ON A MONODROMY THEOREM FOR SHEAVES OF LOCAL FIELDS AND APPLICATIONS

JONATAN HERRERA, MIGUEL ÁNGEL JAVALOYES, AND PAOLO PICCIONE

Abstract. We prove monodromy theorem for local vector fields belonging to a sheaf satisfying the unique continuation property. In particular, in the case of *admissible regular* sheaves of local fields defined on a simply connected manifold, we obtain a global extension result for *every* local field of the sheaf. This generalizes previous works of Nomizu [20] for semi-Riemannian Killing fields, of Ledger–Obata [17] for conformal fields, and of Amores [1] for fields preserving a G-structure of finite type. The result applies to Finsler and pseudo-Finsler Killing fields. Some applications are discussed.

Contents

Date: October 29th, 2014.

1. Introduction

Introduction of this new version needs a thorough revision.

The question of global extension of locally defined Killing, or conformal fields of a semi-Riemannian manifold appears naturally in several contexts. For instance, the property of global extendability of local Killing fields in simply connected Lorentzian manifolds plays a crucial role in the celebrated result that the isometry group of a compact simply connected analytic Lorentz manifold is compact, see [4]. Nomizu's result is often used in connection with de Rham splitting theorem to obtain that a real-analytic complete Riemannian manifold M admitting a local parallel field has universal covering $\widetilde{M}$ which splits metrically as $N \times \mathbb{R}$, see [5, Theorem C] for an example. Recently, the question of (unique) extensibility of local Killing vector fields has been studied in the context of Ricci-flat Lorentz manifolds, in the setting of the black hole rigidity problem, see [11]. The Killing extension property is also one of the crucial steps in the proof of the extensibility of locally homogeneous Lorentz metrics in dimension 3, see [6]. It is to be expected that an analogous extension property should play an important role also for more general sheaves of local vector fields defined on a differentiable manifold. The purpose of the present paper is to explore this question in geometric problems that have not been studied in the literature as of yet, like for instance in Finsler or pseudo-Finsler manifolds.

Let us recall that the first result in the literature, concerning the existence of a global extension of local Killing fields defined on a simply-connected real-analytic Riemannian manifold, was proved by Nomizu in [20]. The same proof applies to the semi-Riemannian case. The question of extension of local conformal fields has been studied in Ledger–Obata [17]. A more conceptual proof of the extendability property was proved by Amores in [1] for local vector fields whose flow preserves a G-structure *of finite type* of an n-dimensional manifold M; here G is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. When G is the orthogonal or the conformal group of some nondegenerate metric on $\mathbb{R}^n$, then Amores' result reproduces the classic semi-Riemannian or conformal case of Nomizu and Ledger–Obata.

Although the class of G-structure automorphisms is quite general, restriction to those of finite type leaves several important examples of geometric structures out of the theory developed in [1]. Most notably..

Examples of admissible sheaves that do not come from G-structures of finite type.

- Finlser and pseudo-Finsler
- See [15, Note 13, p. 332], see also [24]. Examples in Theorem 3 (pseudo-conformal fields on a real hypersurface of $\mathbb{C}^n$ which is non-degenerate) and Theorem 4 (holomorphic fields of a hyperbolic manifold) in [15, p. 332] may work.

Moreover, it is interesting to observe that the sheaf of local vector fields whose flow preserves any G-structure has always a Lie algebra structure (see Proposition 5.3), which seems to be unnecessary for the extendability problem we aim at. Thus, it is natural to extend the results of [1, 17, 20] to arbitrary vector space valued sheaves of local vector fields.

The starting point of the theory developed here is the observation that a necessary condition for the extension property of some class of local vector fields on a manifold M, is that the space of germs of this class of fields must have the same dimension at each point of M. Namely, if the extension property holds, then the space of germs at each point has the same dimension as the space of globally defined vector fields of the given class. This condition is called *regularity* in the paper (Definition 3.4); the reader should be warned that this notion of regularity for sheaves of vector fields is different from the one given in reference [25, Definition 1.3]. Our main abstract result (Theorem 3.6) is that, for a certain class of sheaves of local vector fields, called *admissible*, defined on a simply connected manifold, then the regularity condition is also sufficient to guarantee that local fields extend (uniquely) to globally defined vector fields. By admissible, we mean sheaves of vector fields that have two basic properties: bounded rank and unique continuation, see Subsection 3.1. A sheaf has bounded rank if the space of germs of its local field has uniformly bounded dimension; it has the unique continuation property if two of its local fields coincide on their common domain when they coincide on some non-empty open set.

Mention a discussion of the relations between the two notions

A proof of Theorem 3.6 is obtained by defining a notion of transport of germs along curves (Definition 2.1), and then proving existence and fixed endpoints homotopy invariance of the transport under the regularity assumption (Proposition 3.5). The statement and the proof of Theorem 3.6 have strong analogies with the classical *monodromy theorem* for analytic functions on the plane, see for instance [18]. It is interesting to observe that a somewhat similar procedure had been described in the original article of Nomizu [20], using a differential equation along smooth curves satisfied by Killing vector fields. In the abstract framework considered in the present paper, no such differential equation is available, and one has to resort to purely topological arguments for the existence of transport of germs along curves.

The statement of Theorem 3.6 generalizes Nomizu's result [20] for semi-Riemannian Killing fields, of Ledger–Obata [17] for conformal fields, and of

Amores [1] for fields preserving a G-structure of finite type. Namely, the bounded rank and unique continuation property are proven to hold in the case of semi-Riemannian Killing fields, in the case of conformal fields, and also in the case of vector fields whose flow consists of local automorphisms of a G-structure of finite type (Section 4). It is worth mentioning that the case of G-structures of finite type is studied here with different methods than the ones employed in in [1]. More precisely, in the present paper we determine an explicit characterization of G-fields in terms of compatible connections (see Proposition 4.1), ...

In addition, the result of Theorem 3.6 applies also to situations of field preserving some geometric structure which cannot be described in terms of a G-structure of finite type. The two main examples discussed in this paper are the case of Finsler and pseudo-Finsler structures, see Section 6 and Section 7.

- Find more examples of G-structures of infinite type that satisfy the bounded rank and unique continuation property. Homothetic fields? Affine fields? Projective fields? Curvature collineations? Fields preserving distributions?

2. An abstract extension problem for sheaves of vector fields

We will discuss here an abstract theory concerning the global extension property for sheaves of vector fields on differentiable manifolds. Some of the ideas employed here have their origin in reference [20], where similar results were obtained in the case of Killing vector fields in Riemannian manifolds, although our approach does not use differential equations. A central notion for our theory is a property called unique continuation for sheaves of local vector fields on a manifold.

2.1. Sheaves with the unique continuation property.

Let M be a differentiable manifold with $\dim(M) = n$, and for all open subset $U \subset M$, let $\mathfrak{X}(U)$ denote the sheaf of smooth vector field on U. By smooth we mean of class C^∞, although many of the results of the present paper hold under less regularity assumptions. Given $V \subset U$ and $K \in \mathfrak{X}(U)$, we will denote by $K \mapsto K\big|_V \in \mathfrak{X}(V)$ the restriction map.

For every connected open subset U, choose $\mathcal{F}(U) \subset \mathfrak{X}(U)$, and assume that this family is a vector sub-sheaf, namely, if $K \in \mathcal{F}(U)$ and $V \subset U$, then $K|_V \in \mathcal{F}(V)$ and if a vector field $K \in \mathfrak{X}(U)$, $U = \bigcup_{i\in I} U_i$ with $K|_{U_i} \in \mathcal{F}(U_i)$, then $K \in \mathcal{F}(U)$. Elements of $\mathcal{F}$ will be called *local $\mathcal{F}$-fields.*

We say that $\mathcal{F}$ has the *unique continuation property* if, given an open connected set $U \subset M$ and given $K \in \mathcal{F}(U)$, there exists a non empty open subset $V \subseteq U$ such that $K|_V = 0$, then $K = 0$.

The unique continuation property implies that, given connected open sets $U_1 \subset U_2 \subset M$, then the map $\mathcal{F}(U_2) \ni K \mapsto K\big|_{U_1} \in \mathcal{F}(U_1)$ is injective, and therefore one has a monotonicity property:

$$\dim\big(\mathcal{F}(U_1)\big) \geqslant \dim\big(\mathcal{F}(U_2)\big), \qquad \text{for } U_1 \subset U_2 \text{ connected.} \tag{1}$$

Given $p \in M$, we define $\mathfrak{G}_p^{\mathcal{F}}$ as the space of *germs at* p of local $\mathcal{F}$-vector fields[1], also called $\mathcal{F}$-*germs at* p. More precisely, $\mathfrak{G}_p^{\mathcal{F}}$ is defined as the quotient of the set:

$$\bigcup_U \mathcal{F}(U),$$

where U varies in the family of connected open subsets of M containing p, by the following equivalence relation $\cong_p$: for $K_1 \in \mathcal{F}(U_1)$ and $K_2 \in \mathcal{F}(U_2)$, $K_1 \cong_p K_2$ if $K_1 = K_2$ on a non empty connected open subset $V \subseteq U_1 \cap U_2$. For $K \in \mathcal{F}(U)$, with $p \in U$, we will denote by $\mathrm{germ}_p(K)$ the germ of K at p, which is defined as the $\cong_p$-equivalence class of K. It is easy to see that $\mathfrak{G}_p^{\mathcal{F}}$ is a vector space for all p; moreover, for all $p \in M$ and all connected open neighborhood U of p, the map:

$$\mathcal{F}(U) \ni K \longmapsto \mathrm{germ}_p(K) \in \mathfrak{G}_p^{\mathcal{F}} \tag{2}$$

is linear and injective, since if $\mathrm{germ}_p(K) = 0$, then by definition there exists $V \subseteq U$ such that $K|_V = 0$ and by the unique continuation property, we have that $K = 0$. In particular, if two local $\mathcal{F}$-fields have the same germ at some $p \in M$, they coincide in any connected open neighborhood of p contained in their common domain.

