G-STRUCTURE PRESERVING AFFINE AND ISOMETRIC IMMERSIONS

Joint work with Daniel V. Tausk, USP

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Workshop on Differential Geometry and PDEs

Outline.

1. G-structures

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2. Principal spaces
3. Principal fiber bundles

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4. Connections
5. Inner torsion of a *G*-structure
6. Infinitesimally homogeneous affine manifolds with *G*-structure

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3. Principal fiber bundles
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5. Inner torsion of a *G*-structure
6. Infinitesimally homogeneous affine manifolds with *G*-structure
7. Immersion theorems

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Given a G-structure P on X and a G-structure Q on Y, a map $f : X \to Y$ is *G-structure preserving* if $f \circ p \in Q$ for all $p \in P$.

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Given a G-structure $P \subset \mathrm{Bij}(X_0, X)$ and a subgroup $H \subset G$, there are $[G : H]$ strengthening H-structures of P.

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$\mathrm{Ver}_p = \mathrm{Ker}\big(\mathrm{d}\Pi_p\big) \subset T_pP$ *vertical space*;

canonical isomorphism $\mathrm{d}\beta_p(1) : \mathfrak{g} \xrightarrow{\cong} \mathrm{Ver}_pP$.

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- *pull-backs*: $\Pi : P \to M$ principal fiber bundle, $f : M' \to M$ smooth map, $f^*P = \bigcup_{y \in M'} \left(\{y\} \times P_{f(y)}\right)$.

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$$\boxed{\mathrm{FR}_{E_0}(E) \times_{\mathrm{GL}(E_0)} E_0 \cong E, \quad [p, e_0] \mapsto p(e_0) \in E}$$

Associated and vector bundles

G Lie group, $\Pi : P \to M$ a G-principal bundle, N a differential G-space

Associated bundle: $P \times_G N = \bigcup_{x \in M} P_x \times_G N$

Definition

A *vector bundle* consists of:

- a set E (*total space*)
- a differentiable manifold M (*base manifold*)
- a map $\pi : E \to M$ (*projection*)
- a finite dimensional vector space E_0 (*typical fiber*)
- a vector space structure on each fiber $E_x = \pi^{-1}$ isomorphic to E_0
- a maximal atlas of admissible local sections of the fiber bundle $\mathrm{FR}_{E_0}(E) = \bigcup_{x \in M} \mathrm{FR}_{E_0}(E_x)$.

$$\boxed{\mathrm{FR}_{E_0}(E) \times_{\mathrm{GL}(E_0)} E_0 \cong E, \quad [p, e_0] \mapsto p(e_0) \in E}$$

Def.: A *G-structure on E* is a G-principal subbundle of $\mathrm{FR}(E)$.

Outline

1. *G*-structures
2. Principal spaces
3. Principal fiber bundles
4. Connections
5. Inner torsion of a *G*-structure
6. Infinitesimally homogeneous affine manifolds with *G*-structure
7. Immersion theorems

Principal connections

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$$\boxed{\mathrm{Ker}(\omega_p) = \mathrm{Hor}_p, \quad \omega_p|_{\mathrm{Ver}_p} = \mathrm{d}\beta_p(1)^{-1} : \mathrm{Ver}_p \xrightarrow{\cong} \mathfrak{g}}$$

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G-principal bundles $\Pi : P \to M$, $\Pi' : Q \to M$ with connections $\mathrm{Hor}(P)$ and $\mathrm{Hor}(Q)$ and a morphism of principal bundles $\phi : P \to Q$, then ϕ is *connection preserving* if:

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Properties of principal connections

- can be *pushed forward*
- induce connections on all associated bundles

Connections on vector bundles

Definition

A *connection* on the vector bundle E is an $\mathbb{R}$-bilinear map
$\nabla : \Gamma(TM) \times \Gamma(E) \ni (X, \epsilon) \longmapsto \nabla_X \epsilon \in \Gamma(E)$

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- $\boxed{\text{Connections on } E} \quad \Longleftrightarrow \quad \boxed{\text{Principal connections on } \mathrm{FR}(E)}$

Curvature and torsion

Curvature tensor of ∇: $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$

$$\boxed{R(X,Y)\epsilon = \nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon - \nabla_{[X,Y]}\epsilon}$$

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Given $\iota : TM \to E$ vector bundle morphism, ι-*torsion tensor*:
$T^\iota : \Gamma(TM) \times \Gamma(TM) \to \Gamma(E)$

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When $E = TM$, *torsion*: $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

$\boxed{\nabla \text{ is } \textit{symmetric} \text{ if } T = 0}$

Gauss, Codazzi and Ricci equations

$\pi_1 : E^1 \to M$ and $\pi_2 : E^2 \to M$ vector bundle.

Whitney sum: $\pi : E = E_1 \oplus E_2 \to M$, $\mathrm{pr}_i : E \to E^i$ projection.

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Gauss equation:

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Ricci equation

$$\mathrm{pr}_2\big(R(X, Y)\epsilon_2\big) = R_2(X, Y)\epsilon_2 + \alpha^2\big(X, \alpha^1(Y, \epsilon_2)\big) - \alpha^2\big(Y, \alpha^1(X, \epsilon_2)\big)$$

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The inner torsion $\mathfrak{I}^P_x : T_xM \to \mathfrak{gl}(E_x)/\mathfrak{g}_x$ is:

$$\begin{tikzcd} T_xM \arrow[r, "\Gamma_x"'] \arrow[rr, bend left, "\mathfrak{I}^P_x"] & \mathfrak{gl}(E_x) \arrow[r, "\text{quotient}"'] & \mathfrak{gl}(E_x)/\mathfrak{g}_x \end{tikzcd}$$

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Lemma

$\mathfrak{I}_x^P = 0$ iff g is ∇-parallel and F is parallel (i.e., covariant derivative of sections of F are in F).

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Lemma

$\mathfrak{I}^P_x = 0$ iff g and J are ∇-parallel.

Outline

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Theorem

(M, ∇, P) infinitesimally homogeneous $\implies$ (M, ∇, P) locally homogeneous. If (M, ∇) is geodesically complete and M is simply connected, then (M, ∇, P) is globally homogeneous.

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Proof.

An application of the Cartan–Ambrose–Hicks theorem! $\square$

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- More generally, product of infinitesimally homogeneous manifolds is infinitesimally homogeneous.

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(M, g) 3-dimensional homogeneous manifold with 4-dim. isometry group (includes: Berger spheres, Heisenberg space Nil_3, $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, products $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$)

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Infinitesimally homogeneous $\mathrm{SO}(n-1)$-structure with non vanishing $\mathfrak{I}^{\overline{P}}$

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Lemma (Uniqueness)

If M is connected, given (f^1, L^1) and (f^2, L^2) with $f^1(x_0) = f^2(x_0)$ and $L^1(x_0) = L^2(x_0)$, then $(f^1, L^1) = (f^2, L^2)$.

The immersion theorem – 1

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Then, for all $x_0 \in M$, $y_0 \in M$, $\sigma_0 : \widehat{E}_x \to T_{y_0}\overline{M}$ G-structure preserving, there exist a locally defined affine immersion (f, L) of (M, E, ∇) into $(\overline{M}, \overline{\nabla})$ with $f(x_0) = y_0$, $L(x_0) = \sigma_0$, and such that L is G-structure preserving.

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If M is simply connected and $(\overline{M}, \overline{\nabla})$ is geodesically complete, then the affine immersion is global.

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Applications: isometric immersion theorem into:

- space forms,
- Kähler manifolds with constant holomorphic curvature,
- all homogeneous geometries in dimension 3,
- Lie groups,
- products, etc.