

Woodall's conjecture on packings of dijoins

Paulo Feofiloff *

<https://www.ime.usp.br/~pf/dijoins/>

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Abstract

In every digraph, the size of a minimum directed cut is equal to the maximum number of pairwise disjoint dijoins. This is Woodall's conjecture. Discussion of the conjecture was dormant for two decades until Cornuéjols and Guenin took up the subject. The present paper¹ presents Woodall's conjecture, its capacitated version, Schrijver's counterexample, Cornuéjols and Guenin's counterexamples, and some results of Williams.

1 Introduction

A *digraph* is a pair (V, A) where V is a finite set and A is a set of ordered pairs of elements of V . The elements of V are called *vertices* and those of A are called *arcs*. For each arc vw , the vertex v is the *positive end* and w is the *negative end* of the arc. The sets of vertices and arcs of a digraph D are denoted by $V(D)$ and $A(D)$ respectively. The *transpose*, or *directional dual*, of a digraph D is the digraph obtained by replacing each arc vw by the pair wv .

Cuts. An arc vw *exits* a subset X of $V(D)$ if $v \in X$ and $w \notin X$. The arc vw *enters* X if $v \notin X$ and $w \in X$. A *source* is any subset S of $V(D)$ such that no arc enters S . The sources \emptyset and $V(D)$ are *trivial*. A *sink* is a source in the transpose of D . A *source vertex* is any vertex s such that $\{s\}$ is a source; a *sink vertex* is a source vertex in the transpose of D .

For any set X of vertices, ∂X is the set of arcs that have one end in X and the other outside X . A *directed cut*, or simply *cut*, is any set of the form ∂S where S is either nontrivial source or a nontrivial sink. We say that S is a *positive margin* of the cut and $V(D) \setminus S$ is a *negative margin*. We also say that ∂S is the cut *associated* to S . A cut is *minimal* if none of its proper subsets is a cut.

* Department of Computer Science, Institute of Mathematics and Statistics, University of São Paulo, Brazil.

¹ This paper is a revised version of a talk by the author in 2005 at the Institute of Mathematics and Statistics of the University of São Paulo. A complete survey of Woodall's conjecture and its capacitated version can be seen in Marcel Kenji's master's thesis [Ken07, chap.5].

A digraph is *connected* if \emptyset is not a cut. In a connected digraph, every cut has a unique positive margin and a unique negative margin.

Joins. A *dijoin*, or simply *join*, is any set of arcs that intersects every cut, i.e., any subset J of $A(D)$ such that $J \cap C \neq \emptyset$ for each cut C . A join is *minimal* if none of its proper subsets is a join.

A digraph has a join if and only if \emptyset is not a cut. On the other hand, \emptyset is a join if and only if the digraph has no cut.

The following characterization is useful: a set J of arcs is a join if and only if for every pair (s, t) of vertices there is a path from s to t whose forward arcs² belong to J . This characterization can also be formulated as follows: a set J of arcs is a join if and only if the contraction of all the arcs of J makes the digraph strongly connected.³

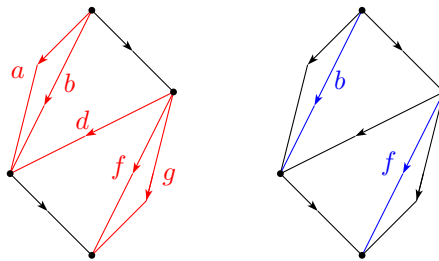


Figure 1: In the digraph on the left, the set of arcs $\{a, b, d, f, g\}$ is a cut. In the digraph on the right, the set of arcs $\{b, f\}$ is a join. (This example clashes with our definition of a digraph because it has parallel arcs. We could “subdivide” the parallel arcs to conform to the definition; but this would make the example too heavy.)

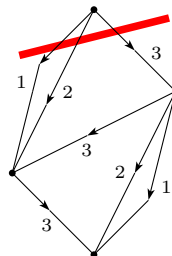


Figure 2: The colored line indicates a cut of size 3. The labels 1, 2 and 3 indicate a packing of three joins.