We will also use the evaluation map:

$$\mathrm{ev}_p : \mathfrak{G}_p^{\mathcal{F}} \longrightarrow T_pM, \tag{3}$$

defined by $\mathrm{ev}_p(\mathfrak{g}) = K_p$, where K is any local $\mathcal{F}$-field around p such that $\mathrm{germ}_p(K) = \mathfrak{g}$.

2.2. **Transport of germs of $\mathcal{F}$-fields.** We will henceforth assume that $\mathcal{F}$ is a sheaf of local vector fields that satisfies the unique continuation property. We will now define the notion of *transport* of germs of $\mathcal{F}$-vector fields along curves. The reader should note the analogy between this definition and the classical notion of analytic continuation along a curve in elementary complex analysis.

Definition 2.1. Let $\gamma : [a, b] \to M$ be a curve, and let $\mathfrak{g}$ be a distribution of $\mathcal{F}$-local fields along γ, i.e., a map $[a, b] \ni t \mapsto \mathfrak{g}_t \in \mathfrak{G}_{\gamma(t)}^{\mathcal{F}}$. We say that $\mathfrak{g}$ is a *transport of $\mathcal{F}$-germs along* γ if the following holds: for all $t_* \in [a, b]$, there exists $\varepsilon > 0$, an open neighborhood U of $\gamma(t_*)$ and $K \in \mathcal{F}(U)$ such that $\gamma(t) \in U$ and $\mathfrak{g}_t = \mathrm{germ}_{\gamma(t)}(K)$ for all $t \in \,]t_* - \varepsilon, t_* + \varepsilon[\, \cap [a, b]$.

It is easy to see that initial conditions identify uniquely transports of $\mathcal{F}$-germs along continuous curves.

[1]The space $\mathfrak{G}_p^{\mathcal{F}}$ is also called the *stalk* of the sheaf $\mathcal{F}$ at p.

Lemma 2.2. *If $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ are transports of $\mathcal{F}$-germs along a continuous path $\gamma : [a, b] \to M$ such that $\mathfrak{g}^{(1)}_{t_*} = \mathfrak{g}^{(2)}_{t_*}$ for some $t_* \in [a, b]$, then $\mathfrak{g}^{(1)} = \mathfrak{g}^{(2)}$.*

Proof. The set $\mathcal{A} = \big\{t \in [a, b] : \mathfrak{g}^{(1)}_t = \mathfrak{g}^{(2)}_t\big\}$ is open, by the very definition of transport, and non empty, because $t_* \in \mathcal{A}$. Let us show that it is closed. Denote by $\overline{\mathcal{A}}$ the closure of $\mathcal{A}$ in $[a, b]$, and let $t_0 \in \overline{\mathcal{A}}$; let U be a sufficiently small connected open neighborhood of $\gamma(t_0)$. There exist $K_1, K_2 \in \mathcal{F}(U)$ such that, for t sufficiently close to t_0:

$$\mathrm{germ}_{\gamma(t)} K_1 = \mathfrak{g}^{(1)}_t \quad \text{and} \quad \mathrm{germ}_{\gamma(t)} K_2 = \mathfrak{g}^{(2)}_t. \tag{4}$$

Since t_0 is in the closure of $\mathcal{A}$, we can find $t \in \mathcal{A}$ such that $\gamma(t) \in U$ and for which (4) holds; in particular, the germs of K_1 and K_2 at such t coincide. It follows that $K_1 = K_2$ in U, and therefore $\mathfrak{g}^{(1)}_{t_0} = \mathrm{germ}_{\gamma(t_0)} K_1 = \mathrm{germ}_{\gamma(t_0)} K_2 = \mathfrak{g}^{(2)}_{t_0}$, i.e., $t_0 \in \mathcal{A}$ and $\mathcal{A}$ is closed. □

2.3. **Homotopy invariance.** We will now show that the notion of transport of germs in invariant by fixed endpoints homotopies.

Proposition 2.3. *Let $\mathcal{F}$ a sheaf of local fields on a manifold M; assume that $\mathcal{F}$ satisfy the unique continuation property. Let $\gamma : [0, 1] \times [a, b] \to M$ be a continuous map such that $\gamma(s, a) = p$ and $\gamma(s, b) = q$ for all $s \in [0, 1]$, with $p, q \in M$ two fixed points. For all $s \in [0, 1]$ set $\gamma_s := \gamma(s, \cdot) : [a, b] \to M$. Fix $\mathfrak{g}_a \in \mathfrak{G}^{\mathcal{F}}_p$, and assume that for all $s \in [0, 1]$ there exists a transport $\mathfrak{g}^{(s)}$ of $\mathcal{F}$-germs along γ_s, with $\mathfrak{g}^{(s)}_a = \mathfrak{g}_a$ for all s. Then, $\mathfrak{g}^{(s)}_b \in \mathfrak{G}^{\mathcal{F}}_q$ does not depend on s.*

Proof. It suffices to show that $s \mapsto \mathfrak{g}^{(s)}_b$ is locally constant on $[0, 1]$. Fix $s \in [0, 1]$ and choose $N \in \mathbb{N}$, $U_1, \dots U_N$ connected open subsets of U, and $K_i \in \mathcal{F}(U_i)$, for $i = 1, \dots, N$, such that, setting $t_i = a + i\frac{b-a}{N}$, $i = 0, \dots, N$:

- $\gamma_s\big([t_{i-1}, t_i]\big) \subset U_i$;
- $\mathrm{germ}_{\gamma(t)}(K_i) = \mathfrak{g}_t$ for all $t \in [t_{i-1}, t_i]$,

for all $i = 1, \dots, N$. Then, given s' sufficiently close to s, $\gamma_{s'}\big([t_{i-1}, t_i]\big) \subset U_i$ for all $i = 1, \dots, N$. This implies that the local fields K_i can be used to define a transport $\mathfrak{g}^{(s')}$ of $\mathfrak{g}_a$ along $\gamma_{s'}$. In particular, $\mathfrak{g}^{(s')}_b = K_N\big(\widetilde{\gamma}(b)\big) = K_N\big(\gamma(b)\big) = \mathfrak{g}^{(s)}_b$. This concludes the proof. □

2.4. **A monodromy theorem.** We are now ready to prove the aimed extension of the classical monodromy theorem in Complex Analysis for sheaves of local vector fields satisfying the unique continuation property.

Proposition 2.4. *Let $\mathcal{F}$ a sheaf of local fields on a manifold M; assume that $\mathcal{F}$ satisfy the unique continuation property. Let $p \in M$ be fixed, let U be a simply connected open neighborhood of p, and let $\mathfrak{g}_p \in \mathcal{G}^{\mathcal{F}}_p$ be a germ of $\mathcal{F}$-field at p such that for all continuous curve $\gamma : [a, b] \to U$ with $\gamma(a) = p$,* *there exists a transport of* $\mathfrak{g}_p$ *along* γ. *Then, there exists (a necessarily unique)* $K \in \mathcal{F}(U)$ *such that* $\mathrm{germ}_p(K) = \mathfrak{g}_p$.

Proof. Since U is simply connected, any two continuous curves in U between two given points are fixed endpoints homotopic. This allows to define a vector field K on U as follows. Given $q \in U$, choose any continuous path $\gamma : [a, b] \to U$ with $\gamma(a) = p$ and $\gamma(b) = q$; let $\mathfrak{g}^\gamma$ be the unique transport of $\mathcal{F}$-germs along γ such that $\mathfrak{g}^\gamma_a = \mathfrak{g}_p$, which exists by assumption. By Proposition 2.3, $\mathfrak{g}^\gamma_b$ does not depend on γ. Then, we set $K_q := \mathrm{ev}_q(\mathfrak{g}^\gamma_b)$, where ev is the evaluation map defined in (3). It is easy to see that $K \in \mathcal{F}(U)$, using the definition of transport. $\square$

Remark 2.5. It is not hard to show that the conclusion of Proposition 2.4 still holds under the weaker assumption that a transport of $\mathfrak{g}_p$ exists along a family of curves starting at p which is dense, for the compact-open topology, in the set of continuous paths in U with fixed initial point.

3. Admissible and regular sheaves

We will now study a class of sheaves for which every germ can be transported along any curve. This class is defined in terms of two properties: rank boundedness and regularity.

3.1. **Sheaves with bounded rank.** We say that $\mathcal{F}$ has *bounded rank* if there exists a positive integer $N^{\mathcal{F}}$ such that:

$$\dim\big(\mathcal{F}(U)\big) \leqslant N^{\mathcal{F}},$$

for all connected open subset $U \subset M$. Let us call *admissible* a sheaf which has both unique continuation and rank boundedness properties. Examples of admissible sheaves will be discussed in Section 4. Observe that admissibility is inherited by sub-sheaves.

Lemma 3.1. *Let* $\mathcal{F}$ *be an admissible sheaf of vector fields on* M. *Then, given any* $p \in M$, *there exists a connected open neighborhood* U *of* p *such that the map* (2) *is an isomorphism. In particular:*

$$\dim(\mathfrak{G}^{\mathcal{F}}_p) = \dim\big(\mathcal{F}(U)\big) \leqslant N^{\mathcal{F}}.$$

Proof. Let us set:

$$\kappa^{\mathcal{F}}_p := \max\big\{\dim\big(\mathcal{F}(U)\big) : U \text{ connected open subset of } M \text{ containing } p\big\}. \tag{5}$$

If U is any connected neighborhood for which $\dim\big(\mathcal{F}(U)\big) = \kappa^{\mathcal{F}}_p$, then the map (2) is an isomorphism. This follows easily from the following monotonicity property (1). $\square$

We will assume henceforth that $\mathcal{F}$ is an admissible sheaf of local fields on the manifold M.

