$$e^{\Gamma(Z)\big([\Gamma(X),\Gamma(Y)]-\Gamma([X,Y])\big)}e^{-\Gamma(Z)} \in \mathfrak{so}(h), \quad \forall\, X, Y, Z \in \mathfrak{g}.$$

Lemma

The condition in the above theorem is equivalent to:

$$\mathrm{ad}^n_{\Gamma(Z)}\big([\Gamma(X),\Gamma(Y)]-\Gamma([X,Y])\big)\in\mathfrak{so}(h),\quad\forall\,X,Y,Z\in\mathfrak{g}.$$

Lemma

The condition in the above theorem is equivalent to:

$$\mathrm{ad}^n_{\Gamma(Z)}\big([\Gamma(X),\Gamma(Y)]-\Gamma([X,Y])\big)\in\mathfrak{so}(h),\quad \forall\, X,Y,Z\in\mathfrak{g}.$$

Since Lie groups are real analytic, and so are left-invariant connections:

Corollary

If G is simply connected, then in the above theorem one has the existence of a globally defined extension of h to a semi-Riemannian metric tensor on G whose Levi–Civita connection is ∇.

Constant connections in $\mathbb{R}^n$

In the special case $G = \mathbb{R}^n$, a constant connection ∇ has curvature: $R(v, w) = \big[\Gamma(v), \Gamma(w)\big] \in \mathrm{Lin}(\mathbb{R}^n)$, where $\nabla_X Y = \mathrm{d}Y(X) + \Gamma(X, Y)$.

Constant connections in $\mathbb{R}^n$

In the special case $G = \mathbb{R}^n$, a constant connection ∇ has curvature: $R(v, w) = \big[\Gamma(v), \Gamma(w)\big] \in \mathrm{Lin}(\mathbb{R}^n)$, where $\nabla_X Y = \mathrm{d}Y(X) + \Gamma(X, Y)$.

Theorem

Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *a symmetric bilinear map, and let* $\mathcal{A}$ *be the image of the map* $\mathbb{R}^n \ni v \mapsto \Gamma(v) \in \mathrm{Lin}(\mathbb{R}^n)$.

Constant connections in $\mathbb{R}^n$

In the special case $G = \mathbb{R}^n$, a constant connection ∇ has curvature: $R(v,w) = \big[\Gamma(v), \Gamma(w)\big] \in \mathrm{Lin}(\mathbb{R}^n)$, where $\nabla_X Y = \mathrm{d}Y(X) + \Gamma(X,Y)$.

Theorem

Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ a symmetric bilinear map, and let $\mathcal{A}$ be the image of the map $\mathbb{R}^n \ni v \mapsto \Gamma(v) \in \mathrm{Lin}(\mathbb{R}^n)$.
A nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^n$ extends to a semi-Riemannian metric on $\mathbb{R}^n$ whose Levi–Civita connection ∇ above iff:

$$(\mathrm{ad}_X)^k[Y,Z] \in \mathfrak{so}(g_0), \quad \forall X, Y, Z \in \mathcal{A}, \ \forall k \geq 0.$$

Constant connections in $\mathbb{R}^n$

In the special case $G = \mathbb{R}^n$, a constant connection ∇ has curvature: $R(v, w) = \big[\Gamma(v), \Gamma(w)\big] \in \mathrm{Lin}(\mathbb{R}^n)$, where $\nabla_X Y = \mathrm{d}Y(X) + \Gamma(X, Y)$.

Theorem

Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ a symmetric bilinear map, and let $\mathcal{A}$ be the image of the map $\mathbb{R}^n \ni v \mapsto \Gamma(v) \in \mathrm{Lin}(\mathbb{R}^n)$.
A nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^n$ extends to a semi-Riemannian metric on $\mathbb{R}^n$ whose Levi–Civita connection ∇ above iff:

$$(\mathrm{ad}_X)^k[Y, Z] \in \mathfrak{so}(g_0), \quad \forall\, X, Y, Z \in \mathcal{A},\ \forall\, k \geq 0.$$

Corollary

Denote by $\mathfrak{g} \subset \mathrm{Lin}(\mathbb{R}^n)$ the Lie algebra generated by $\mathcal{A}$, and set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The conclusion of the Theorem above holds if $\mathfrak{g}' \subset \mathfrak{so}(g_0)$.

Constant connections in $\mathbb{R}^n$

In the special case $G = \mathbb{R}^n$, a constant connection ∇ has curvature: $R(v, w) = \big[\Gamma(v), \Gamma(w)\big] \in \mathrm{Lin}(\mathbb{R}^n)$, where $\nabla_X Y = \mathrm{d}Y(X) + \Gamma(X, Y)$.

Theorem

Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ a symmetric bilinear map, and let $\mathcal{A}$ be the image of the map $\mathbb{R}^n \ni v \mapsto \Gamma(v) \in \mathrm{Lin}(\mathbb{R}^n)$.
A nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^n$ extends to a semi-Riemannian metric on $\mathbb{R}^n$ whose Levi–Civita connection ∇ above iff:

$$(\mathrm{ad}_X)^k[Y, Z] \in \mathfrak{so}(g_0), \quad \forall\, X, Y, Z \in \mathcal{A},\ \forall\, k \geq 0.$$

Corollary

Denote by $\mathfrak{g} \subset \mathrm{Lin}(\mathbb{R}^n)$ the Lie algebra generated by $\mathcal{A}$, and set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The conclusion of the Theorem above holds if $\mathfrak{g}' \subset \mathfrak{so}(g_0)$. If $n = 2$, the condition $\mathfrak{g}' \subset \mathfrak{so}(g_0)$ is also necessary.