Cuts versus packings of joins. A set \mathcal{P} of joins is *disjoint* if the elements of \mathcal{P} are pairwise disjoint. In other words, \mathcal{P} is disjoint if each arc of the digraph belongs to at most one element of \mathcal{P} . A *packing of joins* is the same as a disjoint set of joins. There is no harm in assuming that the joins that make up a packing are minimal.

There is an obvious relationship between the size of a cut and the size of a packing of joins:

² An arc vw of a path is *forward* if the path traverses the arc from v to w and *backward* if the path traverses the arc from w to v .

³ A digraph is *strongly connected* if for each ordered pair (s, t) of its vertices there is a path from s to t without backward arcs.

Lemma 1.1 For any packing \mathcal{P} of joins and any cut C the inequality $|\mathcal{P}| \leq |C|$ holds. ■

The following conjecture of Woodall [Woo78a, Woo78b, Sch03] remains open:

Conjecture 1 (Woodall) If a digraph has a cut then it has a packing \mathcal{P} of joins and a cut C such that $|\mathcal{P}| = |C|$.

This conjecture is dual to the Lucchesi–Younger theorem [LY78], according to which every connected digraph has a packing \mathcal{C} of cuts and a join J such that $|\mathcal{C}| = |J|$.

Every arc of a digraph belongs to either a cut or a directed circuit,⁴ but not both. In particular, $C \cap A(Z) = \emptyset$ for every cut C and every directed circuit Z , where $A(Z)$ is the set of arcs of Z . It follows from this observation that we can restrict the study of Conjecture 1 to DAGs, that is, to digraphs free of directed circuits.

2 Minimum cut and maximum packing of joins

A cut C is *minimum* if there is no cut C' such that $|C'| < |C|$. A packing \mathcal{P} of joins is *maximum* if there is no packing \mathcal{P}' of joins such that $|\mathcal{P}'| > |\mathcal{P}|$. Woodall's conjecture leads us to consider the following pair of optimization problems:

Problem 1 Find a minimum cut of a digraph.

Problem 2 Find a maximum packing of joins of a digraph.

There is a polynomial algorithm for Problem 1 (it is a variant of the Max-flow Min-cut algorithm). No polynomial algorithm is known for Problem 2, but there is no evidence that the problem is NP-hard.

It is convenient to adopt a notation for the size of the objects that the two problems deal with. For any digraph D , we denote by

$$\nu(D)$$

the size of a maximum packing of joins of D and by

$$\tau(D)$$

the size of a minimum cut of D . If D has no cut then $\tau(D) = \infty$ and $\nu(D) = \infty$ (since an unbounded number of copies of \emptyset is a packing of joins). If D has a cut then $\tau(D)$ and $\nu(D)$ are finite. If D is disconnected then $\tau(D) = 0$ (since \emptyset is a cut) and $\nu(D) = 0$ (since there are no joins). If D consists of a path with at least one arc then $\tau(D) = 1$ and $\nu(D) = 1$ (since $A(D)$ is a join).

It follows immediately from Lemma 1.1 that $\nu(D) \leq \tau(D)$ for every digraph D . Conjecture 1 can then be formulated as follows:

Conjecture 2 (Woodall) Every digraph D satisfies the equality $\nu(D) = \tau(D)$.

⁴ A circuit is *directed* if it has no backward arcs.

We say that a digraph D satisfies Woodall's conjecture if $\nu(D) = \tau(D)$. If $\tau(D) \leq 1$ then it is obvious that D satisfies Woodall's conjecture. It is less obvious that D satisfies the conjecture if $\tau(D) = 2$ [Sch03, p.968]. It is also known that every DAG with a single source vertex (or a single sink vertex) satisfies the conjecture [FY87, Sch82].

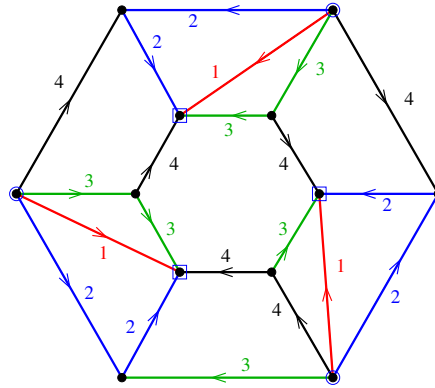


Figure 3: This digraph satisfies Woodall's conjecture; it has $\nu = 4$ and $\tau = 4$. The colors (and the numerical labels) indicate a packing of 4 joins. The digraph is a DAG. The source vertices are marked by circles and the sink vertices by squares.