3.2. **$\mathcal{F}$-special neighborhoods.** A connected open neighborhood U of p as in Lemma 3.1 will be said to be $\mathcal{F}$-*special* for p. If U is $\mathcal{F}$-special for p, then given any $\mathfrak{g}_p \in \mathfrak{G}_p^{\mathcal{F}}$ there exists a unique $K \in \mathcal{F}(U)$ such that $\mathrm{germ}_p(K) = \mathfrak{g}_p$. Observe that the number $\kappa_p^{\mathcal{F}}$ defined in (5) is also given by:

$$\kappa_p^{\mathcal{F}} = \dim(\mathfrak{G}_p^{\mathcal{F}}).$$

Lemma 3.2. *Let $\mathcal{F}$ be an admissible sheaf of vector fields on M. The following statements hold:*

(i) If U is $\mathcal{F}$-special for p, and $U' \subset U$ is a connected open subset containing p, then also U' is $\mathcal{F}$-special for p. In particular, every point admits arbitrarily small *$\mathcal{F}$-special connected open neighborhoods.*

(ii) Assume that $p \in U$ and that $\kappa_q^{\mathcal{F}}$ is constant for $q \in U$. Then, there exists $U' \subset U$ containing p which is $\mathcal{F}$-special for all its points.

Proof. For proof of (i), note that if $p \in U' \subset U$, with U' connected, then:

$$\kappa_p^{\mathcal{F}} \geqslant \dim\big(\mathcal{F}(U')\big) \overset{\text{by (1)}}{\geqslant} \dim\big(\mathcal{F}(U)\big) = \kappa_p^{\mathcal{F}},$$

i.e., $\kappa_p^{\mathcal{F}} = \dim\big(\mathcal{F}(U')\big)$, and U' is $\mathcal{F}$-special for p.

As to the proof of (ii), let $U' \subset U$ be any $\mathcal{F}$-special open connected neighborhood of p. Then, for all $q \in U'$:

$$\kappa_q^{\mathcal{F}} = \kappa_p^{\mathcal{F}} = \dim\big(\mathcal{F}(U')\big),$$

i.e., U' is $\mathcal{F}$-special for q. $\square$

The interest in open sets which are $\mathcal{F}$-special for all their points is the following immediate consequence of the definition:

Proposition 3.3. *If U is $\mathcal{F}$-special for all its points, then given any $p \in U$ and any $\mathfrak{g}_* \in \mathcal{F}_p$, there exists a unique $K \in \mathcal{F}(U)$ such that $\mathrm{germ}_p(K) = \mathfrak{g}_*$. Moreover, if $V \subset U$ is any connected open subset, then for all $K \in \mathcal{F}(V)$ there exists a unique $\widetilde{K} \in \mathcal{F}(U)$ such that $\widetilde{K}|_V = K$.* $\square$

Definition 3.4. A connected open subset U of M on which $\kappa_p^{\mathcal{F}}$ is constant will be called $\mathcal{F}$-*regular*. An admissible sheaf $\mathcal{F}$ will be called regular if $\kappa_p^{\mathcal{F}}$ is constant for all $p \in M$.

If U is a connected open set which is $\mathcal{F}$-special for all its points, then U is $\mathcal{F}$-regular. We will now determine under which circumstances the converse of this statement holds.

3.3. **Existence of transport.** As to the existence of a transport of $\mathcal{F}$-germs with given initial conditions, one has to assume regularity for the sheaf $\mathcal{F}$.

Proposition 3.5. *Let $\gamma : [a, b] \to M$ be a continuous path, and assume that $\kappa^{\mathcal{F}}$ is constant along γ. Then, given any $\mathfrak{g}_* \in \mathfrak{G}_{\gamma(a)}^{\mathcal{F}}$ there exists a unique transport $[a, b] \ni t \mapsto \mathfrak{g}_t \in \mathfrak{G}_{\gamma(t)}^{\mathcal{F}}$ of $\mathcal{F}$-germs along γ such that*

$$\mathfrak{g}_a = \mathfrak{g}_*. \tag{6}$$

Proof. Define:

$$\mathcal{A} = \Big\{ t \in \,]a,b] : \exists \text{ transport of } \mathcal{F}\text{-germs } \mathfrak{g} \text{ along } \gamma|_{[a,t]} \text{ satisfying (6)} \Big\}.$$

Let us show that $\mathcal{A} = \,]a,b]$. Since $\kappa^{\mathcal{F}}$ is constant in a neighborhood of $\gamma(a)$, then by Lemma 3.1, there exists an open neighborhood U_a of $\gamma(a)$ which is $\mathcal{F}$-special for $\gamma(a)$. Then, there exists $K \in \mathcal{F}(U_a)$ such that $\mathrm{germ}_{\gamma(a)} K = \mathfrak{g}_*$. This says that $\mathcal{A}$ contains an interval of the form $]a, \varepsilon]$, for some $\varepsilon > 0$.

Arguing as in Lemma 2.2, we can show that $\mathcal{A}$ is both open and closed in $]a,b]$. Again, openness follows readily from the very definition of transport.

In order to show that $\mathcal{A}$ is closed, assume that $t_0 \in \overline{\mathcal{A}}$ (the closure of $\mathcal{A}$), and let U be an open neighborhood of $\gamma(t_0)$ which is $\mathcal{F}$-special for $\gamma(t_0)$. Choose $t_1 \in \mathcal{A}$ such that $\gamma(t_1) \in U$, then there exists $K \in \mathcal{F}(U)$ such that $\mathrm{germ}_{\gamma(t)}(K) = \mathfrak{g}_t$ for all t sufficiently close to t_1. This is because U is also $\mathcal{F}$-special for all these points $\gamma(t)$ due to the assumption that $\kappa^{\mathcal{F}}$ is constant along γ. Then, $\mathrm{germ}_{\gamma(t)}(K) = \mathfrak{g}_t$ for all t sufficiently close to t_0 (more generally for all t such that $\gamma(t) \in U$), hence $t_0 \in \mathcal{A}$ and $\mathcal{A}$ is closed. Thus, $\mathcal{A} = \,]a,b]$, and the existence of transport is proved.

Uniqueness follows directly from Lemma 2.2. □

As a corollary, we can now state the main result of this section.

Theorem 3.6. *Let M be a (connected and) simply connected differentiable manifold, and let $\mathcal{F}$ be an admissible and regular sheaf of local fields on M. Given any connected open subset $U \subset M$ and any $\widetilde{K} \in \mathcal{F}(U)$, there exists a unique $K \in \mathcal{F}(M)$ such that $K\big|_U = \widetilde{K}$. Similarly, given any $p \in M$ and any $\mathfrak{g}_p \in \mathfrak{G}^{\mathcal{F}}_p$, then there exists a unique $K \in \mathcal{F}(M)$ such that $\mathrm{germ}_p(K) = \mathfrak{g}_p$.*

Proof. The first statement follows easily from the second, which is proved as follows. Given any $p \in M$, any germ $\mathfrak{g}_p \in \mathfrak{G}^{\mathcal{F}}_p$, and and any continuous curve γ in M starting at p, by regularity (Proposition 3.5) there exists a transport of $\mathfrak{g}_p$ along γ. The existence of the desired field $K \in \mathcal{F}(M)$ follows from the monodromy theorem, Proposition 2.4. □

4. On rank boundedness and unique continuation property

4.1. The classical cases: Killing and conformal fields on a semi-Riemannian manifold.

4.2. **Infinitesimal symmetries of a G-structure.** Let G be a Lie subgroup of $GL(n)$, denote by $\mathfrak{g}$ its Lie algebra, and let M^n be a manifold with a G-structure $\mathcal{P}$. Recall that this is a G-principal sub-bundle of the frame bundle $FR(TM)$ of TM.

4.2.1. G-*fields.* It is well known that there exists a connection ∇ compatible with the G-structure,[2] i.e., such that the parallel transport of frames of $\mathcal{P}$ belong to $\mathcal{P}$. A (local) vector field K on M is said to be an *infinitesimal symmetry* of the G-structure $\mathcal{P}$, or a G-*field*, if its flow preserves the G-structure. One can characterize G-fields using a compatible connection; to this aim, we need to recall a few facts about the lifting of vector fields to the frame bundle.

4.2.2. *Lifting* G-*fields to the frame bundle.* Let us recall a notion of lifting of (local) vector fields defined on a manifold M to (local) vector field in the frame bundle $\mathrm{FR}(TM)$. First, diffeomorphisms of M (or between open subsets of M) can be naturally lifted to diffeomorphisms of $\mathrm{FR}(TM)$ (or between the corresponding open subsets of $\mathrm{FR}(TM)$); this notion of lifting preserves the composition of diffeomorphisms. Thus, given a vector field X in M, the flow of X can be lifted to a flow in $\mathrm{FR}(M)$; this is the flow of a vector field $\widetilde{X}$ in $\mathrm{FR}(TM)$. An explicit formula for $\widetilde{X}$ can be written using a connection ∇ on TM; for this general computation, we don't require that ∇ is compatible with $\mathcal{P}$. Observe that ∇ need not be symmetric either, and we will denote by T its torsion.

