ON THE EXISTENCE OF LOCAL CMC TIME FUNCTIONS WITH PRESCRIBED TIME SLICE

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ABSTRACT. Given a spacetime (M, g) and a compact connected spacelike nondegenerate constant mean curvature (CMC) hypersurface Σ, we introduce the notion of CMC deformation modulus h^Σ of Σ. We prove that when h^Σ does not vanish on Σ, there exists a local CMC time function with Σ as a level set, while Σ is not a leaf of any local CMC foliation when h^Σ changes sign on Σ.

1. INTRODUCTION

Let (M, g) be a spacetime and consider a (local) time function $\tau \colon U \to \mathbb{R}$, defined on an open subset U of M. We say that τ is a *(local) CMC time function* if its level sets, or time slices, $\Sigma_t = \tau^{-1}(t)$ are constant mean curvature (CMC) spacelike hypersurfaces of (M, g). In this paper, we study when a given compact spacelike CMC hypersurface Σ of (M, g) is a level set of a local CMC time function. More precisely, we provide a sufficient condition for Σ to be a leaf of a local foliation by spacelike CMC hypersurfaces Σ_t, whose corresponding mean curvature H_t varies smoothly with t. In this case, H_t determines a smooth local CMC time function near Σ, having prescribed time slice Σ. With this purpose, we introduce the concept of *CMC deformation modulus* h^Σ of Σ, which is a real-valued smooth function on Σ that contains linearized information on CMC deformations of Σ.

Theorem. *Let M be a time-oriented Lorentz manifold with a nondegenerate connected, compact, spacelike CMC hypersurface $\Sigma \subset M$. Denote by h^Σ the CMC deformation modulus of Σ (see Section 2 for definitions). Then, the following hold:*

- (i) *If h^Σ does not vanish on Σ, then there exists $\varepsilon > 0$, a neighborhood U of Σ in M, and a smooth CMC time function $\tau \colon U \to (-\varepsilon, \epsilon)$ such that $\Sigma = \tau^{-1}(0)$. Moreover, if τ and τ' are smooth CMC time functions satisfying the above, then $\tau' = \phi \circ \tau$ where ϕ is a local diffeomorphism of $\mathbb{R}$.*
- (ii) *If h^Σ changes sign on Σ, then there does not exist any local CMC foliation $\mathcal{F}$ around Σ that contains Σ as a leaf.*

The proof of the above result uses normal variations of Σ, and an Implicit Function Theorem for CMC embeddings, see Section 3. By adapting these techniques to an equivariant setup, we also show that (i) holds for a special class of *degenerate* spacelike CMC hypersurfaces, for which the degeneracy originates from ambient isometries, see Remark 5.1.

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We stress that no global causal assumptions on the spacetime (M, g) are required, given the *local* character of the question studied in this paper. However, if (M, g) satisfies certain global properties (such as Σ being a Cauchy surface), then stronger conclusions on local CMC time functions having Σ as a time slice can be derived along the same lines.

The question of existence of foliations of spacetimes by (compact) spacelike CMC hypersurfaces has been intensively studied in the literature. Some of the main contributions to this problem were given by Gerhardt in the series of papers [14, 15, 16, 17]. In [15], an Implicit Function Theorem is used to prove that, for a given CMC hypersurface foliation around Σ, the mean curvature defines a smooth function in a neighborhood of Σ, using the assumption that either Σ is not totally geodesic, or that $\mathrm{Ric}_g(\vec{n}_\Sigma, \vec{n}_\Sigma) > 0$ somewhere on Σ. Either one of these assumptions implies that Σ is nondegenerate, see (2.1). It is observed in [18] that the (local) strong energy condition implies that the constant mean curvature increases monotonically with time through CMC foliations. Bartnik [10] proved the existence of CMC compact Cauchy hypersurfaces in *cosmological* spacetimes[1], extending previous results in [14]. In [16], it is proved that one can find closed spacelike hypersurfaces with prescribed constant mean curvature in a globally hyperbolic Lorentzian manifold with a compact Cauchy hypersurface. The existence of a foliation by spacelike CMC surfaces of a 3-dimensional globally hyperbolic spacetime with constant nonpositive curvature is proved in [8]. A similar result is proved in [9] for 3-dimensional globally hyperbolic spacetimes locally modelled on the anti-de Sitter space. In [17], the existence of a foliation by spacelike CMC hypersurfaces of a future end of a globally hyperbolic spacetime having a compact Cauchy hypersurface is proved, under the assumption either that Ric_g is bounded from below on timelike vectors, or that there exists a future mean curvature barrier. More recently, the existence of foliations by spacelike CMC hypersurfaces has been studied in globally hyperbolic spacetimes of constant curvature in [7], see also [5] for the case of flat spacetimes.