3 Linear programs

Let \mathcal{J} the set of all the minimal joins of a digraph $D = (V, A)$ and M be the matrix indexed by $\mathcal{J} \times A$ whose rows are the characteristic vectors of the elements of \mathcal{J} . Consider the following dual pair of linear programs:

$$\text{maximize } y1 \text{ under the constraints } y \in \mathbb{R}_+^{\mathcal{J}} \text{ and } yM \leq 1, \tag{1}$$

$$\text{minimize } 1x \text{ under the constraints } x \in \mathbb{R}_+^A \text{ and } Mx \geq 1. \tag{2}$$

(The "1" represents a vector whose elements are all equal to 1; the vector is indexed by \mathcal{J} or by A , depending on the context. Of course $\mathbb{R}_+^{\mathcal{J}}$ is the set of nonnegative real vectors indexed by \mathcal{J} and \mathbb{R}_+^A is the set of nonnegative real vectors indexed by A .)

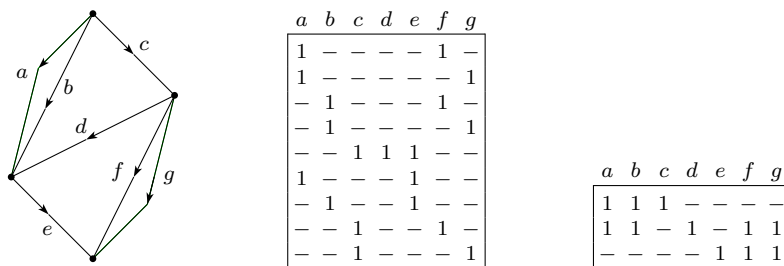


Figure 4: The rows of the first matrix are the characteristic vectors of the minimal joins of the digraph. The digraph is a DAG and has only one source and only one sink. The rows of the second matrix are the characteristic vectors of the minimal cuts.

If we replace " $y \in \mathbb{R}_+^{\mathcal{J}}$ " with " $y \in \{0, 1\}^{\mathcal{J}}$ " in the linear program (1) we will have an integer program that represents Problem 2. Every vector y in this program will represent a packing of joins and $y1$ will be the size of the packing. The optimum value of the integer program will be $\nu(D)$.

If we replace " $x \in \mathbb{R}_+^A$ " with " $x \in \{0, 1\}^A$ " in the linear program (2) we will have an integer program that represents Problem 1. Every x in this program will be the characteristic vector of a cut (since a cut is the same as a set of arcs that intersects all the joins) and $1x$ will be the size of the cut. The optimum value of the integer program will be $\tau(D)$.

As already noted, Woodall's conjecture is dual to the Lucchesi–Younger theorem [LY78]. It follows from that theorem (although this is not obvious) that all vertices of the polyhedron $\{x : x \in \mathbb{R}_+^A \text{ and } Mx \geq 1\}$ are integer and therefore every solution of the linear program (2) belongs to $\{0, 1\}^A$. It follows that $\tau(D) = \nu^*(D)$, where $\nu^*(D)$ is the optimum value of the linear program (1).

4 Max-flow analogy

To some extent, Woodall's conjecture is similar to the Max-flow Min-cut theorem [Sch03, chap.10]. This theorem applies to any digraph and any pair (s, t) of its vertices and guarantees that the size of a max-flow from s to t is equal to the size of a minimum half-cut among those separating s from t . Here, a *flow* is a set of directed paths⁵ from s to t with no common arcs; and a *half-cut* is the set of arcs that exit some set X of vertices that contains s but does not contain t .

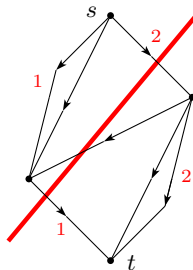


Figure 5: A maximum flow (labels 1 and 2) and a minimum semicut (colored line).

