INTRODUCTION TO SYMPLECTIC MECHANICS: LECTURE IV

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4 Hamiltonian Mechanics

Physically speaking, Hamiltonian mechanics is a paraphrase (and generalization!) of Newton's second law, popularly expressed as "*force equals mass times acceleration*[1]". The symplectic formulation of Hamiltonian mechanics can be retraced (in embryonic form) to the work of Lagrange between 1808 and 1811; what we today call "Hamilton's equations" were in fact written down by Lagrange who used the letter H to denote the "Hamiltonian" to honor Huygens[2] – not Hamilton, who was still in his early childhood at that time! It is however undoubtedly Hamilton's great merit to have recognized the importance of these equations, and to use them with great efficiency in the study of planetary motion, and of light propagation.

4.1 Hamiltonian flows

We will call "Hamiltonian function" (or simply "Hamiltonian") any real function $H \in C^\infty(\mathbb{R}^{2n}_z \times \mathbb{R}_t)$ (although most of the properties we will prove remain valid under the assumption $H \in C^k(\mathbb{R}^{2n}_z \times \mathbb{R}_t)$ with $k \geq 2$: we leave to the reader as an exercise in ordinary differential equations to state minimal smoothness assumptions for the validity of our results).

The *Hamilton equations*

$$\dot{x}_j(t) = \partial_{p_j} H(x(t), p(t), t) \quad , \quad \dot{p}_j(t) = -\partial_{x_j} H(x(t), p(t), t); \tag{1}$$

associated with H are form a (generally non-autonomous) system of $2n$ differential equations. The conditions of existence of the solutions of Hamilton's equations, as well as for which initial points they are defined, are determined by the theory of ordinary differential equations (or "dynamical systems", as it is now called).

The equations (1) can be written economically as

$$\dot{z} = J\partial_z H(z, t) \tag{2}$$

where J is the standard symplectic matrix. Defining the *Hamilton vector field* by

$$X_H = J\partial_z H = (\partial_p H, -\partial_x H) \tag{3}$$

[1] This somewhat unfortunate formulation is due to Kirchhoff.
[2] See Lagrange's *Mécanique Analytique*, Vol. I, pp. 217–226 and 267–270.

(the operator $J\partial_z$ is often called the *symplectic gradient*), Hamilton's equations are equivalent to

$$\sigma(X_H(z,t),\cdot) + d_z H = 0. \tag{4}$$

In fact, for every $z' \in \mathbb{R}^{2n}_z$,

$$\sigma(X_H(z,t),z') = -\langle \partial_x H(z,t), x'\rangle - \langle \partial_p H(z,t), p'\rangle = -\langle \partial_z H(z,t), z'\rangle$$

which is the same thing as (4). This formula is the gate to Hamiltonian mechanics on symplectic manifolds. In fact, formula (4) can be rewritten concisely as

$$i_{X_H}\sigma + dH = 0 \tag{5}$$

where $i_{X_H(\cdot,t)}$ is the contraction operator:

$$i_{X_H(\cdot,t)}\sigma(z)(z') = \sigma(X_H(z,t),z').$$

The interest of formula (5) comes from the fact that it is intrinsic (*i.e.* independent of any choice of coordinates), and allows the definition of Hamilton vector fields on symplectic manifolds: if (M,ω) is a symplectic manifold and $H \in C^\infty(M \times \mathbb{R}_t)$ then, by definition, the Hamiltonian vector field $X_H(\cdot,t)$ is the vector field defined by (5).

One should be careful to note that when the Hamiltonian function H is effectively time-dependent (which is usually the case) then X_H is not a "true" vector field, but rather a family of vector fields on $\mathbb{R}^{2n}_z$ depending smoothly on the parameter t. We can however define the notion of flow associated to X_H:

Definition 1 *Let $t \longmapsto z_t$ be the solution of Hamilton's equations for H passing through a point z at time $t=0$, and let f^H_t be the mapping $\mathbb{R}^{2n}_z \longrightarrow \mathbb{R}^{2n}_z$ defined by $f^H_t(z) = z_t$. The family $(f^H_t) = (f^H_t)_{t\in\mathbb{R}}$ is called the "flow determined by the Hamiltonian function H" or the "flow determined by the vector field X_H".*

A *caveat*: the usual group property

$$f^H_0 = I \quad , \quad f^H_t \circ f^H_{t'} = f^H_{t+t'} \quad , \quad (f^H_t)^{-1} = f^H_{-t} \tag{6}$$

of flows only holds when H is time-independent; in general $f^H_t \circ f^H_{t'} \neq f^H_{t+t'}$ and $(f^H_t)^{-1} \neq f^H_{-t}$ (but of course we still have the identity $f^H_0 = I$).