We also study singularities of CMC foliations around a *degenerate* spacelike CMC hypersurface. In Section 4, we briefly discuss this question on Robertson–Walker spacetimes, which provide physically interesting examples of spacetimes foliated by spacelike CMC (in fact, totally umbilical) hypersurfaces. A natural assumption in General Relativity is the so-called *timelike convergence condition* (TCC), i.e., $\mathrm{Ric}_g(v, v) \geq 0$ for all timelike vectors $v \in TM$. Assuming the Einstein equations $\mathrm{Ric}_g - \frac{1}{2}Rg = 8\pi T$ hold, this condition is equivalent to the strong energy condition $T(V, V) \geq \frac{1}{2}(\operatorname{tr} T)g(V, V)$, and expresses the physical fact that, on average, gravity attracts. Under the (TCC) assumption, it follows easily from (2.1) that the Jacobi operator J^Σ of any spacelike CMC hypersurface Σ of M is positive-semidefinite. Moreover, it is positive-definite (and, in particular, non-singular) if either $\mathrm{Ric}_g(\vec{n}_\Sigma, \vec{n}_\Sigma) > 0$ somewhere along Σ, or if Σ is not totally geodesic. In the class of Robertson–Walker spacetimes, if one assumes (TCC), then by a result of Alías, Romero and Sánchez [1], the unique spacelike CMC hypersurfaces are time slices. In this situation, the worst scenario is described by an example of Gerhardt [15], see Example 4.3, which shows that local CMC time functions may fail to exist around degenerate slices, despite the existence of a global smooth CMC foliation.

[1] i.e., globally hyperbolic spacetimes having a compact Cauchy surface and satisfying the timelike convergence condition.

If (TCC) is no longer assumed, interesting *bifurcation* phenomena may occur at degenerate time slices. In Proposition 4.4, we give a bifurcation criterion that allows to construct examples of spacelike CMC hypersufaces (that are not time slices) collapsing onto degenerate time slices.

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2. CMC deformation modulus

2.1. **CMC hypersurfaces.** Let (M^{n+1}, g) be a spacetime, i.e., a time-oriented Lorentz manifold, and let $\Sigma^n \subset M$ be a smooth connected compact spacelike hypersurface without boundary. Denote by $\vec{n}_\Sigma$ the unit future-pointing normal field to Σ, and by g_Σ the induced Riemannian metric on Σ. For all $p \in \Sigma$, the second fundamental form of Σ at p is the symmetric bilinear form II_p^Σ on $T_p\Sigma$ defined by $\mathrm{II}_p^\Sigma(X, Y) := g_p\big(\nabla_X Y, \vec{n}_\Sigma(p)\big)$, where ∇ is the Levi–Civita connection of g. The sum of its eigenvalues defines a function $H^\Sigma(p) := -\mathrm{tr}_{g_\Sigma}\big(\mathrm{II}_p^\Sigma\big)$, called the *mean curvature*[2] of Σ. If $H^\Sigma(p) = 0$ for all p, then Σ is said to be a *maximal* spacelike hypersurface. More generally, if H^Σ is constant, then Σ is a *constant mean curvature* (CMC) spacelike hypersurface.