The case $n = 2$

Lemma

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero linear map. There exists a nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^2$ with $A \in \mathfrak{so}(g_0)$ if and only if $\operatorname{tr} A = 0$ and $\det A \neq 0$; moreover, g_0 is positive definite (resp., has index 1) if and only if $\det A > 0$ (resp., $\det A < 0$).

The case $n = 2$

Lemma

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero linear map. There exists a nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^2$ with $A \in \mathfrak{so}(g_0)$ if and only if $\operatorname{tr} A = 0$ and $\det A \neq 0$; moreover, g_0 is positive definite (resp., has index 1) if and only if $\det A > 0$ (resp., $\det A < 0$).

Corollary

In the case $n = 2$, the conclusion of the Theorem above holds if and only if either $\mathfrak{g}' = 0$ or if $\mathfrak{g}'$ has dimension 1 and it is spanned by an invertible 2×2 matrix.

An explicit analysis of 2-dimensional and 3-dimensional Lie algebras $\mathfrak{g}$ with 1-dimensional commutator subalgebra $\mathfrak{g}'$ leads to the following:

An explicit analysis of 2-dimensional and 3-dimensional Lie algebras $\mathfrak{g}$ with 1-dimensional commutator subalgebra $\mathfrak{g}'$ leads to the following:

Corollary

Let $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *be a symmetric bilinear map and let* $\mathcal{A} \subset \mathrm{Lin}(\mathbb{R}^2)$ *be the range of the linear map* $v \mapsto \Gamma(v, \cdot)$.

An explicit analysis of 2-dimensional and 3-dimensional Lie algebras $\mathfrak{g}$ with 1-dimensional commutator subalgebra $\mathfrak{g}'$ leads to the following:

Corollary

Let $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a symmetric bilinear map and let $\mathcal{A} \subset \mathrm{Lin}(\mathbb{R}^2)$ be the range of the linear map $v \mapsto \Gamma(v, \cdot)$. Then the conclusion of the Theorem above holds if and only if $[X, Y] = 0$, for all $X, Y \in \mathcal{A}$. In this case, a semi-Riemannian metric g on $\mathbb{R}^2$ with the required property can be chosen with an arbitrary value g_0 at the origin.

Outline

Affine manifolds

Let M, N be manifolds endowed with connections ∇^M and ∇^N.
T^N, T^M, R^M, R^N the torsion and the curvature tensors of ∇^M and ∇^M.

Affine manifolds

Let M, N be manifolds endowed with connections ∇^M and ∇^N.
T^N, T^M, R^M, R^N the torsion and the curvature tensors of ∇^M and ∇^M.

A smooth map $f : M \to N$ is *affine* if: $\boxed{\mathrm{d}f_x\big(\nabla^M_v X\big) = \nabla^N_v\big(\mathrm{d}f \circ X\big).}$

Affine manifolds

Let M, N be manifolds endowed with connections ∇^M and ∇^N.
T^N, T^M, R^M, R^N the torsion and the curvature tensors of ∇^M and ∇^M.

A smooth map $f : M \to N$ is *affine* if: $\boxed{\mathrm{d}f_x\big(\nabla^M_v X\big) = \nabla^N_v\big(\mathrm{d}f \circ X\big).}$

Equivalently, f is affine if for every parallel vector field V along a curve γ, $\mathrm{d}f \circ V$ is parallel along $f \circ \gamma$.

Affine manifolds

Let M, N be manifolds endowed with connections ∇^M and ∇^N.
T^N, T^M, R^M, R^N the torsion and the curvature tensors of ∇^M and ∇^M.

A smooth map $f : M \to N$ is *affine* if: $\boxed{\mathrm{d}f_x\big(\nabla^M_v X\big) = \nabla^N_v\big(\mathrm{d}f \circ X\big).}$

Equivalently, f is affine if for every parallel vector field V along a curve γ, $\mathrm{d}f \circ V$ is parallel along $f \circ \gamma$.

Example

If $M \subset N$, then the inclusion $\mathfrak{i} : M \to N$ is affine iff:

- M is totally geodesic in N;
- ∇^M is the restriction of ∇^N.

Affine maps as parallel sections

Consider the vector bundle $E = \mathrm{Lin}(TM, TN)$ over $M \times N$, with fiber $E_{(m,n)} = \mathrm{Lin}(T_mM, T_nN)$.

Affine maps as parallel sections

Consider the vector bundle $E = \mathrm{Lin}(TM, TN)$ over $M \times N$, with fiber $E_{(m,n)} = \mathrm{Lin}(T_mM, T_nN)$. $\boxed{E = \pi_1^*(TM^*) \otimes \pi_2^*(TN)}$ where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the projections.

Affine maps as parallel sections

Consider the vector bundle $E = \mathrm{Lin}(TM, TN)$ over $M \times N$, with fiber $E_{(m,n)} = \mathrm{Lin}(T_mM, T_nN)$. $\boxed{E = \pi_1^*(TM^*) \otimes \pi_2^*(TN)}$ where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the projections. By

functoriality, ∇^M and ∇^N induce a connection ∇ on E:

$\boxed{\big(\nabla_{(v,w)}\sigma\big)(X) = \nabla^N_{(v,w)}\big(\sigma(X)\big) - \sigma\big(\nabla^M_v X\big)}$, X vector field on M, $v \in TM$, $w \in TN$, $\sigma : M \times N \to E$ section.