For notational and expository simplicity we will implicitly assume (unless otherwise specified) that for every $z_0 \in \mathbb{R}^{2n}_z$ there exists a unique solution $t \longmapsto z_t$ of the system (2) passing through z_0 at time $t=0$. The modifications to diverse statements when global existence (in time or space) does not hold are rather obvious and are therefore left to the reader.

As we noted in previous subsection the flow of a time-dependent Hamiltonian vector field is not a one-parameter group; this fact sometimes leads to technical complications when one wants to perform certain calculations. For this reason it is helpful to introduce two (related) notions, those of *suspended Hamilton* *flow* and *time-dependent Hamilton flow*. We begin by noting that Hamilton's equations $\dot{z} = J\partial_z H(z,t)$ can be rewritten as

$$\frac{d}{dt}(z(t),t) = \tilde{X}_H(z(t),t) \tag{7}$$

where

$$\tilde{X}_H = (J\partial_z H, 1) = (\partial_p H, -\partial_x H, 1). \tag{8}$$

Definition 2 *(i) The vector field $\tilde{X}_H$ on the "extended phase space"*

$$\mathbb{R}^{2n+1}_{z,t} \equiv \mathbb{R}^{2n}_z \times \mathbb{R}_t$$

is called the "suspended Hamilton vector field"; its flow $(\tilde{f}^H_t)$ is called the the "suspended Hamilton flow" determined by H.

(ii) The two-parameter family of mappings $\mathbb{R}^{2n}_z \longrightarrow \mathbb{R}^{2n}_z$ defined by the formula

$$(f^H_{t,t'}(z'),t) = \tilde{f}^H_{t-t'}(z',t') \tag{9}$$

is called the "time-dependent flow" determined by H.

Notice that by definition $\tilde{f}^H_t$ thus satisfies

$$\tfrac{d}{dt}\tilde{f}^H_t = \tilde{X}_H(\tilde{f}^H_t). \tag{10}$$

The point with introducing $\tilde{X}_H$ is that it is a true vector field on extended phase-space the while X_H is, as pointed out above, rather a family of vector fields parametrized by t as soon as H is time-dependent. The system (10) being autonomous in its own right, the mappings $\tilde{f}^H_t$ satisfy the usual group properties:

$$\tilde{f}^H_t \circ \tilde{f}^H_{t'} = \tilde{f}^H_{t+t'} \quad , \quad (\tilde{f}^H_t)^{-1} = \tilde{f}^H_{-t} \quad , \quad \tilde{f}^H_0 = I. \tag{11}$$

Notice that the time-dependent flow has the following immediate interpretation: $f^H_{t,t'}$ is the mapping $\mathbb{R}^{2n}_z \longrightarrow \mathbb{R}^{2n}_z$ which takes the point z' at time t' to the point z at time t, the motion occurring along the solution curve to Hamilton equations $\dot{z} = J\partial_z H(z,t)$ passing through these two points. Formula (9) is equivalent to

$$\tilde{f}^H_t(z',t') = (f^H_{t+t',t'}(z'), t+t'). \tag{12}$$

Note that it immediately follows from the group properties (11) of the suspended flow that we have:

$$f^H_{t,t'} = I \quad , \quad f^H_{t,t'} \circ f^H_{t',t''} = f^H_{t,t''} \quad , \quad (f^H_{t,t'})^{-1} = f^H_{t',t} \tag{13}$$

for all times t, t' and t''. When H does not depend on t we have $f^H_{t,t'} = f^H_{t-t'}$; in particular $f^H_{t,0} = f^H_t$.

Let H be some (possibly time-dependent) Hamiltonian function and $f^H_t = f^H_{t,0}$. We say that f^H_t is a *free symplectomorphism* at a point $z_0 \in \mathbb{R}^{2n}_z$ if $Df^H_t(z_0)$ is a free symplectic matrix. Of course f^H_t is never free at $t=0$ since f^H_0 is the identity. In Proposition 4 we will give a necessary and sufficient condition for the symplectomorphisms $f^H_{t,t'}$ to be free. Let us first prove the following Lemma, the proof of which makes use of the notion of generating function:

Lemma 3 *The symplectomorphism $f : \mathbb{R}_z^{2n} \longrightarrow \mathbb{R}_z^{2n}$ is free in a neighborhood $\mathcal{U}$ of $z_0 \in \mathbb{R}_z^{2n}$ if and only if $Df(z')$ is a free symplectic matrix for $z' \in \mathcal{U}$, that is, if and only if* $\det(\partial x/\partial p') \neq 0$.