If Σ is a compact spacelike CMC hypersurface, let $\Delta^\Sigma = -\mathrm{div}_{g_\Sigma}\mathrm{grad}$ be the *positive definite* Laplace–Beltrami operator of the metric g_Σ, and define the Jacobi operator J^Σ on the space of C^2-functions $\psi\colon \Sigma \to \mathbb{R}$ by

$$J^\Sigma(\psi) := \Delta^\Sigma\psi + \big(\|\mathrm{II}^\Sigma\|^2 + \mathrm{Ric}_g(\vec{n}_\Sigma, \vec{n}_\Sigma)\big)\psi, \tag{2.1}$$

where Ric_g is the Ricci curvature of g, and $\|\mathrm{II}^\Sigma\|$ is the Hilbert–Schmidt norm of II^Σ. It is well-known that J^Σ is a self-adjoint elliptic operator, and that for all $k \geq 2$ and $\alpha \in (0, 1)$, $J^\Sigma\colon C^{k,\alpha}(\Sigma) \to C^{k-2,\alpha}(\Sigma)$ is a Fredholm operator of index 0. A C^2-map ψ on Σ is a *Jacobi field* along Σ if $J^\Sigma\psi = 0$. By elliptic regularity, Jacobi fields are smooth. A compact spacelike CMC hypersurface Σ is said to be *nondegenerate* if $\psi = 0$ is the only Jacobi field along Σ, i.e., if $\ker J^\Sigma = \{0\}$. By Fredholmness, this is equivalent to J^Σ being invertible.

2.2. **Definition of the CMC deformation modulus.** If Σ is a nondegenerate compact spacelike CMC hypersurface, then J^Σ is invertible, hence there exists a unique (smooth) function h^Σ on Σ satisfying $J^\Sigma(h^\Sigma) = 1$, which we call the *CMC deformation modulus* of Σ.

A geometric interpretation of h^Σ can be given as follows. Given a smooth 1-parameter family $\{\Sigma_s\}_{s\in(-\varepsilon,\varepsilon)}$ of spacelike CMC hypersurfaces of M, with $\Sigma_0 = \Sigma$, let $V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\Sigma_s$ be the corresponding variational field along Σ, and let $\psi_V = g(V, \vec{n}_\Sigma)$ be its normal component. Denoting by h_s the mean curvature of Σ_s, we have $J^\Sigma\psi_V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}h_s$. Thus, h^Σ can be interpreted as the normal component ψ_V of the variational vector field V associated to a variation of Σ by spacelike hypersurfaces Σ_s that, to first order, have constant mean curvature $h_s = H^\Sigma + s$.

[2] We use the convention of a negative sign for the definition of the mean curvature, see for instance [2, p. 470]. This choice is irrelevant in the paper.

2.3. **CMC time functions.** A smooth (local) function T on M is a *time function* if its gradient $\nabla\tau$ is everywhere timelike and future directed. In particular, $\mathrm{d}\tau \neq 0$ everywhere, and the level sets $\tau^{-1}(t)$ are embedded spacelike hypersurfaces of M, called *time slices*. A time function τ is called a *CMC time function* (cf. [7]) if its time slices are CMC hypersurfaces.

Given a foliation $\mathcal{F}$ by spacelike CMC surfaces of an open subset U of M, one defines the mean curvature function $H_{\mathcal{F}}$ associated to $\mathcal{F}$ on U by setting $H_{\mathcal{F}}(p)$ equal to the mean curvature of the unique leaf of $\mathcal{F}$ through p. This is a continuous mapping, however, as observed in [15], it may fail to be a time function even in spacetimes where the strong energy condition holds.

3. Proof of Main Theorem

The set $\mathcal{E}^{2,\alpha}(\Sigma, M)$ of $C^{2,\alpha}$-submanifolds of M that are diffeomorphic to Σ, or, equivalently, the set of *unparametrized embeddings*[3] of Σ in M, has the structure of a topological manifold. A sufficiently small neighborhood of Σ in $\mathcal{E}^{2,\alpha}(\Sigma, M)$ is parametrized by maps f in a neighborhood $\mathcal{U}$ of 0 in the Banach space $C^{2,\alpha}(\Sigma)$, using the correspondence $f \mapsto x_f$, where