If F_t denotes the flow of X and $\widetilde{F}_t$ its lifting, for fixed $x \in M$ and $p \in \mathrm{FR}(TM)$ with $\pi(p) = x$, we have:

$$\widetilde{X}(p) = \tfrac{d}{dt}\big|_{t=0}\widetilde{F}_t(p) = \tfrac{d}{dt}\big|_{t=0}[dF_t(x) \circ p]$$

for all $p \in \mathrm{FR}(TM)$ and all $x \in M$. Observe that $\widetilde{F}_t(p) = dF_t(x) \circ p$ is a frame at the point $F_t(x)$.

The horizontal component of $\widetilde{X}(p)$ is:

$$\begin{aligned}[\widetilde{X}(p)]_{\mathrm{hor}} = d\pi_p\big(\widetilde{X}(p)\big) &= \tfrac{d}{dt}\big|_{t=0}\big[\pi(dF_t(x) \circ p)\big] \\ &= \tfrac{d}{dt}\big|_{t=0}[F_t(x)]| = X(x),\end{aligned} \tag{7}$$

i.e., $\widetilde{X}$ is indeed a lifting of X. The vertical component of $\widetilde{X}(p)$ is:

$$[\widetilde{X}(p)]_{\mathrm{vert}} = \tfrac{D}{dt}\big|_{t=0}[dF_t(x) \circ p] = \tfrac{D}{dt}\big|_{t=0}[dF_t(x)] \circ p; \tag{8}$$

for this equality, we have used the fact that $\frac{D}{dt}$ commutes with right-composition with p. We have

$$\tfrac{D}{dt}\big|_{t=0}[dF_t(x)v] = \tfrac{D}{dt}\big|_{t=0}\,\tfrac{d}{ds}\big|_{s=0}\big[F_t\big(x(s)\big)\big],$$

[2]Such compatible connection is not symmetric in general. For instance, if G is the complex general linear subgroup $\mathrm{GL}(n, \mathbb{C})$, so that the corresponding G-structure is an almost complex structure, then there exists a symmetric compatible connection if and only if the structure is in fact complex.

where $s \mapsto x(s)$ is a smooth curve such that $x(0) = x$ and $x'(0) = v$. Then:

$$\begin{aligned}\tfrac{D}{dt}\big|_{t=0}\tfrac{d}{ds}\big|_{s=0}&\big[F_t\big(x(s)\big)\big]\\ &= \tfrac{D}{ds}\big|_{s=0}\tfrac{d}{dt}\big|_{t=0}\big[F_t\big(x(s)\big)\big] + T\left(\tfrac{d}{dt}\big[F_t\big(x(s)\big)\big], \tfrac{d}{ds}\big[F_t\big(x(s)\big)\big]\right)\big|_{s=t=0}\\ &= \nabla_v X(x) + T\big(X(x), v\big).\end{aligned}$$

Therefore, the vertical component of $\widetilde{X}(p)$ is:

$$[\widetilde{X}(p)]_{\mathrm{vert}} = \nabla X(x) \circ p + T\big(X(x), p\ \cdot\ \big). \tag{9}$$

Using formulas (7) and (8), we can write:

$$\widetilde{X} = L(X \circ \pi), \tag{10}$$

where $\pi : \mathrm{FR}(TM) \to M$ is the canonical projection and L is a linear first order differential operator from sections of $\pi^*(TM)$ to sections of $T\big(\mathrm{FR}(TM)\big)$.

4.2.3. *Characterization of* G*-fields.* For all $x \in M$, let $\mathfrak{g}_x$ be the Lie subalgebra of $\mathrm{gl}(T_xM)$ that corresponds to the Lie group G_x of automorphisms of the G-structure of the tangent space T_xM. Clearly, $\mathfrak{g}_x$ is isomorphic to $\mathfrak{g}$ for all x. Using the lifting of vector fields to the frame bundle, it is not hard to prove the following:

Proposition 4.1. *Let ∇ be a connection on M compatible with the G-structure $\mathcal{P}$, and let T denote its torsion. A (local) vector field K is a G-field if and only if $\nabla K(x) + T(K, \cdot) \in \mathfrak{g}_x$ for*[3] *all x.*

Proof. K is a G-field if and only if the flow of its lifting $\widetilde{K}$ preserves the G-structure $\mathcal{P}$, that is, if and only if $\widetilde{K}$ is everywhere tangent to $\mathcal{P}$. Using the fact that ∇ is compatible with $\mathcal{P}$, one sees that the tangent space to $\mathcal{P}$ at some point $p \in \mathcal{P}$ is given by the space of vectors whose vertical component has the form $h \circ p$, with $h \in \mathfrak{g}_x$. Therefore, using formula (9) for the vertical component of liftings, we have that $\widetilde{K}$ is tangent to $\mathcal{P}$ if and only if:

$$\nabla K(x) + T\big(K(x), .\big) \in \mathfrak{g}_x,$$

for all $x \in M$. □

4.2.4. *Admissibility of the sheaf of* G*-fields for* G*-structures of finite type.*

Definition 4.2. Given an open subset $U \subset M$, let $\mathcal{F}^{\mathcal{P}}(U)$ be the subspace of $\mathfrak{X}(U)$ consisting of all vector fields K satisfying $\nabla K(x) + T\big(K(x), .\big) \in \mathfrak{g}_x$ for all $x \in U$. This is a sheaf of vector fields in M that will be called the sheaf of G-*vector fields* (shortly, G-*fields*).

It is not hard to prove that that the sheaf of G-fields consists of Lie algebras. We will give an indirect proof of this fact in Proposition 5.2.

[3] i.e., the map $T_xM \ni v \mapsto \nabla_v K + T(K_x, v) \in T_xM$ belongs to $\mathfrak{g}_x$.

Given $x \in M$, for all $j \geqslant 1$, define the j-*th prolongation* $\mathfrak{g}_x^{(j)}$ of $\mathfrak{g}_x$ as the Lie algebra:

$$\mathfrak{g}_x^{(j)} = \big\{ L : T_xM^{(j+1)} \longrightarrow T_xM \text{ multilinear and symmetric} : \\ \forall\,(v_1,\dots,v_j) \in T_xM^{(j)}, \\ \text{the map } T_xM \ni v \mapsto L(v, v_1,\dots,v_j) \in T_xM \text{ belongs to } \mathfrak{g}_x \big\}. \quad (11)$$

In particular, $\mathfrak{g}_x^{(0)} = \mathfrak{g}_x$. The G-structure is said to be of *finite type* if $\mathfrak{g}_x^{(j)} = \{0\}$ for some $j \geqslant 1$; this condition is independent of x. The *order* of a finite type G structure is the minimum j for which $\mathfrak{g}_x^{(j)} = \{0\}$. For instance, when G is some orthogonal group $O(n,k)$ (in which case a G-structure is a semi-Riemannian structure in M), then the G-structure is of finite type and it has order 1. If the group G is $CO(n,k)$ (in which case a G-structure is a semi-Riemannian conformal structure in M), is of finite type and it has order 2.

Proposition 4.3. *Let $\mathcal{P}$ be a finite type G-structure on M^n, with order $N \geqslant 1$. Then, the sheaf $\mathcal{F}^{\mathcal{P}}$ has rank bounded by $n + \sum_{j=0}^{N-1} \dim\big(\mathfrak{g}_x^{(j)}\big)$.*

Proof. It follows immediately from the corresponding statement on the dimension of the automorphism group of a G-structure of finite type. This is a very classical result, first stated in [7], see also [26, Corollary 4.2, p. 348], [24], or [15, Theorem 1, p. 333]. □

Proposition 4.3 holds also in the case of G-structures on orbifolds, see [3].

Proposition 4.4. *Let $\mathcal{P}$ be a $G \subset GL(n)$ structure of order N on a differentiable manifold M^n, and let X be a G-field. If the N-th order jet of X vanishes at some point $x \in M$, then X vanishes in a neighborhood of x. If M is connected, then X vanishes identically.*

Proof. The proof is by induction on N.

When $N = 0$, then $G = \{1\}$ and $\mathcal{P}$ is a global frame $(X_1,\dots,X_n)$ of M. In this case, the property of being a G-field for X means that X commutes with each one of the X_i's, i.e., X is invariant by the flow of each X_i. In this case, the set of zeroes of X is invariant by the flow of the X_i's. Now, if $F_t^{X_i}$ denotes the flow of X_i at time t, then, for fixed $x \in M$, the map:

$$(t_1,\dots,t_n) \longmapsto F_{t_1}^{X_1} \circ \ldots \circ F_{t_n}^{X_n}(x)$$

is a diffeomorphism of a neighborhood of 0 in $\mathbb{R}^n$ onto a neighborhood of x in M. In order to prove this, it suffices to observe that its derivative at 0 sends the canonical basis of $\mathbb{R}^n$ to $X_1(x),\dots,X_n(x)$. Thus, if $X(x) = 0$ and X is invariant by the flows F^{X_i}, then X vanishes in a neighborhood of x.

For the induction step, we need the notion of lifting of vector fields in M to vector fields in the frame bundle $FR(M)$ that was recalled in Subsection 4.2.2. Consider the lift $\widetilde{X}$ of X to a vector field in $FR(M)$. From formula

(10) it follows readily that if the N-the jet of X vanishes at x, then the jet of order $(N-1)$ of $\widetilde{X}$ vanishes at every $p \in \mathrm{FR}(T_xM)$.