Proof. Set $z = f(z')$; we have

$$Df(z') = \begin{bmatrix} \dfrac{\partial x}{\partial x'}(z') & \dfrac{\partial x}{\partial p'}(z') \\ \dfrac{\partial p}{\partial x'}(z') & \dfrac{\partial p}{\partial p'}(z') \end{bmatrix}$$

and the symplectic matrix $Df(z')$ is thus free for $z' \in \mathcal{U}$ if and only if

$$\det \frac{\partial x}{\partial p'}(z') \neq 0.$$

We next make the following crucial observation: since f is a symplectomorphism we have $dp \wedge dx = dp' \wedge dx'$ and this is equivalent, by Poincaré's lemma to the existence of a function $G \in C^\infty(\mathbb{R}_z^{2n})$ such that

$$pdx = p'dx' + dG(x', p').$$

Assume now that $Df(z')$ is free for $z' \in \mathcal{U}$; then the condition $\det(\partial x/\partial p') \neq 0$ implies, by the implicit function theorem, that we can locally solve the equation $x = x(x', p')$ in p', so that $p' = p'(x, x')$ and hence $G(x', p')$ is, for $(x', p') \in \mathcal{U}$, a function of x, x' only: $G(x', p') = G(x', p'(x, x'))$. Calling this function W:

$$W(x, x') = G(x', p'(x, x'))$$

we thus have

$$pdx = p'dx' + dW(x, x') = p'dx' + \partial_x W(x, x')dx + \partial_{x'} W(x, x')dx'$$

which requires $p = \partial_x W(x, x')$ and $p' = -\partial_{x'} W(x, x')$ and f is hence free in $\mathcal{U}$. The proof of the converse goes along the same lines and is therefore left to the reader. ■

We will use the notations H_{pp}, H_{xp}, and H_{xx} for the matrices of second derivatives of H in the corresponding variables.

Proposition 4 *There exists $\varepsilon > 0$ such that f_t^H is free at $z_0 \in \mathbb{R}_z^{2n}$ for $0 < |t - t_0| \leq \varepsilon$ if and only if $\det H_{pp}(z_0, t_0) \neq 0$. In particular there exists $\varepsilon > 0$ such that $f_t^H(z_0)$ is free for $0 < |t| \leq \varepsilon$ if and only if $\det H_{pp}(z_0, 0) \neq 0$.*

Proof. Let $t \longmapsto z(t) = (x(t), p(t))$ be the solution to Hamilton's equations

$$\dot{x} = \partial_p H(z, t) \quad , \quad \dot{p} = -\partial_x H(z, t)$$

with initial condition $z(t_0) = z_0$. A second order Taylor expansion in t yields

$$z(t) = z_0 + (t - t_0)X_H(z_0, t_0) + O((t - t_0)^2);$$

and hence

$$x(t) = x_0 + (t - t_0)\partial_p H(z_0, t_0) + O((t - t_0)^2).$$

It follows that

$$\frac{\partial x(t)}{\partial p} = (t - t_0)H_{pp}(z_0, t_0) + O((t - t_0)^2).$$

hence there exists $\varepsilon > 0$ such that $\partial x(t)/\partial p$ is invertible in $[t_0 - \varepsilon, t_0[\cap]t_0, t_0 + \varepsilon]$ if and only if $H_{pp}(z_0, t_0)$ is invertible; in view of Lemma 3 this is equivalent to saying that f_t^H is free at z_0. ■

Example 5 *The result above applies when the Hamiltonian H is of the "physical type"*

$$H(z, t) = \sum_{j=1}^{n} \frac{1}{2m_j} p_j^2 + U(x, t)$$

since we have

$$H_{pp}(z_0, t_0) = \operatorname{diag}[\tfrac{1}{2m_1}, ..., \tfrac{1}{2m_n}].$$

In this case f_t^H is free for small non-zero t near each z_0 where it is defined.

4.2 The variational equation

An essential feature of Hamiltonian flows is that they consist of symplectomorphisms. We are going to give an elementary proof of this property; it relies on the fact that the mapping $t \longmapsto Df_{t,t'}^H(z)$ is, for fixed t', the solution of a differential equation, the *variational equation*, and which plays an important role in many aspects of Hamiltonian mechanics (in particular the study of periodic Hamiltonian orbits).

Proposition 6 *For fixed z set $S_{t,t'}^H(z) = Df_{t,t'}^H(z)$.*

(i) The function $t \longmapsto S_{t,t'}(z)$ satisfies the variational equation

$$\frac{d}{dt} S_{t,t'}^H(z) = JD^2H(f_{t,t'}^H(z), t)S_{t,t'}^H(z) \quad , \quad S_{t,t}^H(z) = I \tag{14}$$

where $D^2H(f_{t,t'}^H(z))$ is the Hessian matrix of H calculated at $f_{t,t'}^H(z)$;

(ii) We have $S_{t,t'}^H(z) \in \operatorname{Sp}(n)$ for every z and t, t' for which it is defined, hence $f_{t,t'}^H$ is a symplectomorphism.