($\sigma(X)$ is seen as a section of the pull back bundle $\pi_2^*(TN)$ over $M \times N$)

Affine maps as parallel sections

Consider the vector bundle $E = \mathrm{Lin}(TM, TN)$ over $M \times N$, with fiber $E_{(m,n)} = \mathrm{Lin}(T_mM, T_nN)$. $\boxed{E = \pi_1^*(TM^*) \otimes \pi_2^*(TN)}$ where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the projections. By

functoriality, ∇^M and ∇^N induce a connection ∇ on E:
$\boxed{\big(\nabla_{(v,w)}\sigma\big)(X) = \nabla^N_{(v,w)}\big(\sigma(X)\big) - \sigma\big(\nabla^M_v X\big)}$, X vector field on M, $v \in TM$, $w \in TN$, $\sigma : M \times N \to E$ section.
($\sigma(X)$ is seen as a section of the pull back bundle $\pi_2^*(TN)$ over $M \times N$)

Given a smooth function $f : M \to N$, the differential is a section of E along the map $M \ni x \mapsto \big(x, f(x)\big) \in M \times N$, so that it makes sense $\nabla \mathrm{d}f$.

Affine maps as parallel sections

Consider the vector bundle $E = \mathrm{Lin}(TM, TN)$ over $M \times N$, with fiber $E_{(m,n)} = \mathrm{Lin}(T_mM, T_nN)$. $\boxed{E = \pi_1^*(TM^*) \otimes \pi_2^*(TN)}$ where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the projections. By

functoriality, ∇^M and ∇^N induce a connection ∇ on E:
$\boxed{\big(\nabla_{(v,w)}\sigma\big)(X) = \nabla^N_{(v,w)}\big(\sigma(X)\big) - \sigma\big(\nabla^M_v X\big)}$, X vector field on M, $v \in TM$, $w \in TN$, $\sigma : M \times N \to E$ section.
($\sigma(X)$ is seen as a section of the pull back bundle $\pi_2^*(TN)$ over $M \times N$)

Given a smooth function $f : M \to N$, the differential is a section of E along the map $M \ni x \mapsto \big(x, f(x)\big) \in M \times N$, so that it makes sense $\nabla \mathrm{d}f$.

Lemma

A smooth map $f : M \to N$ is affine iff the differential $\mathrm{d}f$ is ∇-parallel.

Affine maps as horizontal sections

Consider the submersion $\pi : E \to M$ given by the composition of the projection $E \mapsto M \times N$ and $\pi_1 : M \times N \to M$.

Affine maps as horizontal sections

Consider the submersion $\pi : E \to M$ given by the composition of the projection $E \mapsto M \times N$ and $\pi_1 : M \times N \to M$.

Given $\sigma \in \mathrm{Lin}(T_x M, T_y N)$, the tangent space $T_\sigma E$ is the direct sum of:

- $T_x M \oplus T_y N$ (the horizontal space of ∇)
- $\mathrm{Lin}(T_x M, T_y N)$ (the vertical space, tangent to the fiber).

Affine maps as horizontal sections

Consider the submersion $\pi : E \to M$ given by the composition of the projection $E \mapsto M \times N$ and $\pi_1 : M \times N \to M$.

Given $\sigma \in \mathrm{Lin}(T_xM, T_yN)$, the tangent space $T_\sigma E$ is the direct sum of:

- $T_xM \oplus T_yN$ (the horizontal space of ∇)
- $\mathrm{Lin}(T_xM, T_yN)$ (the vertical space, tangent to the fiber).

Define a distribution $\mathcal{D} \subset TE$:

$$\mathcal{D}_\sigma = \mathrm{Graph}(\sigma) \oplus \{0\} \subset \big(T_xM \oplus T_yN\big) \oplus \mathrm{Lin}(T_xM, T_yN).$$

Affine maps as horizontal sections

Consider the submersion $\pi : E \to M$ given by the composition of the projection $E \mapsto M \times N$ and $\pi_1 : M \times N \to M$.

Given $\sigma \in \mathrm{Lin}(T_x M, T_y N)$, the tangent space $T_\sigma E$ is the direct sum of:

- $T_x M \oplus T_y N$ (the horizontal space of ∇)
- $\mathrm{Lin}(T_x M, T_y N)$ (the vertical space, tangent to the fiber).

Define a distribution $\mathcal{D} \subset TE$:

$$\mathcal{D}_\sigma = \mathrm{Graph}(\sigma) \oplus \{0\} \subset \big(T_x M \oplus T_y N\big) \oplus \mathrm{Lin}(T_x M, T_y N).$$

Lemma

Let $s : U \subset M \to E$ be a smooth local section, $\boxed{s(x) = \big(f(x), \sigma(x)\big)}$*, where $f : U \to N$ and $\sigma(x) \in \mathrm{Lin}(T_x M, T_{f(x)} N)$.*

Affine maps as horizontal sections

Consider the submersion $\pi : E \to M$ given by the composition of the projection $E \mapsto M \times N$ and $\pi_1 : M \times N \to M$.

