Teoria de Morse para Geodésicas Periódicas em Variedades de Lorentz

Joint work with L. Biliotti (Univ. Parma) and F. Mercuri (*Unicamp*)

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Outline.

1. The celebrated result of Gromoll and Meyer
2. Some literature
3. On the Lorentzian result
4. Variational framework
5. Equivariant Morse theory

Closed geodesics in Riemannian manifolds

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- Bumpy metrics are generic
  (Abraham 1970, B. White Indiana J. Math. 1991)

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- McCleary & Ziller (Amer. J. Math., 1987, 1991) $\sup_k \beta_k(\Lambda M, \mathbb{Z}_2) = +\infty$ if M is homotopically equivalent to a compact simply connected *homogeneous space* not diffeomorphic to a symmetric space of rank 1.

Gromoll–Meyer type results

- **Grove, Halperin, Vigué–Poirrier (1978–1982): multiplicity of geodesics satisfying more general boundary conditions ($\big(\gamma(0), \gamma(1)\big) \in \mathcal{S} \subset M \times M$).**

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- Matthias (1980): closed geodesics in *Finsler* manifolds
- **Guruprasad, Haefliger (Topology 2006): closed geodesics in *orbifolds***

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- **Masiello (J. Diff. Eq. 1993) there exists one closed spacelike geodesic in standard stationary spacetimes with a compact base.**

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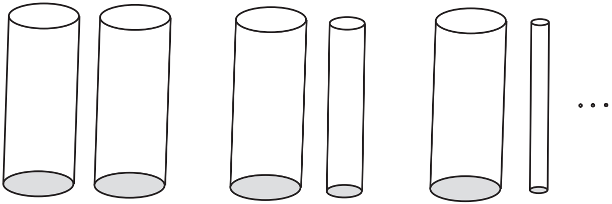

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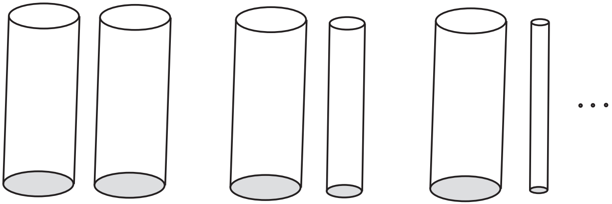

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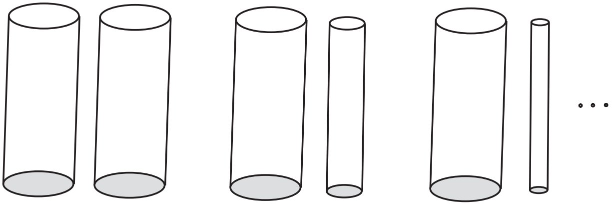

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- quotient out the $\mathbb{R}$-action (by considering curves starting on the Cauchy surface)
- use equivariant Morse theory to count critical $\mathrm{O}(2)$-orbits coming from distinct prime closed geodesics.

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Open problem: What kind of bounded sequences arise from $\mu(\gamma^N)$? It is clear how to construct examples with $\mu(\gamma^N) = 0$ for all N.

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Assume there is only a finite number of distinct prime closed geodesics in M. Then, there exists a finite number of closed geodesics (not necessarily geometrically distinct) $\gamma_1, \dots, \gamma_s$ in M such that:

- every closed geodesic γ is the iterate of some γ_i
- $\mathrm{nul}(\gamma) = \mathrm{nul}(\gamma_i)$.

Proof. Purely arithmetical.

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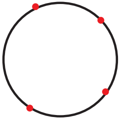

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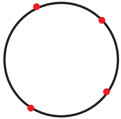

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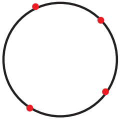

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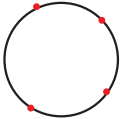

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Example of applications. (Ballmann, Thorbergsson, Ziller) *If* $\exists\, a \in \pi_1(M) \setminus \{1\}$ *with* $a^k = 1$*, s.t. every closed geodesic freely homotopic to some* a^q *is hyperbolic,* $\Longrightarrow \infty$ *closed geo's.*

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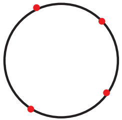

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Conjecture. (V. Bangert, N. Hingston) *If $\pi_1(M)$ is infinite abelian, then there are infinitely many distinct closed geodesics.*

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Closed sublevel: $\mathfrak{f}^c = \big\{x \in \mathcal{M} : \mathfrak{f}(x) \leq c\big\}$.

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Shifting theorem (G & M, Topology 1969)

$\mu(p)$ = Morse index of $\mathfrak{f}$ at p $\Longrightarrow$ $\boxed{\mathfrak{H}_{k+\mu(p)}(\mathfrak{f}, p; \mathbb{F}) \cong \mathfrak{H}_k^0(\mathfrak{f}, p; \mathbb{F})}$

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Theorem

If $\mathfrak{f}^{-1}(c)$ contains a finite number of critical orbits $Gp_1, \ldots, Gp_r$:

$$H_*(\mathfrak{f}^{c+\varepsilon}, \mathfrak{f}^{c-\varepsilon}; \mathbb{F}) \cong \bigoplus_{i=1}^{r} \mathfrak{H}_*(\mathfrak{f}, Gp_i; \mathbb{F})$$

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$$\boxed{\mathrm{Geo}(\mathbb{S}^1, M_0 \times \mathbb{R}) \times \mathrm{Met}(M_0) \times \mathfrak{X}(M_0) \times C^\infty(M_0) \ni \big[\gamma, g_0, \delta, \beta\big] \mapsto [g_0, \delta, \beta]}$$

is a Fredholm nonlinear map with null index? If yes, apply Sard–Smale.

Abstract Morse relations

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Given sequences $(\mu_k)_{k\geq 0}$ and $(\beta_k)_{k\geq 0}$ in $\mathbb{N}\bigcup\{+\infty\}$, they *satisfy the Morse relations* if $\exists$ a formal power series $Q(t) = \sum_{k\geq 0} q_k t^k$ with coefficients in $\mathbb{N}\bigcup\{+\infty\}$ such that:

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- Apply the Morse inequalities to the filtration $\Lambda M = \bigcup_{n\geq 1} f^{c_n}$ to get a uniform upper bound on the Betti numbers of ΛM, getting a contradiction.

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This point does not work in the degenerate case

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QED

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OBRIGADO!!

Notas disponíveis na miha página web:
`http://www.ime.usp.br/~piccione`