Proof. *Proof of (i).* It is sufficient to consider the case $t' = 0$. Set $f_{t,0}^H = f_t^H$ and $S_{t,t'}^H = S_t$. Taking Hamilton's equation into account the time-derivative of the Jacobian matrix $S_t(z)$ is

$$\frac{d}{dt} S_t(z) = \frac{d}{dt}(Df_t^H(z)) = D\left(\frac{d}{dt} f_t^H(z)\right)$$

that is

$$\frac{d}{dt}S_t(z) = D(X_H(f_t^H(z))).$$

Using the fact that $X_H = J\partial_z H$ together with the chain rule, we have

$$\begin{aligned} D(X_H(f_t^H(z))) &= D(J\partial_z H)(f_t^H(z), t) \\ &= JD(\partial_z H)(f_t^H(z), t) \\ &= J(D^2 H)(f_t^H(z), t)Df_t^H(z) \end{aligned}$$

hence $S_t(z)$ satisfies the variational equation (14), proving (i). *Proof of* (ii). Set $S_t = S_t(z)$ and $A_t = (S_t)^T JS_t$; using the product rule together with (14) we have

$$\begin{aligned} \frac{dA_t}{dt} &= \frac{d(S_t)^T}{dt} JS_t + (S_t)^T J\,\frac{dS_t}{dt} \\ &= (S_t)^T D^2 H(z,t)S_t - (S_t)^T D^2 H(z,t)S_t \\ &= 0. \end{aligned}$$

It follows that the matrix $A_t = (S_t)^T JS_t$ is constant in t, hence $A_t(z) = A_0(z) = J$ so that $(S_t)^T JS_t = J$ proving that $S_t \in \mathrm{Sp}(n)$. ■

A related result is:

Proposition 7 *Let $t \longmapsto X_t$ be a C^∞ mapping $\mathbb{R} \longrightarrow \mathfrak{sp}(n)$ and $t \longmapsto S_t$ a solution of the differential system*

$$\frac{d}{dt}S_t = X_t S_t \quad , \quad S_0 = I.$$

We have $S_t \in \mathrm{Sp}(n)$ for every $t \in \mathbb{R}$.

Proof. The condition $X_t \in \mathfrak{sp}(n)$ is equivalent to JX_t being symmetric. Hence

$$\frac{d}{dt}(S_t^T JS_t) = S_t^T X_t^T JS_t + S_t^T JX_t S_t = 0$$

so that $S_t^T JS_t = S_0^T JS_0 = J$ and $S_t \in \mathrm{Sp}(n)$ as claimed. ■

Hamilton's equations are covariant (*i.e.*, they retain their form) under symplectomorphisms. Let us begin by proving the following general result about vector fields which we will use several times. If X is a vector field and f a diffeomorphism we denote by $Y = f^* X$ the vector field defined by

$$Y(u) = D(f^{-1})(f(u))X(f(u)) = [Df(u)]^{-1}X(f(u)). \tag{15}$$

($f^* X$ is called the "pull-back" of the vector field X by the diffeomorphism f.)

Lemma 8 *Let X be a vector field on $\mathbb{R}^m$ and (φ_t^X) its flow. Let f be a diffeomorphism $\mathbb{R}^m \longrightarrow \mathbb{R}^m$. The family (φ_t^Y) of diffeomorphisms defined by*

$$\varphi_t^Y = f^{-1} \circ \varphi_t^X \circ f \tag{16}$$

is the flow of the vector field $Y = (Df)^{-1}(X \circ f)$.

Proof. We obviously have $\varphi_0^Y = I$; in view of the chain rule

$$\begin{aligned}\frac{d}{dt}\varphi_t^Y(x) &= D(f^{-1})(\varphi_t^X(f(x)))X(\varphi_t^X(f(x)) \\ &= (Df)^{-1}(\varphi_t^Y(x))X(f(\varphi_t^Y(x)))\end{aligned}$$

and hence $\frac{d}{dt}\varphi_t^Y(x) = Y(\varphi_t^Y(x))$ which we set out to prove. ■

Specializing to the Hamiltonian case, this lemma yields:

Proposition 9 *Let f be a symplectomorphism.*
(i) We have

$$X_{H\circ f}(z) = [Df(z)]^{-1}(X_H \circ f)(z). \tag{17}$$

(ii) The flows (f_t^H) and $(f_t^{H\circ f})$ are conjugate by f:

$$f_t^{H\circ f} = f^{-1} \circ f_t^H \circ f \tag{18}$$

and thus $f^* X_H = X_{H\circ f}$ when f is symplectic.