The similarity between Woodall's conjecture and the Max-flow Min-cut theorem is only partial. In the theorem, there are two fixed vertices and the paths are directed. In the conjecture, there are no fixed vertices, the paths that represent joins are not necessarily directed, and only the forward arcs of the paths are taken into account.

The Max-flow Min-cut theorem admits a generalization in which each arc a has a capacity (or upper-bound) u_a in the set \mathbb{Z}_+ of nonnegative integers. An arc a cannot be used more than u_a times by the flow and contributes u_a to the size of each half-cut that contains it. It is difficult to imagine that the Max-flow Min-cut theorem could hold without its capacitated generalization also holding.

The similarity between the Max-flow Min-cut theorem and Woodall's conjecture suggests studying the capacitated generalization of the conjecture.

⁵ A path is *directed* if it has no backward arcs.

5 Capacitated generalization of Woodall's conjecture

A *capacitated digraph* is a pair (D, u) where D is a digraph and u is a vector indexed by $A(D)$ with values in $\mathbb{Z}_+ \cup \{\infty\}$. This vector assigns a *capacity* u_a to each arc a of D . We say that a is *null* if $u_a = 0$ and *infinite* if $u_a = \infty$.

Assigning capacity ∞ to an arc has the same effect as contracting the arc, provided we allow infinite arcs to be traversed in either direction. Hence, the concepts of directed path and directed circuit need to be redefined as follows: a path and a circuit are *directed in* (D, u) , or *u -directed*, if all their backward arcs are infinite.

The presence of infinite arcs also calls for a redefinition of the terms "source" and "cut". Thus, a *source* of a capacitated digraph (D, u) is a source S of D such that ∂S contains no infinite arcs, and a *cut* of (D, u) is a cut of D without infinite arcs. In other words, a cut of (D, u) is a set ∂S where S is a nontrivial source of (D, u) .

For the updated definition of cut, the following statement is true: every noninfinite arc of (D, u) belongs to a cut of (D, u) or to a directed circuit of (D, u) , but not both.

The *capacity* of a cut C of (D, u) is the number $u(C) := \sum_{a \in C} u_a$. A cut C is *minimum* if there exists no cut C' of (D, u) such that $u(C') < u(C)$.

The term "join" must also be redefined. Thus, a *join* of (D, u) , or *u -join*, is a set of arcs that intersects all the cuts of (D, u) and contains no infinite arcs and no null arcs. Of course any join of D without infinite and null arcs is also a join of (D, u) . A set J of arcs is a join of (D, u) if and only if for every pair (s, t) of vertices there is a path from s to t each of whose forward arcs belongs to J or is infinite.

For the updated definitions of cut and join, the following statement is true: a capacitated digraph (D, u) has a join if and only if $u(C) > 0$ for every cut C of (D, u) .

In the context of capacitated digraphs, it is natural to use *collections* of joins in place of the sets of joins of Section 1. A collection is a "set" that can contain multiple copies of the same element, each copy contributing 1 to the size of the collection. A collection \mathcal{P} of joins of (D, u) is *disjoint in* (D, u) if

$$|\mathcal{P}(a)| \leq u_a$$

for each arc a , where $\mathcal{P}(a) := \{J \in \mathcal{P} : J \ni a\}$ is the collection of the joins that contain a . In other words, \mathcal{P} is disjoint if each arc a belongs to at most u_a elements of \mathcal{P} . If a is null then no element of \mathcal{P} may contain a .

A *packing* of joins *in* (D, u) is a disjoint collection of joins of (D, u) . The following relation between packings and cuts generalizes Lemma 1.1:

Lemma 5.1 *In any capacitated digraph (D, u) , for any packing \mathcal{P} of joins and any cut C ,*

$$|\mathcal{P}| \leq u(C).$$

Furthermore, if $|\mathcal{P}| = u(C)$ then $|J \cap C| = 1$ for each J in \mathcal{P} and $|\mathcal{P}(a)| = u_a$ for each a in C .