$$x_f(p) = \exp_p\big(f(p)\cdot \vec{n}_\Sigma(p)\big), \tag{3.1}$$

and exp is the exponential map of (M, g). Details of this construction can be found in [4]. The map $\mathcal{H}\colon \mathcal{U} \to C^{0,\alpha}(\Sigma)$ that to each $f \in \mathcal{U}$ associates the mean curvature of x_f is smooth and its differential at $f = 0$ is identified with the Jacobi operator $J^\Sigma\colon C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$. The nondegeneracy assumption implies that this operator is invertible, and thus $\mathcal{H}$ is a local diffeomorphism near $f = 0$. Set $h_0 = \mathcal{H}(0)$; this is the mean curvature of Σ. By the Inverse Function Theorem, there exists $\varepsilon > 0$ and a smooth map $(h_0 - \varepsilon, h_0 + \varepsilon) \ni h \mapsto f_h \in \mathcal{U}$ such that $\mathcal{H}(f_h) = h$ and, if f is sufficiently close to 0 and $\mathcal{H}(f) = h \in (h_0 - \varepsilon, h_0 + \varepsilon)$, then $f = f_h$. Using again elliptic regularity, one sees that f_h is smooth for all $h \in (h_0 - \varepsilon, h_0 + \varepsilon)$.

In other words, there is a unique smooth deformation of Σ by a family $h \mapsto \Sigma_h$ of smooth spacelike CMC hypersurfaces of M, with Σ_h having mean curvature equal to h, and $\Sigma_{h_0} = \Sigma$. Consider the variational vector field V along Σ corresponding to the family Σ_h, given by $V = \psi \cdot \vec{n}_\Sigma$, where $\psi = \frac{\mathrm{d}}{\mathrm{d}h}\big|_{h=h_0} f_h$. Then, since $\mathcal{H}(f_h) = h$,

$$1 = \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=h_0} \mathcal{H}(f_h) = J^\Sigma(\psi),$$

hence $\psi = h^\Sigma$ is the CMC deformation modulus of Σ. We now prove (i) and (ii):

(i) Assume that h^Σ *does not vanish on* Σ. Since Σ is compact, the map $\Sigma \times \mathbb{R} \ni (p, h) \mapsto \big(p, f_h(p)\big) \in \Sigma \times \mathbb{R}$ gives a diffeomorphism from a neighborhood of $\Sigma \times \{h_0\}$ in $\Sigma \times \mathbb{R}$ onto a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$. Moreover, by compactness, $\Sigma \times \mathbb{R} \ni (p, t) \mapsto \exp_p\big(t\cdot\vec{n}_\Sigma(p)\big) \in M$ also gives a diffeomorphism from a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$ onto a tubular neighborhood of Σ in M. The composition of these two maps gives a diffeomorphism from $\Sigma \times (h_0 - \varepsilon, h_0 + \varepsilon)$ to a tubular neighborhood U of Σ in M. Under such diffeomorphism, the hypersurfaces Σ_h correspond to the slices $\Sigma \times \{h\}$, and hence form a foliation of U. Moreover, the projection $\Sigma \times \mathbb{R} \to \mathbb{R}$ defines a smooth map $\widetilde{\tau}$ on U whose level hypersurfaces are the CMC submanifolds Σ_h. Clearly, $\mathrm{d}\widetilde{\tau} \neq 0$ and, since Σ_h is spacelike, $\nabla\widetilde{\tau}$ is timelike. If h^Σ is positive,

[3] i.e., embeddings modulo reparametrizations.

then $\nabla\tau$ has the same direction as $\vec{n}_\Sigma$, i.e., future-directed, and therefore $\widetilde{T}$ is a time function. In this case, the desired time function is given by $\tau := \widetilde{\tau} - h_0$. If h^Σ is negative, the desired time function is $\tau := h_0 - \widetilde{\tau}$. Uniqueness follows easily from the uniqueness of a CMC perturbation of Σ, since any two local CMC time functions having Σ as a level hypersurface must have the same level hypersurfaces and are hence equivalent.