Proof. Let us prove (i); the assertion (ii) will follow in view of Lemma 8 above. Set $K = H \circ f$. By the chain rule

$$\partial_z K(z) = [Df(z)]^T (\partial_z H)(f(z))$$

hence the vector field $X_K = J\partial_z K$ is given by

$$X_K(z) = J[Df(z)]^T \partial_z H(f(z)).$$

Since $Df(z)$ is symplectic we have $J[Df(z)]^T = [Df(z)]^{-1}J$ and thus

$$X_K(z) = [Df(z)]^{-1} J\partial_z H(f(z))$$

which is (17). ■

Remark 10 *Proposition 9 can be restated as follows: set $(x', p') = f(x, p)$ and $K = H \circ f$; if f is a symplectomorphism then we have the equivalence*

$$\begin{gathered}\dot{x}' = \partial_{p'} K(x', p') \quad \text{and } \dot{p}' = -\partial_{x'} K(x', p') \\ \Longleftrightarrow \\ \dot{x} = \partial_p H(x, p) \quad \text{and} \quad \dot{p} = -\partial_x H(x, p).\end{gathered} \tag{19}$$

Example 11 *For instance, the change of variables $(x, p) \longmapsto (I, \theta)$ defined by $x = \sqrt{2I}\cos\theta$, $p = \sqrt{2I}\sin\theta$ is symplectic.*

An interesting fact is that there is a wide class of Hamiltonian functions whose time-dependent flows (f_t^H) consist of free symplectomorphisms if t is not too large (and different from zero). This is the case for instance when H is of the "physical" type

$$H(z, t) = \sum_{j=1}^{n} \frac{1}{2m_j}(p_j - A_j(x, t))^2 + U(x, t) \tag{20}$$

($m_j > 0$, A_j and U smooth). More generally:

Lemma 12 *There exists $\varepsilon > 0$ such that f_t^H is free at $z_0 \in \mathbb{R}_z^{2n}$ for $0 < |t| \leq \varepsilon$ if and only if* $\det H_{pp}(z_0, 0) \neq 0$*, and hence, in particular, when H is of the type (20).*

Proof. Let $t \longmapsto z(t)$ be the solution to Hamilton's equations

$$\dot{x} = \partial_p H(z,t) \quad , \quad \dot{p} = -\partial_x H(z,t)$$

with initial condition $z(0) = z_0$. A second order Taylor expansion at time $t = 0$ yields

$$z(t) = z_0 + tX_H(z_0, 0) + O((t)^2)$$

where $X_H = J\partial_z H$ is the Hamiltonian vector field of H; in particular

$$x(t) = x_0 + t\partial_p H(z_0, 0) + O(t^2)$$

and hence

$$\frac{\partial x(t)}{\partial p} = tH_{pp}(z_0, 0) + O(t^2)$$

where H_{pp} denotes the matrix of derivatives of H in the variables p_j. It follows that there exists $\varepsilon > 0$ such that $\partial x(t)/\partial p$ is invertible in $[-\varepsilon, 0[\cap]0, \varepsilon]$ if and only if $H_{pp}(z_0, 0)$ is invertible; in view of Lemma 3 this is equivalent to saying that f_t^H is free at z_0. ■

4.3 The group Ham(n)

The group Ham(n) is the connected component of the group Symp(n) of all symplectomorphisms of $(\mathbb{R}_z^{2n}, \sigma)$. Each of its points is the value of a Hamiltonian flow at some time t. The study of the various algebraic and topological properties of the group Ham(n) is a very active area of current research.

We will say that a symplectomorphism f of the standard symplectic space $(\mathbb{R}_z^{2n}, \sigma)$ is *Hamiltonian* if there exists a function $H \in C^\infty(\mathbb{R}_{z,t}^{2n+1}, \mathbb{R})$ and a number $a \in \mathbb{R}$ such that $f = f_a^H$. Taking $a = 0$ it is clear that the identity is a Hamiltonian symplectomorphism. The set of all Hamiltonian symplectomorphisms is denoted by Ham(n). We are going to see that it is a connected and normal subgroup of Symp(n); let us first prove a preparatory result which is interesting in its own right:

Proposition 13 *Let (f_t^H) and (f_t^K) be Hamiltonian flows. Then:*

$$f_t^H f_t^K = f_t^{H\#K} \quad \textit{with} \quad H\#K(z,t) = H(z,t) + K((f_t^H)^{-1}(z), t). \tag{21}$$

$$(f_t^H)^{-1} = f_t^{\bar{H}} \quad \textit{with} \quad \bar{H}(z,t) = -H(f_t^H(z), t). \tag{22}$$