Now, observe that is X is a G-field, then $\widetilde{X}$ is tangent to $\mathcal{P}$. Furthermore, if some diffeomorphism of M preserves $\mathcal{P}$, then the lifting of such diffeomorphism to $\mathrm{FR}(TM)$ restricted to $\mathcal{P}$ preserves the $G^{(1)}$-structure $\mathcal{P}^{(1)}$ of $\mathcal{P}$, where $\mathcal{P}^{(1)}$ is the first prolongation of $\mathcal{P}$. It follows that the restriction of $\widetilde{X}$ to $\mathcal{P}$ is a $G^{(1)}$-field for the $G^{(1)}$-structure $\mathcal{P}^{(1)}$.

For the induction step, assume that $\mathcal{P}$ has order k, and that X is a G-field for $\mathcal{P}$ with vanishing N-th jet at x. The field $\widetilde{X}$ restricted to $\mathcal{P}$ is a $G^{(1)}$-field for $\mathcal{P}^{(1)}$ which has vanishing jet of order $(N-1)$ at p, for arbitrary $p \in \mathrm{FR}(T_xM)$. Since $\mathcal{P}^{(1)}$ has order $N-1$, the induction hypotheses gives us that $\widetilde{X}$ vanishes identically in a neighborhood of p in $\mathcal{P}$. It follows that X vanishes in a neighborhood of x in M (the projection onto M of the neighborhood of p). This concludes the proof of the first statement.

In particular, the result proves that the set of points at which the N-th order jet of X vanishes is open. Evidently, it is also closed, and therefore if M is connected, then X vanishes identically. □

Corollary 4.5. *The sheaf of local G-fields of a finite order G-structure has the unique continuation property.* □

Remark 4.6 (Local uniqueness vs. analyticity)**.** It is an interesting question to establish for which G-structures of infinite type, the corresponding sheaf of local G-fields has the unique continuation property. A natural guess would be to consider real-analytic structures, this seems to have been claimed in [1, Proposition 3.2, p. 5]. However, there exist real-analytic G-structures of infinite type whose sheaf of local G-fields does not have the unique continuation property. Consider for instance a real-analytic symplectic manifold (M^{2n}, ω). In this case, G is the symplectic group $\mathrm{Sp}(2n)$. Given any smooth local function $H : U \subset M \to \mathbb{R}$, then its Hamiltonian field $\vec{H}$ has flow that preserves ω. In particular, two distinct smooth function that coincide on some non-empty open subset define distinct Hamiltonian fields, that coincide on that open set.

Observe that, by Corollary 4.5, for G-structure of finite type, local uniqueness is independent of analyticity, thus, the two properties seem to be logically independent.

5. On the regularity condition

The central assumption of Theorem 3.6, i.e., that the space of germs of the admissible sheaf $\mathcal{F}$ should have constant dimension along M, is in general hard to establish. We will present here two situations where such regularity assumption is satisfied, namely the *real-analytic* and the *transitive* case.

5.1. **The real-analytic case.**

Here we should prove that real-analyticity implies κ constant for bounded rank sheaves.

5.2. **The transitive case.** Given a sheaf of vector fields $\mathcal{F}$ on the manifold M, let $\mathcal{D}(\mathcal{F})$ denote the associated pseudo-group of local diffeomorphisms of M. Recall that this is the pseudo-group generated by the family of all local diffeomorphisms given by the local flow of elements of $\mathcal{F}$.

Definition 5.1. The sheaf $\mathcal{F}$ is said to be *full* if for every $X \in \mathcal{F}$ and any $\varphi \in \mathcal{D}(\mathcal{F})$, the pull-back $\varphi^*(X)$ belongs to $\mathcal{F}$. Given a complete sheaf $\mathcal{F}$, the pseudo-group $\mathcal{D}(\mathcal{F})$ is said to be *transitive* if it acts transitively on M, i.e., if given any two points $x, y \in M$, there exists $\varphi \in \mathcal{D}(\mathcal{F})$ such that $\varphi(x) = y$.

As an immediate consequence of the definition, we have the following:

Proposition 5.2. *Let $\mathcal{F}$ be a full sheaf of local vector fields on M which has bounded rank. If $\mathcal{D}(\mathcal{F})$ is transitive, then $\mathcal{F}$ is regular.*

Proof. Given $x, y \in M$, and $\varphi \in \mathcal{D}(\mathcal{F})$ such that $\varphi(x) = y$, then the pull-back φ^* induces an isomorphism between the space of germs $\mathfrak{G}_y^{\mathcal{F}}$ and $\mathfrak{G}_x^{\mathcal{F}}$. □

An interesting consequence of fullness is the following result:

Proposition 5.3. *Let $\mathcal{F}$ be a full sheaf of smooth local vector fields on a manifold M. Assume that $\mathcal{F}$ is closed by C^∞-convergence on compact sets. Then, $\mathcal{F}$ is closed by Lie brackets, i.e., $\mathcal{F}(U)$ is a Lie algebra for every open set $U \subset M$. This is the case, in particular, for the sheaf of local fields whose flow preserve a G-structure on M.*

Proof. See [25, Remark (b), p. 10]. □

The sheaf $\mathcal{F}$ is called *transitive* if the evaluation map $\mathrm{ev}_x : \mathfrak{G}_x^{\mathcal{F}} \to T_xM$ is surjective for all x. In other words, $\mathcal{F}$ is transitive if for all $x \in M$ and all $v \in T_xM$, there exists a local $\mathcal{F}$-field K defined around x such that $K_x = v$. When $\mathcal{F}$ is a Lie algebra sheaf, the transitivity of a $\mathcal{F}$ implies the transitivity of $\mathcal{D}(\mathcal{F})$.

Proposition 5.4. *If $\mathcal{F}$ is a full Lie algebras sheaf of local fields on M which is transitive, then $\mathcal{D}(\mathcal{F})$ is transitive.*

Proof. See [25, Proposition 1.1 and § 1.3]. □

Proposition 5.4 applies in particular to the sheaf of local vector fields that are infinitesimal symmetries of a G-structure, see Section 4.2.

5.3. **More examples that do not fit into the case of G-structures of finite type.**

- Infinitesimal symmetries of affine manifolds. This can be done using the fact that affine connections correspond to 1-structures in the frame bundle $\mathrm{Fr}(TM)$, see [24, Section 8.3].
- Infinitesimal symmetries of quasi-complex manifolds. These are examples of elliptic G-structures.
- Infinitesimal projective symmetries. See below.

An affine connection defines a global parallelism on the frame bundle. Let M be an n-dimensional manifold endowed with an affine connection ∇. Let $\mathrm{Fr}(TM)$ denote the frame[4] bundle of M, which is the total space of a $\mathrm{GL}(n)$-principal bundle on M, and it has dimension equal to n^2+n. Denote by $\pi : \mathrm{Fr}(TM) \to M$ the canonical projection. Let us recall how to define a global parallelism on $\mathrm{Fr}(TM)$, i.e., a $\{1\}$-structure, using the connection ∇.

Given $p \in \mathrm{Fr}(TM)$, i.e., $p : \mathbb{R}^n \to T_xM$ is an isomorphism for some $x \in M$, then $T_p\big(\mathrm{Fr}(TM)\big)$ admits a splitting into a vertical and a horizontal space, denoted respectively Ver_p and Hor_p. The vertical space Ver_p is canonical, and canonically identified with $\mathrm{Lin}(\mathbb{R}^n, T_xM)$. The horizontal space Hor_p is defined by the connection ∇, and it is isomorphic to T_xM via the differential $\mathrm{d}\pi(p) : \mathrm{Hor}_p \to T_xM$.

Fix a frame $q : \mathbb{R}^{n^2} \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n)$. A frame for Ver_p is given by $p^{-1} \circ q : \mathbb{R}^{n^2} \to \mathrm{Ver}_p$. A frame for Hor_p is given by

$$\left(\mathrm{d}\pi_p\big|_{\mathrm{Hor}_p}\right)^{-1} \circ p : \mathbb{R}^n \longrightarrow \mathrm{Hor}_p.$$

Putting these two things together, we obtain a global frame of $\mathrm{Fr}(TM)$, which is determined by the choice of the horizontal distribution.

Clearly, not every global frame of $\mathrm{Fr}(TM)$ arises from a connection on M. On the other hand, if two connections define the same global frame, then they coincide. Moreover, a smooth diffeomorphism $f : M \to M$ is affine, i.e., it preserves ∇, if and only if the induced[5] map $\widetilde{f} : \mathrm{Fr}(TM) \to \mathrm{Fr}(TM)$ preserves the associated global frame.

5.4. **Local projective vector fields.** The projective transformations appear as a natural generalization of the affine ones. While in the affine transformations the connection of a pseudo-Riemannian manifold is preserved, the projective transformation focus its attention on the geodesics of such a space. Formally, an automorphism $f : M \to M$ from a pseudo-Riemannian manifold (M, g) is a projective transformation if maps bijectively pre-geodesics onto pre-geodesics. As happen with the affine transformations, the projective ones determine a Lie group of bounded dimension (see [16, Chapter IV, Theorem 6.1]). A vector field K is a (local) projective vector field (or an infinitesimal projective transformation) if its flow defines locally a projective transformation. These vector fields come characterized by the following equation, which is a clear generalization of (**??**)

$$(\mathcal{L}_K\nabla)(X, Y) = \Omega_K(X)Y - \Omega_K(Y)X \tag{12}$$

for all $X, Y \in \mathfrak{X}(M)$ and some one-form Ω_K determined by K. Then, the sheaf of projective vector fields has bounded rank, and we can obtain the following extension result (compare with Theorem **??**).

---

[4]By a *frame* of an N-dimensional vector space V, we mean an isomorphism $q : \mathbb{R}^N \to V$.

[5]The induced map $\widetilde{f}$ is defined by $\widetilde{f}(p) = \mathrm{d}f(x) \circ p$, where $p : \mathbb{R}^n \to T_xM$.

