G-STRUCTURE PRESERVING AFFINE AND ISOMETRIC IMMERSIONS

Joint work with Daniel V. Tausk, USP

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Workshop on Differential Geometry and PDEs

G-structures

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- 2 Principal spaces

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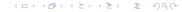
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Given a *G*-structure *P* on *X* and a *G*-structure *Q* on *Y*, a map $f: X \to Y$ is *G*-structure preserving if $f \circ p \in Q$ for all $p \in P$.



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Given a *G*-structure $P \subset \operatorname{Bij}(X_0, X)$ and a subgroup $H \subset G$, there are [G : H] strengthening H-structures of P.



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$$\operatorname{Ver}_p = \operatorname{Ker}(\operatorname{d}\Pi_p) \subset T_p P$$
 vertical space;
canonical isomorphism $d\beta_p(1) : \mathfrak{g} \stackrel{\cong}{\longrightarrow} \operatorname{Ver}_p P$.

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 - ▶ for all $x \in M$, there exists a smooth local section $s : U \to P$ with $x \in U$ and $s(U) \subset Q$.
- *pull-backs*: $\Pi: P \to M$ principal fiber bundle, $f: M' \to M$ smooth map, $f^*P = \bigcup_{y \in M'} (\{y\} \times P_{f(y)})$.



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Def.: A *G-structure on E* is a *G*-principal subbundle of FR(E).

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Connection form of Hor: \mathfrak{g} -valued one form ω on P:

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Properties of principal connections

- can be pushed forward
- induce connections on all associated bundles

Definition

$$\nabla : \Gamma(TM) \times \Gamma(E) \ni (X, \epsilon) \longmapsto \nabla_X \epsilon \in \Gamma(E)$$

Definition

A *connection* on the vector bundle E is an \mathbb{R} -bilinear map

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- Connections on E \iff Principal connections on FR(E)

Curvature and torsion

Curvature tensor of ∇ : $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$

$$R(X,Y)\epsilon = \nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon - \nabla_{[X,Y]} \epsilon$$

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When E = TM, torsion: $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

 ∇ is *symmetric* if T = 0



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Gauss equation:

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Codazzi equations

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$$\operatorname{pr}_{2}(R(X,Y)\epsilon_{2}) = R_{2}(X,Y)\epsilon_{2} + \alpha^{2}(X,\alpha^{1}(Y,\epsilon_{2})) - \alpha^{2}(Y,\alpha^{1}(X,\epsilon_{2}))$$

Outline

- G-structures
- Principal spaces
- Principal fiber bundles
- 4 Connections
- Inner torsion of a G-structure
- Infinitesimally homogeneous affine manifolds with G-structure
- Immersion theorems

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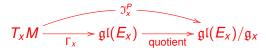
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The inner torsion $\mathfrak{I}_{X}^{P}:T_{X}M\to\mathfrak{gl}(E_{X})/\mathfrak{g}_{X}$ is:



Example 1: 1-structures

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Lemma

$$\mathfrak{I}_{x}^{P}=0$$
 iff ∇ is flat.

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Lemma

 $\mathfrak{I}_{x}^{P}=0$ iff g is ∇ -parallel.



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$$\begin{split} \mathfrak{gl}(E_X)/\mathfrak{g}_X &\cong \text{sym}(E_X) \oplus \text{Lin}(F_X, F_X^{\perp}) \\ T+\mathfrak{g}_X &\longmapsto \left(\frac{1}{2}(T+T^*), \frac{1}{2}\mathfrak{q}_X \circ (T-T^*)|_{F_X}\right) \end{split}$$

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 $q: E \to F^{\perp}$ projection, α is the 2nd fundamental form of F.

 $\pi: E \to M$ vector bundle with a Riemannian metric g

 $F \subset E$ vector subbundle

P set of frames *adapted* to the orthogonal sum $E = F \oplus F^{\perp}$ is a G-structure on E, where $G = O(k_1) \times O(k_2)$.

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Lemma

 $\mathfrak{I}_{\chi}^{P}=0$ iff g is ∇ -parallel and F is parallel (i.e., covariant derivative of sections of F are in F).

Example 4: O(k-1)-structures E vector bundle with Riemannian metric g

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 $\mathfrak{I}_{x}^{P}=0$ iff g and J are ∇ -parallel.

Outline

- G-structures
- Principal spaces
- Principal fiber bundles
- Connections
- Inner torsion of a G-structure
- 6 Infinitesimally homogeneous affine manifolds with G-structure
- Immersion theorems

 (M, ∇) affine manifold,

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 (M, ∇, P) is *infinitesimally homegeneous* if \mathfrak{I}^P , T and R are *constant* in frames of the G-structure P.

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Theorem

 (M, ∇, P) infinitesimally homogeneous \implies (M, ∇, P) locally homogeneous. If (M, ∇) is geodesically complete and M is simply connected, then (M, ∇, P) is globally homogeneous.

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 (M, ∇, P) is *locally homogeneous* if for all $x, y \in M$ and every *G*-structure preserving map $\sigma: T_XM \to T_YM$ there exists neighborhoods $U \ni x$, $V \ni y$ and a smooth G-structure preserving affine diffeomorphism $f: U \to V$ with f(x) = v and $df_x = \sigma$.

Proof.

An application of the Cartan–Ambrose–Hicks theorem!

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- More generally, product of infinitesimally homogeneous manifolds is infinitesimally homogeneous.

(M,g) 3-dimensional homogeneous manifold with 4-dim. isometry group (includes: Berger spheres, Heisenberg space Nil_3 , $\widetilde{PSL_2(\mathbb{R})}$, products $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$)

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- G-structures
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- 4 Connections
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Definition

An affine immersion of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ is a pair (f, L), where $f: M \to \overline{M}$ is a smooth map, $L: \widehat{E} \to f^*T\overline{M}$ is a connection preserving vector bundle isomorphism with: $L_X|_{\mathcal{T}_XM} = \mathrm{d}f_X, \quad \forall \, X \in M.$

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Lemma (Uniqueness)

If M is connected, given (f^1, L^1) and (f^2, L^2) with $f^1(x_0) = f^2(x_0)$ and $L^1(x_0) = L^2(x_0)$, then $(f^1, L^1) = (f^2, L^2)$.

Theorem (part 1)

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If M is simply connected and $(\overline{M}, \overline{\nabla})$ is geodesically complete, then the affine immersion is global.

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Applications: isometric immersion theorem into:

- space forms,
- Kähler manifolds with constant holomorphic curvature,
- all homogeneous geometries in dimension 3,
- Lie groups,
- products, etc.