(ii) Assume that h^Σ *changes sign on* Σ. By the above application of the Inverse Function Theorem, any spacelike CMC hypersurface Σ' diffeomorphic and sufficiently close to Σ must coincide with one of the Σ_h. Thus, any local CMC foliation $\mathcal{F}$ that contains Σ as a leaf must also contain some Σ_h as a leaf, with h arbitrarily close to h_0. We claim that Σ_h and Σ intersect for h close to h_0, and hence such a foliation $\mathcal{F}$ cannot exist. Indeed, the CMC deformation modulus of Σ is $h^\Sigma = \frac{\mathrm{d}}{\mathrm{d}h}\big|_{h=h_0} f_h$, where $\Sigma \ni p \mapsto \exp_p\big(f_h(p)\cdot\vec{n}_\Sigma(p)\big)$ parametrizes the family Σ_h, see (3.1). Since h^Σ changes sign on Σ, it follows that Σ_h intersects Σ for h close to h_0. $\square$

4. Bifurcation in Robertson–Walker spacetimes

We now discuss a phenomenon of lack of local uniqueness of a CMC foliation around a degenerate CMC hypersurface. The appropriate mathematical framework is obtained from bifurcation theory. As a paradigmatic example, let us consider a simple class of models given by *Robertson–Walker spacetimes*. Assume (M_0, g_0) is a compact n-dimensional Riemannian manifold, $I \subset \mathbb{R}$ is an open interval, and $\alpha\colon I \to \mathbb{R}^+$ is a smooth function. The Robertson–Walker spacetime with data (M_0, g_0, I, α) is the product manifold $M = I \times M_0$ endowed with the time-oriented Lorentz metric tensor g defined by:

$$g = -\mathrm{d}t^2 + \alpha(t)^2\, g_0. \tag{4.1}$$

The unit timelike vector ∂_t is a globally defined conformal field, and it is assumed to give the positive time-orientation of (M, g).

Robertson–Walker spacetimes are well-known models in the literature, and provide a class of examples of Lorentz manifolds with a natural foliation by spacelike CMC hypersurfaces, given by the time slices $\Sigma_t = \{t\} \times M_0$, $t \in I$. In fact, Σ_t are *totally umbilical* hypersurfaces (i.e., their second fundamental form is a multiple of the induced metric); in particular, they have constant mean curvature.[4] For all $p \in \Sigma_t$ and $v, w \in T_p\Sigma_t$, one has:

$$\mathrm{I\!I}_p^{\Sigma_t}(v,w) = -\alpha'\alpha\, g_0(v,w) = -\alpha^2(\log\alpha)'\, g_0(v,w).$$

Thus, the mean curvature h_t of Σ_t is given by

$$h_t = -\mathrm{tr}_{\alpha^2 g_0}\big(\mathrm{I\!I}^{\Sigma_t}\big) = n(\log\alpha)', \tag{4.2}$$

and the squared norm of $\mathrm{I\!I}^{\Sigma_t}$ is

$$\big\|\mathrm{I\!I}^{\Sigma_t}\big\|^2 = n\big((\log\alpha)'\big)^2. \tag{4.3}$$

Furthermore, a direct calculation gives

$$\mathrm{Ric}_g(\partial_t, \partial_t) = -n\frac{\alpha''}{\alpha} = -n\left(\big((\log\alpha)'\big)^2 + (\log\alpha)''\right). \tag{4.4}$$

[4] An explicit computation of the second fundamental form and the mean curvature of Σ_t can be found in [2]. See [1] for details on the mean curvature of more general graphs in Robertson–Walker spacetimes.

From (2.1), it follows that the Jacobi operator J^t of Σ_t is given by

$$J^t = \Delta_{\alpha^2 g_0} - n(\log\alpha)'' = \tfrac{1}{\alpha^2}\Delta_{g_0} - n(\log\alpha)''. \tag{4.5}$$

Thus, a time slice Σ_{t_0} is a degenerate spacelike CMC hypersurface if and only if $\lambda_0 = n\big(\alpha''(t_0)\alpha(t_0) - \alpha'(t_0)^2\big)$ is an eigenvalue of the Laplace operator Δ_{g_0}.

Remark 4.1. The timelike convergence condition (TCC) holds in a Robertson–Walker spacetime if and only if

$$\alpha'' \leq 0, \quad \text{and} \quad \mathrm{Ric}_0 \geq (n-1)\big(\alpha\alpha'' - {\alpha'}^2\big),$$

where Ric_0 is the Ricci tensor of the Riemannian manifold (M_0, g_0), see [1]. Under the (TCC) assumption, it is proved in [1] that a compact spacelike CMC hypersurface in *essentially any* Robertson–Walker spacetime must be a time slice Σ_t. It is also worth recalling that, even under the (TCC) assumption, a time slice Σ_t may fail to be nondegenerate (see Example 4.3 below).