Given $\sigma \in \mathrm{Lin}(T_xM, T_yN)$, the tangent space $T_\sigma E$ is the direct sum of:

- $T_xM \oplus T_yN$ (the horizontal space of ∇)
- $\mathrm{Lin}(T_xM, T_yN)$ (the vertical space, tangent to the fiber).

Define a distribution $\mathcal{D} \subset TE$:

$$\mathcal{D}_\sigma = \mathrm{Graph}(\sigma) \oplus \{0\} \subset \big(T_xM \oplus T_yN\big) \oplus \mathrm{Lin}(T_xM, T_yN).$$

Lemma

Let $s : U \subset M \to E$ be a smooth local section, $\boxed{s(x) = \big(f(x), \sigma(x)\big)}$, *where $f : U \to N$ and $\sigma(x) \in \mathrm{Lin}(T_xM, T_{f(x)}N)$.*
Then, s is $\mathcal{D}$-horizontal iff:

- $\sigma(x) = \mathrm{d}f(x)$ *for all x and f is affine.*

The Levi form of $\mathcal{D} = \mathrm{Graph}(\sigma) \oplus \{0\}$

Lemma

The curvature tensor R^E of the connection ∇ of E is given by:

$$R^E_{(x,y)}\big((v_1, w_1), (v_2, w_2)\big)\sigma = R^N_y(w_1, w_2) \circ \sigma - \sigma \circ R^M_x(v_1, v_2),$$

for all $(x, y) \in M \times N$, $v_1, v_2 \in T_xM$, $w_1, w_2 \in T_yN$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$.

The Levi form of $\mathcal{D} = \mathrm{Graph}(\sigma) \oplus \{0\}$

Lemma

The curvature tensor R^E of the connection ∇ of E is given by:

$$R^E_{(x,y)}\big((v_1, w_1), (v_2, w_2)\big)\sigma = R^N_y(w_1, w_2) \circ \sigma - \sigma \circ R^M_x(v_1, v_2),$$

for all $(x, y) \in M \times N$, $v_1, v_2 \in T_xM$, $w_1, w_2 \in T_yN$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$.

Lemma

Given $x \in M$, $y \in N$, $\sigma \in \mathrm{Lin}(T_xM, T_yN)$, the Levi form of $\mathcal{D}$ at the point $\sigma \in E$ is given by:

$$\mathfrak{L}^{\mathcal{D}}_\sigma(v_1, v_2) = \Big(\sigma\big(T^M(v_1, v_2)\big) - T^N\big(\sigma(v_1), \sigma(v_2)\big),$$
$$\sigma \circ R^M_x(v_1, v_2) - R^N_y\big(\sigma(v_1), \sigma(v_2)\big) \circ \sigma\Big),$$

for all $v_1, v_2 \in T_xM$.

Induced maps between affine manifolds

Given $x_0 \in M$, $y_0 \in N$ and $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$ and a geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = x_0$, one gets a geodesic $\mu : [a, b] \to N$ with $\mu(a) = y_0$ and $\mu'(a) = \sigma_0\big(\gamma'(a)\big)$.

Induced maps between affine manifolds

Given $x_0 \in M$, $y_0 \in N$ and $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$ and a geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = x_0$, one gets a geodesic $\mu : [a, b] \to N$ with $\mu(a) = y_0$ and $\mu'(a) = \sigma_0\big(\gamma'(a)\big)$.

Also, obtain a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$: $\boxed{\sigma = P_\gamma \circ \sigma_0 \circ P_\mu^{-1}}$ where P_γ and P_μ are the parallel transport.

Induced maps between affine manifolds

Given $x_0 \in M$, $y_0 \in N$ and $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$ and a geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = x_0$, one gets a geodesic $\mu : [a, b] \to N$ with $\mu(a) = y_0$ and $\mu'(a) = \sigma_0\big(\gamma'(a)\big)$.

Also, obtain a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$: $\boxed{\sigma = P_\gamma \circ \sigma_0 \circ P_\mu^{-1}}$ where P_γ and P_μ are the parallel transport.

Iterating, given a piecewise geodesic $\gamma : [a, b] \to M$ starting at x_0, one gets a piecewise geodesic $\mu : [a, b] \to N$ starting at y_0, and a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$.

Induced maps between affine manifolds

Given $x_0 \in M$, $y_0 \in N$ and $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$ and a geodesic $\gamma : [a,b] \to M$ with $\gamma(a) = x_0$, one gets a geodesic $\mu : [a,b] \to N$ with $\mu(a) = y_0$ and $\mu'(a) = \sigma_0\big(\gamma'(a)\big)$.

Also, obtain a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$: $\boxed{\sigma = P_\gamma \circ \sigma_0 \circ P_\mu^{-1}}$ where P_γ and P_μ are the parallel transport.

Iterating, given a piecewise geodesic $\gamma : [a,b] \to M$ starting at x_0, one gets a piecewise geodesic $\mu : [a,b] \to N$ starting at y_0, and a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$.

We say that μ and σ are *induced by γ and σ_0*.

Induced maps between affine manifolds

Given $x_0 \in M$, $y_0 \in N$ and $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$ and a geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = x_0$, one gets a geodesic $\mu : [a, b] \to N$ with $\mu(a) = y_0$ and $\mu'(a) = \sigma_0\big(\gamma'(a)\big)$.

Also, obtain a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$: $\boxed{\sigma = P_\gamma \circ \sigma_0 \circ P_\mu^{-1}}$ where P_γ and P_μ are the parallel transport.