Theorem 5.5. *If (M, g) is a (connected and) simply connected analytic pseudo-Riemannian manifold, then any local projective vector field extends globally.*

6. Killing and conformal fields for Finsler manifolds

Let (M, F) be a Finsler manifold with $F : TM \to [0, +\infty)$, namely, a continuous positive homogeneous function which is smooth away from the zero section and such that the fundamental tensor defined as

$$g_v(u, w) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}F(v + tu + sw)^2|_{t=s=0}$$

for every $v \in TM$ and $u, w \in T_{\pi(v)}M$, is positive definite. Let us recall that the Chern connection associated to a Finsler metric can be seen as a family of affine connections, namely, for every non-null vector field V on an arbitrary open subset $U \subset M$, we can define an affine connection ∇^V in U, which is torsion-free and almost g-compatible (see for example [12]). The Chern connection also determines a covariant derivative along any curve $\gamma : [a, b] \to M$, which depends on a non-null reference vector W and which we will denote as D^W_γ. Let us define a local Killing (resp. conformal) vector field of (M, F) as a vector field K such that the flow of K gives local isometries (resp. conformal maps) for every $t \in \mathbb{R}$ where it is defined. Let us begin by giving a characterization of (local) Killing and conformal vector fields.

Proposition 6.1. *Let (M, F) be a Finsler manifold and K a vector field in an open subset $U \subset M$. Then*

(i) K *is Killing if and only if* $g_v(v, \nabla^v_v K) = 0$ *for every* $v \in \pi^{-1}(U)$.

(ii) K *is conformal if and only if there exists a function* $f : U \to (0, +\infty)$ *such that* $g_v(v, \nabla^v_v K) = f(\pi(v))F(v)^2$ *for every* $v \in \pi^{-1}(U)$,

where $\pi : TM \to M$ *is the canonical projection.*

Proof. Given $v \in \pi^{-1}(U)$, let $\gamma : [-1, 1] \to U$ be a curve such that $\dot\gamma(0) = v$. Now let ϕ_s be the flow of K in the instant $s \in \mathbb{R}$ for s small enough in such a way that the flow is defined for that instant along γ. Define the two-parameter $\Lambda(t, s) = \phi_s(\gamma(t))$, denote $= \frac{d}{dt}\Lambda(t, s)$ and $\beta_t(s) = \Lambda(t, s)$. By definition $\dot\beta_t(s) = K$. We have that

$$\frac{d}{ds}F(\dot\gamma_s(t))^2 = 2g_{\dot\gamma_s}(D^{\dot\gamma_s}_{\beta_t}\dot\gamma_s, \dot\gamma_s) = 2g_{\dot\gamma_s}(D^{\dot\gamma_s}_{\gamma_s}K, \dot\gamma_s). \tag{13}$$

Here we have used [12, Proposition 3.2]. Observe that K is Killing if and only if $F(\dot\gamma(t)) = F(\dot\gamma_s(t))$ for every curve γ. By the above equation this is equivalent to $g_v(\nabla^v_v K, v) = 0$ for every $v \in \pi^{-1}(U)$. When K is conformal, we have that $F(\dot\gamma_s(t))^2 = h(s, \gamma(t))F(\dot\gamma(t))^2$ and then

$$2g_v(\nabla^v_v K, v) = f(\pi(v))F(v)^2$$

where $f(p) = \frac{\partial h}{\partial s}(0, p)$. Moreover, if K satisfies an equation of this type for some function f, then if we define $h(s, p) = \int_0^s f(\phi_\mu(p))d\mu$, we deduce that the flow ϕ_s is conformal with $F(\phi_s^*(v))^2 = h(s,\pi(v))F(v)^2$. This concludes the proof of part (ii). $\square$

In order to see that local Killing and conformal vector fields of Finsler manifolds constitute and admissible sheaf, we need to introduce an average Riemannian metric associated to a Finsler metric. There are several notions of Riemannian metric (see [8, 19]). We will consider the following one. Let us define $B_p = \{v \in T_pM : F(v) \leqslant 1\}$ and let Ω be the unique one-form (or density) such that B_p has volume equal to 1 for every $p \in M$. Then we define the volume form on S^n at $u \in S^{n-1}$ as

$$\omega(\eta_1, \eta_2, \dots, \eta_{n-1}) = \Omega(\eta_1, \eta_2, \dots, \eta_{n-1}, u)$$

for every $\eta_1, \eta_2, \dots, \eta_{n-1} \in T_uS^{n-1}$. Then the average Riemannian metric is defined as

$$g_R(v,w) := \int_{u\in S^{n-1}} g_u(v,w)\omega. \tag{14}$$

The first observation about the average Riemannian metric is that if $\phi : U \subset M \to V \subset M$ is a (local) isometry (resp. conformal map) of (M,F), then it is also a (local) isometry (resp. conformal map) of (M, g_R), and in particular, if K is a local Killing (resp. conformal) vector field of (M,F), then it is also a local Killing (resp. conformal) vector field of (M, g_R). This implies that the sheaf of local Killing (resp. conformal) vector fields of a Finsler manifold is admissible, so that Theorem 3.6 aplies whenever M is simply-connected and the sheaf of local Killing (resp. conformal) vector fields is regular. This happens for example when the manifold is analytic.

Proposition 6.2. *Let (M,F) be a simply-connected analytic Finslerian manifold. Then every local Killing (resp. conformal) vector field can be extended to a unique Killing (resp. conformal) vector field defined on the whole manifold M.*

Proof. As we have seen above that the sheaf of local Killing (resp. conformal) vector fields of a Finsler manifold is admissible, we only have to show that it is regular and to apply Theorem 3.6. Let $p \in M$ be an arbitrary point and choose an analytic system of coordinates (U, φ) such that $p \in U$ and U is simply-connected. Let $\tilde{X}$ be a local Killing (resp. conformal) vector field of (M,F) in some open subset $\tilde{U} \subset U$. In order to show that the sheaf is regular in p, we only need to show that $\tilde{X}$ extends to a local Killing (resp. conformal) vector field of (M,F) in U. As $\tilde{X}$ is also a local Killing (resp. conformal) vector field of the average Riemannian metric given in (14), then $\tilde{X}$ can be extended to a local Killing (resp. conformal) vector field X of g_R defined on U. Let us show that X is also a local Killing (resp. conformal) vector field of (M,F). Given any vector $v \in \pi^{-1}(U)$, let V be the vector field obtained as the inverse image by φ^* of the constant vector field $\varphi(v)$ in $\varphi(U)$. Now observe that in the case that $\tilde{X}$ is conformal, the function $f : \tilde{U} \to (0, +\infty)$ in part (ii) of Proposition 6.1 is given by

$f(p) = g_V(\nabla^V_V X, V)/F(V)^2$, and thus, f extends analitically to U. Then $g_V(\nabla^V_V X, V) = 0$ (resp. $g_V(\nabla^V_V X, V) = f(\pi(V))F(V)^2$), because it is an analytic equation in U that is zero in $\tilde{U}$, so it has to be zero in all U. As this can be done for any $v \in \pi^{-1}(U)$, we conclude by Proposition 6.1 that X is local Killing (resp. conformal) for (M, F), as required. □

6.1. **Applications.** Thanks to previous results, we are able to give a nice characterization of the homogeneity of Finsler manifolds. For this, we first need the following technical lemma, which is well known on Semi-Riemannian Geometry

Lemma 6.3. *On a complete Finslerian Manifold M every global Killing vector field K is complete.*

The proof follows by using the same arguments as in the Semi-Riemannian case (see [21, Proposition 9.30] for instance), and taking into account that the Jacobi fields are well-enough behaved on the Finslerian settings (see [13, Section 3.4], specially Proposition 3.13 and Lemma 3.14). With this previous result at hand, we are in conditions to prove the following characterization:

Theorem 6.4. *Let M be a (connected and) simply-connected, complete Finslerian manifold. Then, M is homogeneous if, and only if, the following two conditions hold:*

(i) $\kappa^{\mathcal{F}}_p$ *is constant on* M *and;*
(ii) *for some point* $p_0 \in M$ *and all* $v \in T_{p_0}M$ *there exists a local Killing vector field* V *such that* $V(p_0) = v$.

Proof. Let us begin with the right implication. As the Finslerian manifold is homogeneous, the pseudo-group $\mathcal{D}(\mathcal{F})$ associated to the full sheaf $\mathcal{F}$ of local Killing vector fields is transitive, and so, Proposition 5.2 ensures (i). For (ii) recall that, for each $v \in T_{p_0}M$ we can consider $\epsilon > 0$ and a 1-parameter family of isometries $\{\psi_t\}_{t\in(-\epsilon,\epsilon)} : M \to M$ with $\psi_0(p_0) = p_0$ and $\frac{\partial}{\partial t}|_{t=0}(\psi_t(p_0)) = v$. Then, $V(p) = \frac{\partial}{\partial t}|_{t=0}(\psi_t(p))$ is a Killing vector field with $V(p_0) = v$, and so, (ii) is satisfied.

For the left implication, let us assume that (i) and (ii) hold. As $\kappa^{\mathcal{F}}_p$ is constant, the sheaf of local Killing vector fields $\mathcal{F}$, which is an admissible sheaf, is also a regular one. Then, as M is simply-connected, Theorem 3.6 ensures that any local Killing vector field extends globally. Moreover, from previous Lemma, such a global Killing vector fields are complete, and so, the pseudo-group $\mathcal{D}(\mathcal{F})$ is truly a group.