PROOF: Let \mathcal{P} be a packing of joins and C a cut of (D, u) . For each element J of \mathcal{P} there exists an arc a of C such that $\mathcal{P}(a) \ni J$. Therefore,

$$|\mathcal{P}| \leq \sum_{a \in C} |\mathcal{P}(a)| \leq \sum_{a \in C} u_a = u(C).$$

Suppose now that $|\mathcal{P}| = u(C)$. Then the first " \leq " holds as " $=$ " and therefore $|J \cap C| = 1$ for each J in \mathcal{P} . The second " \leq " also holds as " $=$ ", whence $|\mathcal{P}(a)| = u_a$ for each a in C . ■

The definitions of parameters τ and ν needs to be adjusted to take the capacities of the arcs into account. Thus, we denote by $\nu(D, u)$ the size of a maximum packing of joins of (D, u) and by $\tau(D, u)$ the capacity of a minimum cut of (D, u) . The inequality

$$\nu(D, u) \leq \tau(D, u) \quad (3)$$

is satisfied by every capacitated digraph (D, u) as a consequence of Lemma 5.1.

The corresponding generalization of Woodall's conjecture (Conjecture 2) is known as the Edmonds–Giles conjecture [EG77]:

Conjecture 3 (Edmonds–Giles) *Every capacitated digraph (D, u) satisfies the equality $\nu(D, u) = \tau(D, u)$.*

If $\tau(D, u) = 0$ then $\nu(D, u) = 0$ and hence $\nu(D, u) = \tau(D, u)$. If $\tau(D, u) = 1$ then $\nu(D, u) = \tau(D, u)$ since $\{a \in A(D) : 0 < u_a < \infty\}$ is a join. Therefore, the conjecture is correct when restricted to capacitated digraphs in which $\tau(D, u) \leq 1$.

Null arcs. The capacitated generalization of the Max-flow Min-cut theorem (see Section 4) can be reduced to the original, non-capacitated, version. The reduction consists of removing the arcs of capacity 0 and replacing each arc of capacity $k \geq 2$ with k arcs in parallel. At first glance, the same construction could reduce the Edmonds–Giles conjecture to Woodall's conjecture. Indeed, an arc a of capacity $k \geq 2$ can be simulated by k copies of a in parallel, but removing an arc of capacity 0 may create new cuts,⁶ thus changing the instance of the problem. Therefore, the Edmonds–Giles conjecture is not a special case of Woodall's conjecture.

6 Counterexamples

The Edmonds–Giles conjecture is false. The following sections will present several counterexamples. A *counterexample* is any capacitated digraph (D, u) such that $\nu(D, u) < \tau(D, u)$. All known counterexamples have null arcs and therefore do not affect Woodall's conjecture.

We say that a digraph D is *good* if there is no u such that (D, u) is a counterexample. Conjecture 3 could be formulated by saying "every digraph is good". It is known, for example, that

1. every DAG with a single source vertex is good;
2. every source-sink connected⁷ DAG is good.

⁶ The removal of an arc does not create new cuts if and only if the arc is transitive. An arc vw is *transitive* in (D, u) if there is a directed path from v to w in $(D - vw, u')$, where u' is the restriction of u to the set of arcs of $D - vw$.

⁷ A DAG is *source-sink connected* if each source vertex is connected to each sink vertex by a directed path.

The proof of 1 is analogous to that of the Max-flow Min-cut theorem mentioned in Section 4. This proof contains a polynomial algorithm that calculates $\tau(D, u)$. The proof of 2 was obtained by Schrijver [Sch82] and, independently, by F. and Younger [FY87].

7 Schrijver's counterexample

Schrijver [Sch80] found the first counterexample to Conjecture 3. The counterexample is represented in Figure 6 and will be denoted by (D_1, u_1) .

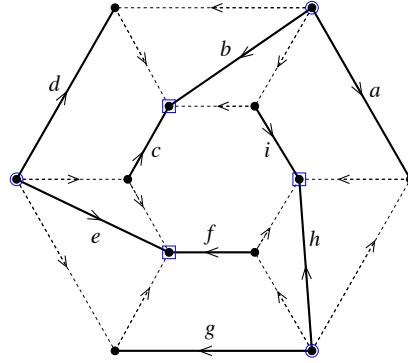


Figure 6: Schrijver's counterexample (D_1, u_1) . The capacity vector u_1 has values in $\{0, 1\}$. The null arcs are indicated by dashed lines; the others are indicated by solid lines. The digraph is a DAG; the source vertices are marked by circles and the sink vertices by squares.