Iterating, given a piecewise geodesic $\gamma : [a, b] \to M$ starting at x_0, one gets a piecewise geodesic $\mu : [a, b] \to N$ starting at y_0, and a linear map $\sigma : T_{\gamma(b)}M \to T_{\mu(b)}N$.

We say that μ and σ are *induced by* γ *and* σ_0.

Observation: If $f : M \to N$ is an affine map with $f(x_0) = y_0$, $\gamma : [a, b] \to M$ is a (piecewise) geodesic with $\gamma(a) = x_0$, then $f\big(\gamma(b)\big) = \mu(b)$ and $\mathrm{d}f\big(\gamma(b)\big) = \sigma$, where μ and σ are the "objects" induced by $\mathrm{d}f(x_0)$ and γ.

A generalized Cartan–Ambrose–Hicks theorem

Problem: Given (M, ∇^M) and (N, ∇^N), $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$, want to find a (local) affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

A generalized Cartan–Ambrose–Hicks theorem

Problem: Given (M, ∇^M) and (N, ∇^N), $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$, want to find a (local) affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.
Candidate: construct f using geodesics induced by σ_0:
$f\big(\gamma(b)\big) = \mu(b)$.

A generalized Cartan–Ambrose–Hicks theorem

Problem: Given (M, ∇^M) and (N, ∇^N), $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$, want to find a (local) affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Candidate: construct f using geodesics induced by σ_0: $f\big(\gamma(b)\big) = \mu(b)$. By the single leaf Frobenius theorem, this works iff the Levi form of $\mathcal{D}$ vanishes along the section $\big(x, f(x)\big)$.

A generalized Cartan–Ambrose–Hicks theorem

Problem: Given (M, ∇^M) and (N, ∇^N), $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$, want to find a (local) affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Candidate: construct f using geodesics induced by σ_0: $f\big(\gamma(b)\big) = \mu(b)$. By the single leaf Frobenius theorem, this works iff the Levi form of $\mathcal{D}$ vanishes along the section $\big(x, f(x)\big)$.

Theorem

$U \subset T_{x_0}M$ open and star-shaped at the origin, $\exp_{x_0} : U \xrightarrow{\cong} V \subset N$. assume $\sigma(U) \subset \mathrm{Dom}(\exp_{y_0})$.

A generalized Cartan–Ambrose–Hicks theorem

Problem: Given (M, ∇^M) and (N, ∇^N), $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$, want to find a (local) affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Candidate: construct f using geodesics induced by σ_0: $f\big(\gamma(b)\big) = \mu(b)$. By the single leaf Frobenius theorem, this works iff the Levi form of $\mathcal{D}$ vanishes along the section $\big(x, f(x)\big)$.

Theorem

$U \subset T_{x_0}M$ open and star-shaped at the origin, $\exp_{x_0} : U \xrightarrow{\cong} V \subset N$. assume $\sigma(U) \subset \mathrm{Dom}(\exp_{y_0})$. For $x \in V$, let $\gamma_x : [0,1] \to M$ be the unique geodesic such that $\gamma_x'(0) \in U$ and $\gamma_x(1) = x$; let $\mu_x : [0,1] \to N$ and $\sigma_x : T_xM \to T_{\mu_x(1)}N$ be the "objects" induced by γ_x and σ_0.

A generalized Cartan–Ambrose–Hicks theorem

Problem: Given (M, ∇^M) and (N, ∇^N), $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$, want to find a (local) affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Candidate: construct f using geodesics induced by σ_0: $f\big(\gamma(b)\big) = \mu(b)$. By the single leaf Frobenius theorem, this works iff the Levi form of $\mathcal{D}$ vanishes along the section $\big(x, f(x)\big)$.

Theorem

$U \subset T_{x_0}M$ open and star-shaped at the origin, $\exp_{x_0} : U \xrightarrow{\cong} V \subset N$. assume $\sigma(U) \subset \mathrm{Dom}(\exp_{y_0})$. For $x \in V$, let $\gamma_x : [0,1] \to M$ be the unique geodesic such that $\gamma_x'(0) \in U$ and $\gamma_x(1) = x$; let $\mu_x : [0,1] \to N$ and $\sigma_x : T_xM \to T_{\mu_x(1)}N$ be the "objects" induced by γ_x and σ_0. If $\forall\, x \in V$, σ_x relates T^M with T^N and R^M with R^N, i.e.:

$$\sigma_x\big(T^M(\cdot,\cdot)\big) = T^N\big(\sigma_x(\cdot), \sigma_x(\cdot)\big), \quad \sigma_x\big(R^M(\cdot,\cdot)\cdot\big) = R^N\big(\sigma_x(\cdot), \sigma_x(\cdot)\big)\sigma_x(\cdot).$$

A generalized Cartan–Ambrose–Hicks theorem

Problem: Given (M, ∇^M) and (N, ∇^N), $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$, want to find a (local) affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.
Candidate: construct f using geodesics induced by σ_0: $f\big(\gamma(b)\big) = \mu(b)$. By the single leaf Frobenius theorem, this works iff the Levi form of $\mathcal{D}$ vanishes along the section $\big(x, f(x)\big)$.