Now, let us define the following subset of M

$$M' = \{p \in M : p = \psi(p_0) \text{ for some } \psi \in \mathcal{D}(\mathcal{F})\},$$

i.e., the orbit of the point p_0 under the action of the group $\mathcal{D}(\mathcal{F})$. As the natural map $\varphi_{p_0} : \mathcal{D}(\mathcal{F}) \to M$ which takes an element $\psi \in \mathcal{D}(\mathcal{F})$ to $\psi(p_0)$ is regular, the subset M' is a submanifold of M. Moreover, from construction, it is also a complete submanifold, as it is homogeneous respect the induced Finsler metric (observe that [21, Remark 9.37] is also true for the Finsler case). Finally, let us prove that M' contains an open set of M. In fact, for all $v \in T_{p_0}M$, let us denote by V the global Killing vector field such that $V(p_0) = v$ (recall (ii)). Then, if we denote by $\psi^V : \mathbb{R} \to M$ the flow of V with $\psi^V(0) = p_0$, the set

$$U := \{p \in M : p = \psi^V(1) \text{ for some } v \in T_{p_0}M\ \}$$

is open for M and it is contained in M', as each ψ^V belongs to $\mathcal{D}(\mathcal{F})$. In conclusion, M' is a complete and homogeneous submanifold of M with dimension n. Therefore, $M' = M$, and so, M is an homogeneous Finslerian manifold.

□

7. Pseudo-Finsler local Killing fields

Let us now consider infinitesimal symmetries of another type of structure that cannot be described in terms of a finite order G-structure: pseudo-Finsler structure. These structures are the indefinite counterpart to Finsler structures, in the same way Lorentzian or pseudo-Riemannian metrics are the indefinite counterpart to Riemannian metrics. A precise definition will be given below. Let us observe here that, unlike the Finsler (i.e., positive definite) case, the construction of average metrics is not possible in this situation, and we have to resort to a different type of construction, based on the notion of Sasaki metrics.

7.1. **Conic pseudo-Finsler structures.**

Let us recall that a *pseudo-Finsler* structure (or a *conic pseudo-Finsler* structure) on a (connected) manifold M consists of an open subset $\mathcal{T} \subset T_0M$, where T_0M denotes the tangent bundle with its zero section removed, and a smooth function $L : \mathcal{T} \to \mathbb{R}$ satisfying the following properties:

(i) for all $p \in M$, the intersection $\mathcal{T}_p = \mathcal{T} \cap T_pM$ is a non empty open cone of the tangent space T_pM;
(ii) $L(tv) = t^2L(v)$ for all $v \in \mathcal{T}$ and all $t > 0$;
(iii) for all $v \in \mathcal{T}$, the Hessian $g_v(u, w) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}L(v + tu + sw)|_{t=s=0}$ nondegenerate.

By continuity, the *fundamental tensor* g_v has constant index, which is called the index of the pseudo-Finsler structure. The case when $\mathcal{T} = T_0M$ and the index of g_v is zero, i.e., g_v is positive definite for all v, is the standard Finsler structure. When g_v does not depend on v, then we have a standard pseudo-Riemannian manifold. As in the classical Finsler metrics, we can define the associated Chern connection as a family of affine connection (see for example [12]).

7.2. **Pseudo-Finsler isometries.** An isometry of the pseudo-Finsler structure $(M, \mathcal{T}, L)$ is a diffeomorphism f of M, with $df(\mathcal{T}) = \mathcal{T}$ and $L \circ df = L$. The notion of local isometry is defined similarly. Clearly, the set $\mathrm{Iso}(M, \mathcal{T}, L)$ of such pseudo-Finsler isometries is a group with respect to composition, and one has a natural action of $\mathrm{Iso}(M, \mathcal{T}, L)$ on M. It is proved in [9] that this action makes $\mathrm{Iso}(M, \mathcal{T}, L)$ into a Lie transformation group of M. In Appendix A we will show that $\dim\left(\mathrm{Iso}(M, \mathcal{T}, L)\right) \leqslant \frac{1}{2}n(n+1)$, where $n = \dim(M)$. By a similar argument, we will show here that the same inequality holds for the dimension of the space of germs of infinitesimal symmetries of a pseudo-Finsler manifold, see Proposition 7.1.

Given a pseudo-Finsler structure $(M, \mathcal{T}, L)$ of index k, there is an associated pseudo-Riemannian metric of index $2k$ on the manifold $\mathcal{T}$, called the *Sasaki metric* of $(M, \mathcal{T}, L)$, that will be denoted by g^L, and which is defined as follows. Let $\pi : TM \to M$ be the canonical projection. The geodesic spray of L defines a *horizontal distribution* on $\mathcal{T}$, i.e., a rank n distribution on $\mathcal{T}$ which is everywhere transversal to the canonical vertical distribution. Equivalently, the horizontal distribution associated to L can be defined using the Chern connection of the pseudo-Finsler structure. In particular, a vector $X \in TTM$ is horizontal iff $D^{\alpha}_{\pi(\alpha)}\alpha(0) = 0$, where D is the covariant derivative induced by the Chern connection, and $\alpha : (-\varepsilon, \varepsilon) \to TM$ is a curve (transversal to the fibers) such that $\dot\alpha(0) = X$. For $p \in M$ and $v \in \mathcal{T}_p$, let us denote by $\mathrm{Ver}_v = \mathrm{Ker}(d\pi_v)$ and Hor^F_v the corresponding subspaces of $T_v\mathcal{T}$. One has a canonical identification $i_v : T_pM \to \mathrm{Hor}^L_v$ (given by the differential at v of the inclusion $T_pM \hookrightarrow TM$); moreover, the restriction of $d\pi_v : \mathrm{Hor}^L_v \to T_pM$ is an isomorphism. The Sasaki metric g^L is defined by the following properties:

- on Ver_v, g^L is the push-forward of the fundamental tensor g_v by the isomorphism $i_v : T_pM \to \mathrm{Ver}_v$;
- on Hor^L_v, g^L is the pull-back of the fundamental tensor g_v by the isomorphism $d\pi_v : \mathrm{Hor}^L_v \to T_pM$;
- Ver_v and Hor^L_v are g^L-orthogonal.

A (local) vector field X on M whose flow consists of (local) isometries for the pseudo-Finsler structure $(M, \mathcal{T}, L)$ will be called a (local) *Killing field of* $(M, \mathcal{T}, L)$.

7.3. **Lifting of isometries and Killing fields.** If f is a (local) isometry of $(M, \mathcal{T}, L)$, then df is a local isometry of the pseudo-Riemannian manifold $\mathcal{T}$ endowed with the Sasaki metric g^L. This is proved in [9], see Appendix A.

The lifting $f \mapsto df$ of pseudo-Finsler isometries preserves the composition of diffeomorphisms. Thus, given a (local) vector field X in M, the flow of X can be lifted to a flow in TM; this is the flow of a (local) vector field $\widetilde{X}$

in TM. If X is a Killing field of $(M, \mathcal{T}, L)$, then $\widetilde{X}$ is tangent to $\mathcal{T}$,[6] and we obtain (by restriction) a vector field on $\mathcal{T}$. Clearly, the flow of $\widetilde{X}$ consists of g^L-isometries, so that $\widetilde{X}$ is a (local) Killing vector field of the pseudo-Riemannian manifold $(\mathcal{T}, g^L)$. An explicit formula for $\widetilde{X}$ can be written using local coordinates and an auxiliary symmetric connection on M. If F_t denotes the flow of X and $\widetilde{F}_t = dF_t$ its lifting, we have:

$$\widetilde{X}(v) = \tfrac{d}{dt}\big|_{t=0}\widetilde{F}_t(v) = \tfrac{d}{dt}\big|_{t=0} dF_t(v)$$

for all $v \in \mathcal{T}$ and all $x \in M$. The curve $t \mapsto dF_t(v)$ projects onto the curve $t \mapsto F_t(x)$ in M, so that the horizontal component of $\widetilde{X}(v)$ is:

$$\big[\widetilde{X}(v)\big]_{\mathrm{hor}} = X(x). \tag{15}$$

Using the auxiliary connection, let us compute the vertical component of $\widetilde{X}(v)$ by:

$$\big[\widetilde{X}(v)\big]_{\mathrm{ver}} = \tfrac{D}{dt}\big|_{t=0}\big[dF_t(v)\big] = \nabla_v X. \tag{16}$$

7.4. **The sheaf of local pseudo-Finsler Killing fields.** Let $\mathcal{F}^{\mathrm{pF}}$ denote the sheaf of local Killing fields of $(M, \mathcal{T}, L)$. Using the relations between the pseudo-Finsler structure and the associated Sasaki metric, we can prove the following:

Proposition 7.1. *Let $(M, \mathcal{T}, L)$ be a pseudo-Finsler manifold, with $n = \dim(M)$. Then:*

(a) *$\mathcal{F}^{pF}$ has bounded rank: for all open connected subset $U \subset M$*

$$\dim\big(\mathcal{F}^{pF}(U)\big) \leqslant \tfrac{1}{2}n(n+1);$$

(b) *$\mathcal{F}^{pF}$ has the unique continuation property.*

Proof. Denote by $\mathcal{F}^{\mathrm{pR}(g^L)}$ the sheaf of local Killing fields of the pseudo-Riemannian manifold $(\mathcal{T}, g^L)$, where g^L is the Sasaki metric of $(M, \mathcal{T}, L)$. Given a connected open subset $U \subset M$, denote by $\widetilde{U} = \pi^{-1}(U) \cap \mathcal{T}$, where $\pi : TM \to M$ is the canonical projection. Then, $\widetilde{U}$ is a connected open subset of $\mathcal{T}$. Given $X \in \mathcal{F}^{\mathrm{pF}}(U)$, let $\widetilde{X} \in \mathfrak{X}(\widetilde{U})$ be the lifting of X. As we have seen in Subsection 7.3, $\widetilde{X} \in \mathcal{F}^{\mathrm{pR}(g^L)}(\widetilde{U})$, and the maps $X \mapsto \widetilde{X}$ gives an injective linear map from $\mathcal{F}^{\mathrm{pF}}(U)$ to $\widetilde{X} \in \mathcal{F}^{\mathrm{pR}(g^F)}(\widetilde{U})$. Injectivity follows easily from the fact that, for $v \in \widetilde{U}$ and $x = \pi(v)$, then the ~~vertical~~ horizontal component of $\widetilde{X}(v)$ is $X(x)$. Then, rank boundedness and unique continuation property for the sheaf $\mathcal{F}^{\mathrm{pF}}$ follow immediately from the corresponding properties of $\mathcal{F}^{\mathrm{pR}(g^L)}$. $\square$

---

[6] M. A: esto no lo he entendido bien. Creo que el punto crucial aqui es que el flujo de $\tilde{X}$ preserva τ. Que el campo $\tilde{X}$ sea tangente a τ no quiere decir nada, pues τ es un abierto de TM, no?