Fact 7.1 $\nu(D_1, u_1) = 1$ and $\tau(D_1, u_1) = 2$.

PROOF: The vector u_1 is binary, that is, its components are in $\{0, 1\}$. It is easy to verify that $\tau(D_1, u_1) = 2$ and that one of the two margins of each minimum cut of (D_1, u_1) has a single vertex. Let B_1 be the set of *active* arcs, that is, arcs whose capacity is 1. The subdigraph induced by B_1 consists of three paths, each having length 3. We say that these are the *active paths* of the digraph. We also say that a cut is *critical* if it intersects each active path only once. As seen in Figure 7, there are four critical cuts.

Suppose for a moment that $\nu(D_1, u_1) \geq 2$. Then B_1 includes two mutually disjoint joins of (D_1, u_1) , say J and K . The arcs of each active path lie alternately in J and K , since each internal vertex of each active path is a margin of a cut with exactly 2 active arcs. In other words, each active path follows either the pattern (J, K, J) or the pattern (K, J, K) . In the set of three active paths, these two patterns can be combined in only 4 different ways, as shown in Figure 7. However, for each of the 4 combinations, either J or K does not intersect one of the critical cuts. Thus, J or K is not a join of (D_1, u_1) , contrary to what we had supposed. This contradiction shows that $\nu(D_1, u_1) < 2$. Since B_1 is a join of (D_1, u_1) , we have $\nu(D_1, u_1) = 1$. ■

Schrijver's capacitated digraph has the form of a ring of length $2i$, with $i = 3$. The analogous capacitated digraphs with $i = 5, 7, 9, \dots$ (see Figure 8) are also counterexamples. The analogous capacitated digraphs with $i = 2, 4, 6, 8, \dots$ are not counterexamples.

a	b	c	d	e	f	g	h	i
J	K	J	J	K	J	J	K	J
J	K	J	J	K	J	K	J	K
J	K	J	K	J	K	J	K	J
J	K	J	K	J	K	K	J	K

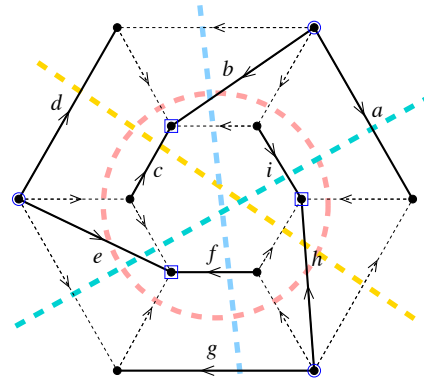


Figure 7: Each row of the table shows a possible arrangement of two potential mutually disjoint joins, J and K , in the capacitated digraph (D_1, u_1) in Figure 6. In each row of the table, one of J and K does not intersect one of the four critical cuts indicated in the drawing. In the first row, for example, J does not intersect the critical cut indicated by the pink circle.

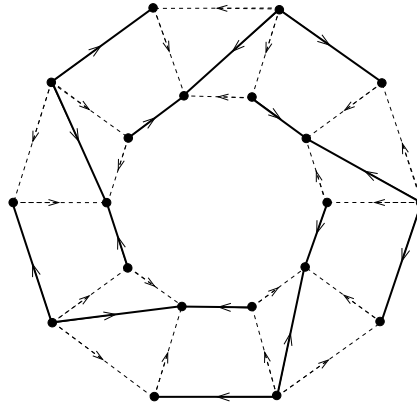


Figure 8: The counterexample (D'_1, u'_1) in this figure is the generalization of (D_1, u_1) of Figure 6 based on a ring of length 2×5 .

7.1 Fractional packing of joins

The following digression is interesting but has no direct bearing on the Edmonds–Giles conjecture. The capacitated digraph (D_1, u_1) in Figure 6 does not have a packing of size 2, but has a “fractional packing” of size 2, as we will show.