Theorem

$U \subset T_{x_0}M$ open and star-shaped at the origin, $\exp_{x_0} : U \xrightarrow{\cong} V \subset N$. assume $\sigma(U) \subset \mathrm{Dom}(\exp_{y_0})$. For $x \in V$, let $\gamma_x : [0,1] \to M$ be the unique geodesic such that $\gamma_x'(0) \in U$ and $\gamma_x(1) = x$; let $\mu_x : [0,1] \to N$ and $\sigma_x : T_xM \to T_{\mu_x(1)}N$ be the “objects” induced by γ_x and σ_0. If $\forall\, x \in V$, σ_x relates T^M with T^N and R^M with R^N, i.e.:
$$\sigma_x\big(T^M(\cdot,\cdot)\big) = T^N\big(\sigma_x(\cdot), \sigma_x(\cdot)\big), \quad \sigma_x\big(R^M(\cdot,\cdot)\cdot\big) = R^N\big(\sigma_x(\cdot), \sigma_x(\cdot)\big)\sigma_x(\cdot).$$
Then map $f : V \to N$ defined by $f(x) = \mu_x(1)$ is affine and $\mathrm{d}f(x) = \sigma_x$ for all $x \in V$; in particular, $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

The global result

Theorem (Cartan–Ambrose–Hicks)

Assume that ∇^N is geodesically complete and that M is connected and simply-connected. Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map. For each piecewise geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = x_0$ denote by $\mu_\gamma : [a, b] \to N$ and by $\sigma_\gamma : T_{\gamma(b)}M \to T_{\mu_\gamma(b)}N$ respectively the piecewise geodesic and the linear map induced by the piecewise geodesic γ and by σ_0.

The global result

Theorem (Cartan–Ambrose–Hicks)

Assume that ∇^N is geodesically complete and that M is connected and simply-connected. Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear map. For each piecewise geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = x_0$ denote by $\mu_\gamma : [a, b] \to N$ and by $\sigma_\gamma : T_{\gamma(b)}M \to T_{\mu_\gamma(b)}N$ respectively the piecewise geodesic and the linear map induced by the piecewise geodesic γ and by σ_0. Assume that for every piecewise geodesic γ the linear map σ_γ relates T^M with T^N and R^M with R^N. Then there exists a smooth affine map $f : M \to N$ such that for every piecewise geodesic $\gamma : [a, b] \to M$ we have $f \circ \gamma = \mu_\gamma$ and $\mathrm{f}(\gamma(b)) = \sigma_\gamma$; in particular, $f(x_0) = y_0$ and $\mathrm{f}(x_0) = \sigma_0$.

The global result

Theorem (Cartan–Ambrose–Hicks)

Assume that ∇^N *is geodesically complete and that* M *is connected and simply-connected. Let* $x_0 \in M$, $y_0 \in N$ *be given and let* $\sigma_0 : T_{x_0}M \to T_{y_0}N$ *be a linear map. For each piecewise geodesic* $\gamma : [a, b] \to M$ *with* $\gamma(a) = x_0$ *denote by* $\mu_\gamma : [a, b] \to N$ *and by* $\sigma_\gamma : T_{\gamma(b)}M \to T_{\mu_\gamma(b)}N$ *respectively the piecewise geodesic and the linear map induced by the piecewise geodesic* γ *and by* σ_0*. Assume that for every piecewise geodesic* γ *the linear map* σ_γ *relates* T^M *with* T^N *and* R^M *with* R^N*. Then there exists a smooth affine map* $f : M \to N$ *such that for every piecewise geodesic* $\gamma : [a, b] \to M$ *we have* $f \circ \gamma = \mu_\gamma$ *and* $\mathrm{f}\big(\gamma(b)\big) = \sigma_\gamma$*; in particular,* $f(x_0) = y_0$ *and* $\mathrm{f}(x_0) = \sigma_0$*.*

Remark. In the statement of the Cartan–Ambrose–Hicks Theorem, if one assumes in addition that σ_0 is an isomorphism, and that ∇^M is geodesically complete then it follows that the affine map $f : M \to N$ is a covering map.

Totally geodesic immersions

Corollary

Let (M, g^M), (N, g^N) be Riemannian manifolds with (N, g^N) complete and M connected and simply-connected. Let $x_0 \in M$, $y_0 \in N$ be given and let $\sigma_0 : T_{x_0}M \to T_{y_0}N$ be a linear isometry onto a subspace of $T_{y_0}N$. For each piecewise geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = x_0$ denote by $\mu_\gamma : [a, b] \to N$ and by $\sigma_\gamma : T_{\gamma(b)}M \to T_{\mu_\gamma(b)}N$ respectively the piecewise geodesic and the linear map induced by the piecewise geodesic γ and by σ_0. Assume that for every piecewise geodesic γ the linear map σ_γ relates R^M with R^N. Then there exists a totally geodesic isometric immersion $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{f}(x_0) = \sigma_0$.

Higher order Cartan–Ambrose–Hicks theorem

Given a tensor field τ on a manifold endowed with a connection ∇, we denote by $\nabla^{(r)}\tau$ its r-th covariant derivative, for $r \geq 1$; we set $\nabla^{(0)}\tau = \tau$.

Higher order Cartan–Ambrose–Hicks theorem

Given a tensor field τ on a manifold endowed with a connection ∇, we denote by $\nabla^{(r)}\tau$ its r-th covariant derivative, for $r \geq 1$; we set $\nabla^{(0)}\tau = \tau$.

Theorem

Let M, N be real-analytic manifolds endowed with real-analytic connections ∇^M and ∇^N. $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$.

