On the isometry group and the geometry of compact stationary Lorentzian manifolds

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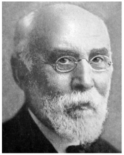
Global geometry: Riemannian vs. Lorentzian



Compact Riemannian manifolds:

- are complete and geodesically complete
- are geodesically connected
- have compact isometry group

Compact Lorentz manifolds:

- may be geodesically incomplete
- may fail to be geodesically connected
- have possibly non compact isometry group

Lack of compactness of $\mathrm{Iso}(M, g)$

Unlike Riemannian isometries, Lorentzian isometries:

- need not be *equicontinuous*
- may generate *chaotic dynamics* on the manifold

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  $\mathrm{Iso}(\mathbb{R}^{n+1}, q) = \mathrm{O}(q) \cong \mathrm{O}(n, 1)$ non compact.

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- q Lorentzian quadratic form in $\mathbb{R}^n$, $\mathrm{Iso}(\mathbb{R}^{n+1}, q) = \mathrm{O}(q) \cong \mathrm{O}(n, 1)$ non compact.
- The orthogonal frame bundle $\mathrm{Fr}(M, g)$ has non compact fibers. $\mathrm{Iso}(M, g)$ is identified topologically with any of its orbits in $\mathrm{Fr}(M, g)$.

The isometry group of a compact Lorentz manifold

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Theorem (Adams, Stuck, Zeghib, 1997)

The identity component $\mathrm{Iso}_0(M, g)$ is direct product:

$$A \times K \times H$$

- *A is abelian*
- *K is compact*
- *H is locally isomorphic to:*
  - $\mathrm{SL}(2, \mathbb{R})$
  - *an oscillator group*
  - *a Heisenberg group.*

Geometry of M via isometry group

Theorem (Zeghib)

If $\mathrm{Iso}_0(M, g)$ *contains a group locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$*, then* $\widetilde{M}$ *is a warped product of* $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ *and a Riemannian manifold.*

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Oscillator groups: characterized as the only simply connected solvable non abelian Lie groups that admit bi-invariant Lorentz metrics (Medina, Revoy, 1985). $\boxed{G = \mathbb{S}^1 \ltimes \mathrm{Heis}}$

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Action of $\mathbb{S}^1$ on the Lie algebra $\mathfrak{heis}$:

- Positivity conditions on the eigenvalues $\Longrightarrow$ existence of bi-invariant Lorentz metrics
- arithmetic conditions $\Longrightarrow$ existence of lattices.

An algebraic criterion for pre-compactness

Theorem

Let G be a connected Lie group, $K \subset G$ a maximal compact subgroup and $\mathfrak{k} \subset \mathfrak{g}$ their Lie algebras. Let $\mathfrak{m}$ be an Ad_K-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$.
Then, $\mathfrak{g}$ has a non empty open cone of vectors that generate precompact 1-parameter subgroups of G if and only if there exists $v \in \mathfrak{k}$ such that the restriction $\mathrm{ad}_v : \mathfrak{m} \to \mathfrak{m}$ is an isomorphism.

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Corollary 1

Let (M, g) be a compact Lorentz manifold that has a Killing vector field which is timelike somewhere. Then, $\mathrm{Iso}_0(M, g)$ is compact unless it contains a group locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group.

Non compact isometry group

Corollary 2

If (M, g) admits a somewhere timelike Killing vector field, then the two conditions are *mutually exclusive*:

(a) $\mathrm{Iso}_0(M, g)$ is not compact;

(b) $\mathrm{Iso}(M, g)$ has infinitely many connected components.

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Proof. Use Corollary 1 and Zeghib's classification:

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Proof. Use Corollary 1 and Zeghib's classification:

If $\mathrm{Iso}_0(M, g)$ contains a group locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group then:

- $\mathrm{Iso}(M, g)$ has only a finite number of connected components;
- M is not simply connected.

Definition

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Obs.: $\rho : \Gamma \to \mathrm{GL}(\mathcal{E})$ of Riemannian type $\Longleftrightarrow$ $\rho(\Gamma)$ precompact.

Compact Lorentz manifolds with *large* isometry group

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Proposition

(M, g) compact Lorentz manifold. If the conjugacy action of $\Gamma = \mathrm{Iso}(M, g)/\mathrm{Iso}_0(M, g)$ on $\mathrm{Iso}_0(M, g)$ is not of post-Riemannian type, then $\mathrm{Iso}_0(M, g)$ has a timelike orbit in M, and $\mathrm{Iso}(M, g)$ has infinitely many connected components.

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Gauss map:

$$\mathcal{G} : M \longrightarrow \mathrm{Sym}\big(\mathfrak{Iso}(M, g)\big)$$

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Proposition

If the action of Γ on $\mathrm{Iso}_0(M,g)$ is not of post-Riemannian type, then $\mathrm{Iso}_0(M,g)$ has somewhere timelike orbits.

Proof: Use $\mathfrak{k}(v,w) = \int_M \mathcal{G}_p(v,w)\,\mathrm{d}p$.

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Theorem (P.P., A. Zeghib)

Compact Lorentzian manifolds with large isometry groups are essentially built up by tori.

The structure result

Theorem

Let (M, g) be a compact Lorentz manifold that has a somewhere timelike Killing vector field, and whose isometry group $\mathrm{Iso}(M, g)$ has infinitely many connected components. Then:

- $\mathrm{Iso}_0(M, g)$ *contains a torus* $\mathbb{T}^d$ *endowed with a Lorentz form q, such that* Γ *is a subgroup of* $\mathrm{O}(q, \mathbb{Z})$*;*
- *up to finite cover, M is:*
  - *either a direct product* $\mathbb{T}^d \times N$*, with N compact Riemannian manifold*
  - *or an amalgamated metric product* $\mathbb{T}^d \times_{\mathbb{S}^1} L$*, where L is a lightlike manifold with an isometric* $\mathbb{S}^1$*-action.*

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Long exact homotopy sequence of the fibration $X \times Y \to (X \times Y)/\mathbb{S}^1$:

$$\mathbb{Z} \cong \pi_1(\mathbb{S}^1) \to \pi_1(X) \times \pi_1(Y) \to \pi_1(Z) \to \pi_0(\mathbb{S}^1) \cong \{1\}$$

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Proposition

If $\pi_1(X) \times \pi_1(Y)$ is not cyclic, then $(X \times Y)/\mathbb{S}^1$ is not simply connected.

Two interesting consequences

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Proof.

When $\mathrm{Iso}_0(M,g)$ contains a group locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or to an oscillator group use Zeghib's classification.

When $\mathrm{Iso}(M,g)$ has infinitely many connected components, use the structure result.