Proof. Let us first prove (21). By the product and chain rules we have

$$\begin{aligned}\frac{d}{dt}(f_t^H f_t^K) &= (\frac{d}{dt}f_t^H)f_t^K + (Df_t^H)f_t^K \frac{d}{dt}f_t^K \\ &= X_H(f_t^H f_t^K) + (Df_t^H)f_t^K \circ X_K(f_t^K)\end{aligned}$$

and it thus suffices to show that

$$(Df_t^H)f_t^K \circ X_K(f_t^K) = X_{K\circ(f_t^H)^{-1}}(f_t^K). \tag{23}$$

Writing

$$(Df_t^H)f_t^K \circ X_K(f_t^K) = (Df_t^H)((f_t^H)^{-1}f_t^H f_t^K) \circ X_K((f_t^H)^{-1}f_t^H f_t^K)$$

the equality (23) follows from the transformation formula (17) in Proposition 9. Formula (22) is now an easy consequence of (21), noting that $(f_t^H f_t^{\bar{H}})$ is the flow determined by the Hamiltonian

$$K(z,t) = H(z,t) + \bar{H}((f_t^H)^{-1}(z),t) = 0;$$

$f_t^H f_t^{\bar{H}}$ is thus the identity, so that $(f_t^H)^{-1} = f_t^{\bar{H}}$ as claimed. ∎

Let us now show that $\mathrm{Ham}(n)$ is a group, as claimed:

Proposition 14 $\mathrm{Ham}(n)$ *is a normal and connected subgroup of the group* $\mathrm{Symp}(n)$ *of all symplectomorphisms of* $(\mathbb{R}_z^{2n}, \sigma)$*.*

Proof. Let us show that if $f, g \in \mathrm{Ham}(n)$ then $fg^{-1} \in \mathrm{Ham}(n)$. We begin by remarking that if $f = f_a^H$ for some $a \neq 0$ then we also have $f = f_1^{H^a}$ where $H^a(z,t) = aH(z,at)$. In fact, setting $t^a = at$ we have

$$\frac{dz^a}{dt} = J\partial_z H^a(z^a,t) \Longleftrightarrow \frac{dz^a}{dt^a} = J\partial_z H(z^a,t^a)$$

and hence $f_t^{H^a} = f_{at}^H$. We may thus assume that $f = f_1^H$ and $g = f_1^K$ for some Hamiltonians H and K. Now, using successively (21) and (22) we have

$$fg^{-1} = f_1^H (f_1^K)^{-1} = f_1^{H\#\bar{K}}$$

hence $fg^{-1} \in \mathrm{Ham}(n)$. That $\mathrm{Ham}(n)$ is a normal subgroup of $\mathrm{Symp}(n)$ immediately follows from formula (18) in Proposition 9: if g is a symplectomorphism and $f \in \mathrm{Ham}(n)$ then

$$f_1^{H\circ g} = g^{-1} f_1^H g \in \mathrm{Ham}(n) \tag{24}$$

so we are done. ∎

The result above motivates the following definition:

Definition 15 *The set* $\mathrm{Ham}(n)$ *of all Hamiltonian symplectomorphisms equipped with the law* $fg = f \circ g$ *is called the group of Hamiltonian symplectomorphisms of the standard symplectic space* $(\mathbb{R}_z^{2n}, \sigma)$*.*

The topology of $\mathrm{Symp}(n)$ is defined by specifying the convergent sequences: we will say that $\lim_{j\to\infty} f_j = f$ in $\mathrm{Symp}(n)$ if and only if for every compact set $\mathcal{K}$ in $\mathbb{R}_z^{2n}$ the sequences $(f_{j|\mathcal{K}})$ and $(D(f_{j|\mathcal{K}}))$ converge uniformly towards $f_{|\mathcal{K}}$

and $D(f_{|\mathcal{K}})$, respectively. The topology of $\mathrm{Ham}(n)$ is the topology induced by $\mathrm{Symp}(n)$.

We are now gong to prove a deep and beautiful result due to Banyaga. It essentially says that a path of time-one Hamiltonian symplectomorphisms passing through the identity at time zero is itself Hamiltonian. It will follow that $\mathrm{Ham}(n)$ is a connected group.

Let $t \longmapsto f_t$ be a path in $\mathrm{Ham}(n)$, defined for $0 \leq t \leq 1$ and starting at the identity: $f_0 = I$. We will call such a a path a *one-parameter family of Hamiltonian symplectomorphisms.* Thus, each f_t is equal to some symplectomorphism $f_1^{H_t}$. A striking – and not immediately obvious! – fact is that each path $t \longmapsto f_t$ is itself is the flow of a Hamiltonian function!