Higher order Cartan–Ambrose–Hicks theorem

Given a tensor field τ on a manifold endowed with a connection ∇, we denote by $\nabla^{(r)}\tau$ its r-th covariant derivative, for $r \geq 1$; we set $\nabla^{(0)}\tau = \tau$.

Theorem

Let M, N be real-analytic manifolds endowed with real-analytic connections ∇^M and ∇^N. $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$. If for all $r \geq 0$ the linear map σ_0 relates $\nabla^{(r)}T^M_{x_0}$ with $\nabla^{(r)}T^N_{y_0}$ and $\nabla^{(r)}R^M_{x_0}$ with $\nabla^{(r)}R^N_{y_0}$

Higher order Cartan–Ambrose–Hicks theorem

Given a tensor field τ on a manifold endowed with a connection ∇, we denote by $\nabla^{(r)}\tau$ its r-th covariant derivative, for $r \geq 1$; we set $\nabla^{(0)}\tau = \tau$.

Theorem

Let M, N be real-analytic manifolds endowed with real-analytic connections ∇^M and ∇^N. $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$. If for all $r \geq 0$ the linear map σ_0 relates $\nabla^{(r)}T^M_{x_0}$ with $\nabla^{(r)}T^N_{y_0}$ and $\nabla^{(r)}R^M_{x_0}$ with $\nabla^{(r)}R^N_{y_0}$ then there exists a real-analytic affine map $f : U \to N$ defined on an open neighborhood U of x_0 in M satisfying $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Theorem

Let M, N be real-analytic manifolds endowed with real-analytic connections ∇^M and ∇^N, respectively. Assume that ∇^N is geodesically complete and that M is (connected and) simply-connected. Then every affine map $f : U \to N$ defined on a nonempty connected open subset U of M extends to an affine map from M to N. In particular, if in addition $x_0 \in M$, $y_0 \in N$, $\sigma_0 \in \mathrm{Lin}(T_{x_0}M, T_{y_0}N)$ satisfy the hypotheses of Theorem above, then there exists an affine map $f : M \to N$ with $f(x_0) = y_0$ and $\mathrm{d}f(x_0) = \sigma_0$.

Affine symmetries

Definition

An *affine symmetry* around a point $x_0 \in M$ is an affine map $f : U \to M$ defined in an open neighborhood U of x_0 with $f(x_0) = x_0$ and $\mathrm{d}f(x_0) = -\mathrm{Id}$.

Affine symmetries

Definition

An *affine symmetry* around a point $x_0 \in M$ is an affine map $f : U \to M$ defined in an open neighborhood U of x_0 with $f(x_0) = x_0$ and $\mathrm{d}f(x_0) = -\mathrm{Id}$.

Applying the higher order Cartan–Ambrose–Hicks theorem to $\sigma_0 = -\mathrm{Id} : T_{x_0}M \to T_{x_0}M$ we get the following curious result:

Affine symmetries

Definition

An *affine symmetry* around a point $x_0 \in M$ is an affine map $f : U \to M$ defined in an open neighborhood U of x_0 with $f(x_0) = x_0$ and $\mathrm{d}f(x_0) = -\mathrm{Id}$.

Applying the higher order Cartan–Ambrose–Hicks theorem to $\sigma_0 = -\mathrm{Id} : T_{x_0}M \to T_{x_0}M$ we get the following curious result:

Corollary

Let M be a real-analytic manifold endowed with a real-analytic connection ∇. Let $x_0 \in M$ be fixed. Then there exists an affine symmetry around x_0 if and only if:

$$\nabla^{(2r)} T_{x_0} = 0, \quad \textit{and} \quad \nabla^{(2r+1)} R_{x_0} = 0, \quad \textit{for all } r \geq 0.$$

Affine symmetries

Definition

An *affine symmetry* around a point $x_0 \in M$ is an affine map $f : U \to M$ defined in an open neighborhood U of x_0 with $f(x_0) = x_0$ and $\mathrm{d}f(x_0) = -\mathrm{Id}$.

Applying the higher order Cartan–Ambrose–Hicks theorem to $\sigma_0 = -\mathrm{Id} : T_{x_0}M \to T_{x_0}M$ we get the following curious result:

Corollary

Let M be a real-analytic manifold endowed with a real-analytic connection ∇. Let $x_0 \in M$ be fixed. Then there exists an affine symmetry around x_0 if and only if:

$$\nabla^{(2r)} T_{x_0} = 0, \quad \textit{and} \quad \nabla^{(2r+1)} R_{x_0} = 0, \quad \textit{for all } r \geq 0.$$

If M is simply-connected and complete, one has the existence of a globally defined affine symmetry $f : M \to M$ around x_0.

Outline

Affine manifold with G-structure

M n-dimensional differentiable manifold, $G \subset \mathrm{GL}(\mathbb{R}^n)$ Lie subgroup

Affine manifold with G-structure

M n-dimensional differentiable manifold, $G \subset \mathrm{GL}(\mathbb{R}^n)$ Lie subgroup
Assume that M is endowed with a connection ∇ and a G-structure $P \subset \mathrm{Ref}(TM)$.