Remark 4.2. Suppose (M^4, g) is a Robertson–Walker spacetime that satisfies the Einstein equations $\mathrm{Ric}_g - \frac{1}{2}Rg = 8\pi T$. The stress-energy tensor T is said to have the form of a *perfect fluid* if there exists a future-directed timelike unit vector field U on M and functions ρ and p on M such that $T = (\rho + p)\omega_U \otimes \omega_U + pg$, where $\omega_U := g(U, \cdot)$ is the 1-form dual to U. The functions ρ and p are respectively called *energy density* and *pressure*. Assume (M_0, g_0) has constant sectional curvature and the Robertson–Walker spacetime (M, g), given by (4.1), is a perfect fluid solution to the Einstein equation satisfying the strong energy condition. If $h_{t_0} > 0$ for some t_0, then the interval I has a finite left-endpoint t_{initial} with $t_0 - \frac{1}{h_{t_0}} < t_{\text{initial}} < t_0$, such that $\alpha(t) > 0$ and $\rho'(t) \leq 0$ on $(t_{\text{initial}}, t_0)$, and every timelike geodesic normal to the time slice $\{t\} \times M_0$ is past incomplete. Moreover, if $\lim_{t \to t_{\text{initial}}} \alpha(t) = 0$ and $\lim_{t \to t_{\text{initial}}} \alpha'(t) = \infty$, then t_{initial} is called a *big bang*, in which case $\lim_{t \to t_{\text{initial}}} \rho(t) = \infty$. The notion of *big crunch* can be analogously defined, corresponding to a finite right-endpoint t_{final} with similar properties.

Example 4.3. The following model is discussed in [15, Section 3] as an example of a spacetime satisfying the big bang and big crush hypotheses and (TCC), which is foliated by spacelike CMC hypersurfaces and has exactly one totally geodesic leaf, on which the derivative of the mean curvature function vanishes. In particular, the mean curvature function fails to be a time function.

The example is a Robertson–Walker spacetime with data:

- (M_0, g_0) is the n-dimensional round sphere, $n \geq 2$;
- $I = (-\varepsilon, \varepsilon)$, with $\varepsilon \in (0, 1]$;
- $\alpha(t) = e^{f(t)}$, with $f(t) = -\displaystyle\int_0^t \frac{s^3}{\varepsilon^2 - s^2}\,\mathrm{d}s = \frac{1}{2}t^2 + \frac{\varepsilon^2}{2}\log(\varepsilon^2 - t^2) - \varepsilon^2 \log\varepsilon.$

Using (4.2), the mean curvature h_t of the slice $\Sigma_t = \{t\} \times S^n$ is given by

$$h_t = n(\log\alpha)' = nf' = -\frac{n\,t^3}{\varepsilon^2 - t^2},$$

which vanishes exactly at $t = 0$. Thus, Σ_0 is totally geodesic, and it is the unique maximal leaf of the foliation. Moreover, since $h'_t = 0$ at $t = 0$, the maximal leaf Σ_0 is degenerate and h_t is not a time function. Notice also that $h'_t > 0$ for $t \neq 0$, hence h_t is a strictly increasing function on $(-\varepsilon, \varepsilon)$.

If Σ_{t_0} is nondegenerate, then the CMC deformation modulus $h^{\Sigma_{t_0}}$ is well-defined. In Robertson–Walker spacetimes, it follows from (4.5) that $h^{\Sigma_{t_0}}$ exists if and only if $\alpha''(t_0)\alpha(t_0) - \alpha'(t_0)^2 \neq 0$, which is equivalent to $h'_{t_0} \neq 0$. In this case, $h^{\Sigma_{t_0}}$ is the constant function $-\frac{\alpha(t_0)^2}{n\left(\alpha''(t_0)\alpha(t_0)-\alpha'(t_0)^2\right)}$. For the spacetime in Example 4.3, a direct computation shows that $\alpha''(0)\alpha(0) - \alpha'(0)^2 = 0$, so the CMC deformation modulus h^{Σ_0} is not defined.