Theorem 16 *Let (f_t) be a one-parameter family in* $\mathrm{Ham}(n)$. *Then $(f_t) = (f_t^H)$ where the Hamilton function H is given by*

$$H(z,t) = -\int_0^1 \sigma(X(uz,t),z)du \quad \textit{with} \quad X = (\tfrac{d}{dt}f_t) \circ (f_t)^{-1}. \tag{25}$$

Proof. By definition of X we have $\frac{d}{dt}f_t = Xf_t$ so that all we have to do is to prove that X is a (time-dependent) Hamiltonian field. For this it suffices to show that the contraction $i_X\sigma$ of the symplectic form with X is an exact differential one-form, for then $i_X\sigma = -dH$ where H is given by (25). The f_t being symplectomorphisms, they preserve the symplectic form σ and hence $\mathcal{L}_X\sigma = 0$. In view of Cartan's homotopy formula we have

$$\mathcal{L}_X\sigma = i_X d\sigma + d(i_X\sigma) = d(i_X\sigma) = 0$$

so that $i_X\sigma$ is closed; by Poincaré's lemma it is also exact. ■

Let $(f_t^H)_{0\leq t\leq 1}$ and $(f_t^K)_{0\leq t\leq 1}$ be two arbitrary paths in $\mathrm{Ham}(n)$. The paths $(f_t^H f_t^K)_{0\leq t\leq 1}$ and $(f_t)_{1\leq t\leq 1}$ where

$$f_t = \begin{cases} f_{2t}^K & \text{when } \ 0 \leq t \leq \frac{1}{2} \\ f_{2t-1}^H f_1^K & \text{when } \ \frac{1}{2} \leq t \leq 1 \end{cases}$$

are homotopic with fixed endpoints. Let us construct explicitly a homotopy of the first path on the second, that is, a continuous mapping

$$h : [0,1] \times [0,1] \longrightarrow \mathrm{Ham}(n)$$

such that $h(t,0) = f_t^H f_t^K$ and $h(t,1) = f_t$. Define h by $h(t,s) = a(t,s)b(t,s)$ where a and b are functions

$$a(t,s) = \begin{cases} I & \text{for } \ 0 \leq t \leq \frac{s}{2} \\ f_{(2t-s)/(2-s)}^H & \text{for } \ \frac{s}{2} \leq t \leq 1 \end{cases}$$

$$b(t,s) = \begin{cases} f_{2t/(2-s)}^K & \text{for } \ 0 \leq t \leq 1 - \frac{s}{2} \\ f_1^K & \text{for } \ \frac{s}{2} \leq t \leq 1 \end{cases}.$$

We have $a(t,0) = f_t^H$, $b(t,0) = f_t^K$ hence $h(t,0) = f_t^H f_t^K$; similarly

$$h(t,1) = \begin{cases} f_{2t}^K & \text{for } \ 0 \leq t \leq \frac{1}{2} \\ f_{2t-1}^H f_1^K & \text{for } \ \frac{1}{2} \leq t \leq 1 \end{cases}$$

that is $h(t,1) = f_t$.

4.4 Hamiltonian periodic orbits

Let $H \in C^\infty(\mathbb{R}_z^{2n})$ be a time-independent Hamiltonian function, and (f_t^H) the flow determined by the associated vector field $X_H = J\partial_z$.

Definition 17 *Let $z_0 \in \mathbb{R}_z^{2n}$; the mapping*

$$\gamma : \mathbb{R}_t \longrightarrow \mathbb{R}_z^{2n} \quad , \quad \gamma(t) = f_t^H(z_0)$$

is called "(Hamiltonian) orbit through z_0". If there exists $T > 0$ such that $f_{t+T}^H(z_0) = f_t^H(z_0)$ for all $t \in \mathbb{R}$ one says that the orbit γ through z_0 is "periodic with period T". [The smallest possible period is called "primitive period"].

The following properties are obvious:

- Let γ, γ' be two orbits of H. Then the ranges $\operatorname{Im}\gamma$ and $\operatorname{Im}\gamma'$ are either disjoint or identical.
- The value of H along any orbit is constant ("theorem of conservation of energy")..

The first property follows from the uniqueness of the solutions of Hamilton's equations, and the second from the chain rule, setting $\gamma(t) = (x(t), p(t))$:

$$\begin{aligned} \frac{d}{dt} H(\gamma(t)) &= \langle \partial_x H(\gamma(t)), \dot{x}(t) \rangle + \langle \partial_p H(\gamma(t)), \dot{p}(t) \rangle \\ &= -\langle \dot{p}(t), \dot{x}(t) \rangle + \langle \dot{x}(t), \dot{p}(t) \rangle \\ &= 0 \end{aligned}$$

where we have taken into account Hamilton's equations.

Assume now that γ is a periodic orbit through z_0. We will use the notation $S_t(z_0) = Df_t^H(z_0)$.