Affine manifold with G-structure

M n-dimensional differentiable manifold, $G \subset \mathrm{GL}(\mathbb{R}^n)$ Lie subgroup
Assume that M is endowed with a connection ∇ and a G-structure $P \subset \mathrm{Ref}(TM)$.
For $x \in M$, let:

- G_x be the Lie subgroup of $\mathrm{GL}(T_xM)$ consisting of G-structure preserving endomorphisms of T_xM,
- $\mathfrak{g}_x \subset \mathfrak{gl}(T_xM)$ the Lie algebra of G_x
- $\delta_x : T_xM \to \mathfrak{gl}(T_xM)/\mathfrak{g}_x$ the inner torsion of the G-structure P.

Affine manifold with G-structure

M n-dimensional differentiable manifold, $G \subset \mathrm{GL}(\mathbb{R}^n)$ Lie subgroup
Assume that M is endowed with a connection ∇ and a G-structure $P \subset \mathrm{Ref}(TM)$.
For $x \in M$, let:

- G_x be the Lie subgroup of $\mathrm{GL}(T_xM)$ consisting of G-structure preserving endomorphisms of T_xM,
- $\mathfrak{g}_x \subset \mathfrak{gl}(T_xM)$ the Lie algebra of G_x
- $\delta_x : T_xM \to \mathfrak{gl}(T_xM)/\mathfrak{g}_x$ the inner torsion of the G-structure P.

The triple (M, ∇, P) will be called an *affine manifold with G-structure*.

Infinitesimally homogenous affine manifolds

Given $x, y \in M$ and a G-structure preserving morphism $\sigma : T_x M \to T_y M$ then the Lie group isomorphism $\mathcal{I}_\sigma : \mathrm{GL}(T_x M) \to \mathrm{GL}(T_y M)$ defined by:

$$\mathcal{I}_\sigma : \mathrm{GL}(T_x M) \ni T \longmapsto \sigma \circ T \circ \sigma^{-1} \in \mathrm{GL}(T_y M)$$

carries G_x onto G_y.

Infinitesimally homogenous affine manifolds

Given $x, y \in M$ and a G-structure preserving morphism $\sigma : T_xM \to T_yM$ then the Lie group isomorphism $\mathcal{I}_\sigma : \mathrm{GL}(T_xM) \to \mathrm{GL}(T_yM)$ defined by:

$$\mathcal{I}_\sigma : \mathrm{GL}(T_xM) \ni T \longmapsto \sigma \circ T \circ \sigma^{-1} \in \mathrm{GL}(T_yM)$$

carries G_x onto G_y.Its differential at the identity $\mathrm{Ad}_\sigma : \mathfrak{gl}(T_xM) \to \mathfrak{gl}(T_yM)$ carries $\mathfrak{g}_x$ onto $\mathfrak{g}_y$ and therefore it induces a linear isomorphism $\overline{\mathrm{Ad}}_\sigma : \mathfrak{gl}(T_xM)/\mathfrak{g}_x \longrightarrow \mathfrak{gl}(T_yM)/\mathfrak{g}_y$.

Definition

An affine manifold with G-structure M is said to be *infinitesimally homogeneous* if for all $x, y \in M$ and all G-structure preserving morphism $\sigma : T_xM \to T_yM$, the following conditions hold:

- $\overline{\mathrm{Ad}}_\sigma \circ \delta_x = \delta_y \circ \sigma$;
- $T_y\big(\sigma(v), \sigma(w)\big) = \sigma\big(T_x(v, w)\big)$, for all $v, w \in T_xM$;
- $R_y\big(\sigma(v), \sigma(w)\big) \circ \sigma = \sigma \circ R_x(v, w)$, for all $v, w \in T_xM$.

Affine immersions

Theorem (Hypotheses)

M, $\overline{M}$ manifolds, $\pi : E \to M$ be a vector bundle over M. Set $\widehat{E} = TM \oplus E$ and denote by $\iota : TM \to \widehat{E}$ the inclusion map. Let $\widehat{\nabla}$ and $\overline{\nabla}$ be connections on $\widehat{E}$ and on $T\overline{M}$ respectively. Let G be a Lie group and assume that $\widehat{E}$ and $T\overline{M}$ are endowed with G-structures $\widehat{P}$ and $\overline{P}$, respectively. Assume that $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous and that for all $x \in M$, $y \in \overline{M}$ and every G-structure preserving morphism $\sigma : \widehat{E}_x \to T_y\overline{M}$, the following conditions hold:

- $\overline{\mathrm{Ad}}_\sigma \circ \hat{\delta}_x = \bar{\delta}_y \circ \sigma|_{T_xM}$;
- $\overline{T}_y\big(\sigma(v), \sigma(w)\big) = \sigma\big(\widehat{T}_x(v, w)\big)$, *for all* $v, w \in T_xM$;
- $\overline{R}_y\big(\sigma(v), \sigma(w)\big) \circ \sigma = \sigma \circ \widehat{R}_x(v, w)$, *for all* $v, w \in T_xM$.

Affine immersions

Theorem (Hypotheses)

Then, for all $x_0 \in M$, all $y_0 \in \overline{M}$ and every G-structure preserving morphism $\sigma : \widehat{E}_{x_0} \to T_{y_0}\overline{M}$ there exists a smooth immersion $f : U \to \overline{M}$ defined on an open neighborhood U of x_0 in M and a G-structure preserving and connection preserving vector bundle isomorphism $L : \widehat{E}|_U \to f^* T\overline{M}$ such that $L|_{TM} = \mathrm{d}f$, $f(x_0) = y_0$ and $L_{x_0} = \sigma_0$.