Let us now give a criterion to establish when local uniqueness of CMC hypersurfaces fails around a degenerate time slice Σ_{t_0}, using bifurcation techniques.

Proposition 4.4. *Let (M^n, g) be a Robertson–Walker spacetime with $I = (-\varepsilon, \varepsilon)$. Suppose that for some $t_0 \in I$ the time slice Σ_{t_0} is a degenerate spacelike CMC hypersurface, i.e., $\lambda_0 = n\big(\alpha''(t_0)\alpha(t_0) - \alpha'(t_0)^2\big)$ is an eigenvalue of the Laplace operator Δ_{g_0}. Furthermore, assume that the following hold:*

(i) *the mean curvature function $t \mapsto h_t$ is locally injective near $t = t_0$;*
(ii) *the function $I \ni t \mapsto \lambda_0 - n\big(\alpha''(t)\alpha(t) - \alpha'(t)^2\big) \in \mathbb{R}$ changes sign at $t = t_0$;*
(iii) *λ_0 has odd multiplicity as an eigenvalue of Δ_{g_0}.*

Then, there exists a bifurcating branch *of spacelike CMC hypersurfaces for the family $\{\Sigma_t\}_{t\in I}$ issuing at Σ_{t_0}, i.e., there are spacelike CMC hypersurfaces of (M, g) that are diffeomorphic to M_0, arbitrarily close to Σ_{t_0}, and that do not belong to the family $\{\Sigma_t\}_{t\in I}$.*

Proof. The result is obtained as an application of a classical bifurcation theorem for families of Fredholm operators, see for instance [19, Theorem II.4.4]. The functional framework is as in Section 3. Namely, a sufficiently small neighborhood of Σ_{t_0} in the space of submanifolds of M diffeomorphic to Σ_{t_0} is identified with a neighborhood $\mathcal{U}$ of the Banach space $C^{2,\alpha}(\Sigma_{t_0})$. To each map $f \in \mathcal{U}$, one associates the embedding $x_f \colon \Sigma_{t_0} \to M$ defined using the normal exponential map along Σ_{t_0}. Consider the nonlinear map $F : \mathcal{U} \times \mathbb{R} \to C^{0,\alpha}(\Sigma_{t_0})$ defined by:

$$F(f, \mu) = H_{x_f} - \mu,$$

where $H_{x_f} \in C^{0,\alpha}(\Sigma_{t_0})$ is the mean curvature function of x_f. Clearly, $F(f, \mu) = 0$ if and only if x_f has constant mean curvature equal to μ.

Consider the 1-parameter family of constant functions $f_t = t - t_0$ on Σ_{t_0}. Recalling (3.1) and keeping in mind that the vector field $\partial_t = \vec{n}_{\Sigma_{t_0}}$ is geodesic, we have $x_{f_t}(\Sigma_{t_0}) = \Sigma_t$ for all t. Assumption (i) guarantees the existence of a continuous map $\mu \mapsto t(\mu)$, giving a bijection from a neighborhood of $\mu_0 = h_{t_0}$ to a neighborhood of t_0 in $\mathbb{R}$, such that $F(f_{t(\mu)}, \mu) = 0$ for all μ near μ_0. Thus, the existence of a bifurcating branch for the family of spacelike CMC hypersurfaces $\{\Sigma_t\}_{t\in I}$ issuing at Σ_{t_0} is equivalent to the existence of a bifurcating branch of solutions to the equation $F(f, \mu) = 0$ for the continuous path $\mu \mapsto f_{t(\mu)}$ issuing at μ_0.