Definition 18 *Let $\gamma : t \longmapsto f_t^H(z_0)$ be a a periodic orbit with period T. The symplectic matrix $S_T(z_0) = Df_T^H(z_0)$ is called "monodromy matrix". The eigenvalues of $S_T(z_0)$ are called the "Floquet multipliers" of γ.*

The following property is well-known in Floquet theory:

Lemma 19 *(i) Let $S_T(z_0)$ be the monodromy matrix of the periodic orbit γ. We have*

$$S_{t+T}(z_0) = S_t(z_0)S_T(z_0) \tag{26}$$

for all $t \in \mathbb{R}$. In particular $S_{NT}(z_0) = S_T(z_0)^N$ for every integer N. (ii) Monodromy matrices corresponding to the choice of different origins on the periodic orbit are conjugate of each other in $\mathrm{Sp}(n)$ *hence the Floquet multipliers do not depend on the choice of origin on the periodic orbit; (iii) Each periodic orbit has an even number > 0 of Floquet multiplier equal to one.*

Proof. *Proof of (i).* The mappings f_t^H form a group, hence, taking into account the equality $f_T^H(z_0) = z_0$:

$$f_{t+T}^H(z_0) = f_t^H(f_T^H(z_0))$$

so that by the chain rule,

$$Df_{t+T}^H(z_0) = Df_t^H(f_T^H(z_0))Df_T^H(z_0)$$

that is (26) since $f_T^H(z_0) = z_0$. *Proof of (ii).*Evidently the orbit through any point $z(t)$ of the periodic orbit γ is also periodic. We begin by noting that if z_0 and z_1 are points on the same orbit γ then there exists t_0 such that $z_0 = f_{t_0}^H(z_1)$. We have

$$f_t^H(f_{t_0}^H(z_1)) = f_{t_0}^H(f_t^H(z_1))$$

hence, applying the chain rule of both sides of this equality,

$$Df_t^H(f_{t_0}^H(z_1))Df_{t_0}^H(z_1) = Df_{t_0}^H(f_t^H(z_1))Df_t^H(z_1).$$

Choosing $t = T$ we have $f_t^H(z_0) = z_0$ and hence

$$S_T(f_{t_0}^H(z_1))S_{t_0}(z_1) = S_{t_0}(z_1)S_T(z_1)$$

that is, since $f_{t_0}^H(z_1) = z_0$,

$$S_T(z_0)S_{t_0}(z_1) = S_{t_0}(z_1)S_T(z_1).$$

It follows that the monodromy matrices $S_T(z_0)$ and $S_T(z_1)$ are conjugate and thus have the same eigenvalues. *Proof of (iii).* We have, using the chain rule together with the relation $f_t^H \circ f_{t'}^H = f_{t+t'}^H$

$$\left.\frac{d}{dt'} f_t^H(f_{t'}^H(z_0))\right|_{t'=0} = Df_t(z_0)X_H(z_0) = X_H(f_t^H(z_0))$$

hence $S_{T_0}(z_0)X_H(z_0) = X_H(z_0)$ setting $t = T$; $X_H(z_0)$ is thus an eigenvector of $S_T(z_0)$ with eigenvalue one; the Lemma follows the eigenvalues of a symplectic matrix occurring in quadruples $(\lambda, 1/\lambda, \bar{\lambda}, 1/\bar{\lambda})$. ■

The following theorem is essentially due to Poincaré:

Theorem 20 *Let* $E_0 = H(z_0)$ *be the value of* H *along a periodic orbit* γ_0. *Assume that* γ_0 *has exactly two Floquet multipliers equal to one. Then there exists a unique smooth* 1*-parameter family* (γ_E) *of periodic orbits of* E *with period* T *parametrized by the energy* E, *and each* γ_E *is isolated on the hypersurface* $\Sigma_E = \{z : H(z) = E\}$ *among those periodic orbits having periods close to the period* T_0 *of* γ_0 *Moreover* $\lim_{E\to E_0} T = T_0$.

One shows, using "normal form" techniques that when the conditions of the theorem above are fulfilled, the monodromy matrix of γ_0 can be written as

$$S_T(z_0) = S_0^T \begin{bmatrix} U & 0 \\ 0 & \tilde{S}(z_0) \end{bmatrix} S_0$$

with $S_0 \in \mathrm{Sp}(n)$, $\tilde{S}(z_0) \in \mathrm{Sp}(n-1)$ and U is of the type $\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ for some real number β; the $2(n-1)\times 2(n-1)$ symplectic matrix $\tilde{S}(z_0)$ is called the stability matrix of the isolated periodic orbit γ_0. It plays a fundamental role not only in the study of periodic orbits, but also in semiclassical mechanics.