Corollary 7.2. *Let* $(M, \mathcal{T}, L)$ *be a simply connected pseudo-Finsler manifold. If* $(M, \mathcal{T}, L)$ *is either locally homogeneous, or real-analytic, then every local Killing field for* $(M, \mathcal{T}, L)$ *admits a global extension.*

Proof. By Proposition 7.1, the sheaf of local Killing fields of (M, τ, L) is admissible. The conclusion will follow from Theorem 3.6, once we show that the sheaf is regular. Consider any point $p \in M$ and an open (simply-connected) subset U which contains p and it admits a coordinate system. Now let $q \in U$ and K a local Killing field of (M, τ, L) defined in $U' \subset U$, with $q \in U'$ and $\tilde{K}$, the lift to TM which is a local Killing field of (TM, g^L). Observe that as the sheaf of local Killing fields of a pseudo-Riemannian manifold is admissible and regular (when all the data is analytic), then $\tilde{K}$ can be extended to a Killing field $\hat{K}$ in $\tilde{U} = \pi^{-1}(U)$. Let us see that $\hat{K}$ is a projectable vector field. Observe that a vector field X in TM is projectable if and only if $\mathcal{H}([X, V]) = 0$ for every vertical vector field V, where $\mathcal{H}$ denotes the projection to the horizontal subspace. Moreover in $\pi^{-1}(U)$ we can consider the vertical partial vectors ∂_{y^i} of the natural coordinates (x, y). Then we know that $\mathcal{H}([\hat{K}, \partial_{y^i}]) = 0$ in $\pi^{-1}(U')$ for $i = 1, \ldots, n$. As $\mathcal{H}([\hat{K}, \partial_{y^i}])$ is analytic, it has to be zero in U, and this implies that $\hat{K}$ is projectable. This means that the flow of $\hat{K}$ sends vertical spaces to vertical spaces, and as it is given by local isometries of (TM, g^L), horizontal spaces to horizontal spaces. This implies that the flow of the projection $\pi^*(\hat{K})$ acts by local isometries and then it is a local Killing field of (M, τ, L) in U that extends K. $\square$

Appendix A. Dimension of the group of pseudo-Finsler isometries

Let $(M, \mathcal{T}, L)$ be a pseudo-Finsler structure of index k, as defined in Section 7, and let g^L be the associated Sasaki metric on $\mathcal{T}$.

It is proved in [9, Lemma 1] that, given any C^2-diffeomorphism $f : M \to M$, one has the following commutative diagrams:

$$\begin{array}{ccc} T_v(TM) & \xrightarrow{d^2 f} & T_{df(v)}(TM) \\ \Big\downarrow{\scriptstyle d\pi} & & \Big\downarrow{\scriptstyle d\pi} \\ T_pM & \xrightarrow[df]{} & T_{f(p)}M \end{array} \qquad \begin{array}{ccc} T_pM & \xrightarrow{df} & T_{f(p)}M \\ \Big\downarrow{\scriptstyle i_v} & & \Big\downarrow{\scriptstyle i_{df(v)}} \\ \mathrm{Ver}_v & \xrightarrow[d^2 f]{} & \mathrm{Ver}_{d^2 f(v)} \end{array} \tag{17}$$

Using this diagram, it is proved in [9] that if f is an isometry of $(M, \mathcal{T}, L)$, then $df : \mathcal{T} \to \mathcal{T}$ is an isometry of g^L.

We have the canonical upper bound on the dimension of the isometry group of a pseudo-Finsler structure.

The map $f \mapsto df$ gives a natural injection of $\mathrm{Iso}(M, \mathcal{T}, L)$ into the isometry group of the Sasaki pseudo-Riemannian metric g^L. Since $\dim(\mathcal{T}) = 2n$, then we have $\dim(\mathrm{Iso}(M, \mathcal{T}, L)) \leqslant n(2n+1)$, where $n = \dim(M)$. However, the following sharper estimate holds:

Proposition A.1. *Let* $(M^n, \mathcal{T}, L)$ *be a pseudo-Finsler manifold. Then, the Lie group* $\mathrm{Iso}(M, \mathcal{T}, L)$ *has dimension less than or equal to* $\frac{1}{2}n(n+1)$.

Proof. Using (17), one sees that using the identifications $i_v : T_pM \xrightarrow{\cong} \mathrm{Ver}_v$ and $d\pi_v : \mathrm{Hor}^F_v \xrightarrow{\cong} T_pM$, for all $f \in \mathrm{Iso}(M, \mathcal{T}, L)$, both restrictions of $d(df)$ to Ver_v and to Hor^F_v of $d(df)_v$ coincide with $df_v : T_pM \to T_{f(p)}M$. This gives $\dim\big(\mathrm{Iso}(M, \mathcal{T}, L)\big) \leqslant \frac{1}{2}n(n+1)$. □

References

[1] A. M. Amores, *Vector fields of a finite type G-structure*, J. Differential Geom. **14** (1979), no. 1, 1–6.

[2] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. (9) **36** (1957), 235–249.

[3] A. V. Bagaev, N. I. Zhukova, *Automorphism groups of* G*-structures of finite type on orbifolds*, Sibirsk. Mat. Zh. **44** (2003), no. 2, 263–278; translation in Siberian Math. J. **44** (2003), no. 2, 213–224.

[4] G. D'Ambra, *Isometry groups of Lorentz manifolds*, Invent. Math. **92** (1988), no. 3, 555–565.

[5] R. G. Bettiol, B. Schmidt, *Three-manifolds with many flat planes*, preprint 2014, arXiv:1407.4165.

[6] S. Dumitrescu, K. Melnick, *Quasihomogeneousthree-dimensional real anaytic lorentz metrics do not exist*, preprint 2014, arXiv:1406.2302v1.

[7] C. Ehresmann, *Sur les pseudogroupes de Lie de type fini*, C. R. Acad. Sci. Paris **246** (1958), 360–362.

[8] R. Gallego Torromé, *Averaged structures associated with a Finsler structure*, arXiv:math/0501058v9 [math.DG].

[9] R. Gallego–Torromé, P. Piccione, *On the Lie group structure of pseudo-Finsler isometries*, arXiv:1311.0199, to appear in Houston J. Math.

[10] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 460–472.

[11] A. Ionescu, S. Klainerman, *On the local extension of Killing vector-fields in Ricci flat manifolds*, J. Amer. Math. Soc. **26** (2013), no. 2, 563–593.

[12] M. A. Javaloyes, *Chern connection of a pseudo-Finsler metric as a family of affine connections*, arXiv:1303.6263.

[13] M. A. Javaloyes and B.L. Soares, *Geodesics and Jacobi fields of pseudo-Finsler manifolds*, arXiv:1401.8149v2.

[14] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Vol. I. Reprint of the 1963 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[15] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Vol. II. Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[16] S. Kobayashi, *Transformation groups in differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70. Springer–Verlag, New York–Heidelberg, 1972.

[17] A. J. Ledger, M. Obata, *Compact riemannian manifolds with essential groups of conformorphisms*, Trans. Amer. Math. Soc. **150** (1970), 645–651.

[18] S. Krantz, *Handbook of complex variables*, Birkhäuser, 1999.

[19] V. S. Matveev, H.-B. Rademacher, M. Troyanov, and A. Zeghib, *Finsler conformal Lichnerowicz-Obata conjecture*, Ann. Inst. Fourier (Grenoble), 59 (2009), pp. 937–949.

[20] K. Nomizu, *On local and global existence of Killing vector fields*, Ann. of Math. (2) 72 (1960), 105–120.

[21] B. O'Neill, *Semi-Riemannian geometry. With applications to relativity*, Pure and Applied Mathematics, 103. Academic Press, New York, 1983.

[22] R. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. 22 (1957).

[23] P. Piccione, D. V. Tausk, *The theory of connections and G-structures. Applications to affine and isometric immersions*, XIV Escola de Geometria Diferencial. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2006.

[24] E. A. Ruh, *On the automorphism group of a* G*-structure*, Comment. Math. Helv. textbf39 (1964), 189–204.

[25] I. M. Singer, S. Sternberg, *The infinite groups of Lie and Cartan. I. The transitive groups.*, J. Analyse Math. **15** (1965), 1–114.

[26] S. Sternberg, *Lectures on differential geometry.* Prentice-Hall, Englewood Cliffs, N. J., (1964)

[27] A. H. Taub, *A characterization of conformally flat spaces*, Bull. Amer. Math. Soc. **55**, (1949). 85–89.

Universidade de São Paulo, Brazil