The result of [19, Theorem II.4.4] applies in this situation to obtain the desired conclusion. The derivative of $\frac{\partial F}{\partial f}(f_{t(\mu)}, \mu) \colon C^{2,\alpha}(\Sigma_{t_0}) \to C^{0,\alpha}(\Sigma_{t_0})$ is identified with the Jacobi operator $J^{t(\mu)}$, which is Fredholm of index 0. This operator is singular at $\mu = \mu_0$. Assumption (ii) implies that for $\mu \neq \mu_0$ near μ_0, then $J^{t(\mu)}$ is non-singular; this follows easily from the fact that the eigenvalues of Δ_{g_0} is a discrete subset of $\mathbb{R}$. Finally, assumptions (ii) and (iii) imply the *odd crossing number* assumption in [19, Theorem II.4.4], which yields the existence of a bifurcating branch of solutions of the equation $F(f, \mu) = 0$. ∎

Example 4.5. Let (M_0, g_0) be a compact Riemannian manifold whose Laplace–Beltrami operator has an infinite number of eigenvalues with odd multiplicity. For instance, the Laplace–Beltrami operator of the round sphere S^n has eigenvalues $\lambda_k = k(k+n-1)$, with multiplicity $\binom{n+k}{n} - \binom{n+k-2}{n}$, see [11, Chapter 3, C], and, for every n, there exist infinitely many λ_k with odd multiplicity, e.g., λ_{2m}, $m \geq 0$, when $n = 3$. Given an interval $I \subset \mathbb{R}$, let $\alpha\colon I \to \mathbb{R}^+$ be a smooth function such that $g(t) = \alpha(t)\alpha''(t) - \alpha'(t)^2$ is strictly increasing on I, with $\inf g = 0$ and $\sup g = +\infty$. Then, the Robertson–Walker spacetime with data (M_0, g_0, I, α) has infinitely many degenerate time slices at which bifurcation occurs, by Proposition 4.4.

5. Final remarks

Remark 5.1. Important classes of spacetimes, such as Bianchi spacetimes, Gowdy spacetimes, or U(1)-symmetric vacuum Einstein spacetimes have a (local) isometry group of positive dimension, see [6] for a review of the global properties of these models. The presence of symmetries of the spacetime (M, g) determines an obstruction to the nondegeneracy assumption of the CMC hypersurface Σ. More precisely, if K is a local Killing vector field defined in a neighborhood of Σ, which is not everywhere tangent to Σ, then the map $\psi_K = g(K, \vec{n}_\Sigma)$ is a nontrivial Jacobi field along Σ. We call such a function ψ_K a *Killing–Jacobi field* along Σ. A compact spacelike CMC hypersurface Σ is called *equivariantly nondegenerate* if every Jacobi field along Σ is Killing–Jacobi.

It is an interesting question to extend part (i) of our main Theorem to the more general case of equivariantly nondegenerate CMC hypersurfaces. This is possible under the further assumptions that Σ is the boundary of a pre-compact open subset Ω of M, and that every Killing-Jacobi field ψ_K along γ is generated by a local Killing field whose domain contains $\overline{\Omega}$. Under these circumstances, one still has solutions for the equation $J^\Sigma(\psi) = 1$. Namely, the constant function 1 is L^2-orthogonal to the kernel of J^Σ, which consists of the Killing–Jacobi fields ψ_K:

$$\int_\Sigma \psi_K \,\mathrm{d}\Sigma = \int_\Sigma g(K, \vec{n}_\Sigma)\,\mathrm{d}\Sigma = \int_\Omega \mathrm{div}_g(K) = 0.$$

Since J^Σ is symmetric with respect to the L^2-inner product, it follows that 1 is in the image of J^Σ, which implies that the set of solutions of the equation $J^\Sigma(\psi) = 1$ is an affine subspace of $C^{2,\alpha}(\Sigma)$, parallel to $\ker J^\Sigma$. It is not hard to show that the proof in Section 3 holds under the assumption that *some* solution ψ of $J^\Sigma(\psi) = 1$ does not vanish on Σ. In this case, a sufficiently small tubular neighborhood of Σ in M is foliated by a smooth family of CMC hypersurfaces. As to the uniqueness, in this situation one has to consider a weaker notion that accounts for the flow (by isometries) of the Killing fields, see [12] for details.

Remark 5.2. In the bifurcation result of Proposition 4.4, assumption (iii) is unnecessary if one assumes that Σ_{t_0} is the boundary of a compact domain of M. Namely, in this situation one can use a variational approach to the CMC problem obtained by extremizing area with respect to volume-preserving variations. For variational bifurcation, it suffices to assume a jump of the Morse index, not necessarily by an odd number, which is guaranteed by assumption (ii) alone.

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