Let’s say that the joins $\{a, c, d, f, h\}$, $\{d, f, g, i, b\}$, $\{g, i, a, c, e\}$ and $\{b, h, e\}$ are *special*. Assign weight $\frac{1}{2}$ to each special join and weight 0 to all other joins of D_1 . Every arc of capacity 1 in (D_1, u_1) belongs to exactly two of the special joins, and every arc of capacity 0 belongs to no special join. Thus, the sum of the weights of all joins that contain a given arc a is no more than the capacity of a . Therefore, we can say that the weighted collection of special joins is “disjoint”. The size of this weighted collection is the sum of the weights of all joins, i.e., $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$. Thus, (D_1, u_1) has a “fractional packing” of size 2.

This example illustrates a general phenomenon. For any capacitated digraph (D, u) , con-

sider the linear programs

$$\text{maximize } y1 \text{ under the constraints } y \in \mathbb{R}_+^{\mathcal{J}} \text{ and } yM \leq u \quad (4)$$

$$\text{minimize } ux \text{ under the constraints } x \in \mathbb{R}_+^A \text{ and } Mx \geq 1 \quad (5)$$

which generalize programs (2) and (1) of Section 3. It can be shown that $\nu^*(D, u) = \tau(D, u)$, where $\nu^*(D, u)$ is the optimum value of program (4) and $\tau(D, u)$ is the optimum value of program (5).

8 Cornuéjols and Guenin's counterexamples

For two decades, (D_1, u_1) was the only known counterexample to Conjecture 3. In 2002, Cornuéjols and his student Guenin [CG02] found two new counterexamples, which we will denote by (D_2, u_2) and (D_3, u_3) . These counterexamples are represented in figures 9 and 10 respectively.

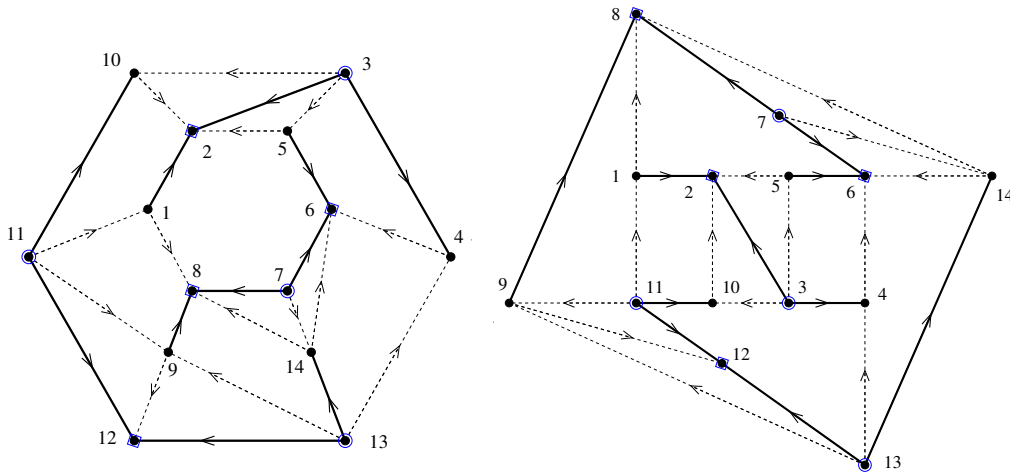


Figure 9: Two drawings of the counterexample (D_2, u_2) of Cornuéjols and Guenin. The capacity vector u_2 has values in $\{0, 1\}$. The null arcs are indicated by dashed lines; the others by solid lines. The digraph is a DAG; the source vertices are marked by circles and the sink vertices by squares.

Fact 8.1 $\nu(D_2, u_2) = 1$ and $\tau(D_2, u_2) = 2$. ■

Fact 8.2 $\nu(D_3, u_3) = 1$ and $\tau(D_3, u_3) = 2$. ■

The proofs of Facts 8.1 and 8.2 are similar to the proof of Fact 7.1. Figure 11 shows the critical cuts used in the proofs. (These are the cuts that intersect each active path only once.)

9 Minimal counterexamples

When looking for new counterexamples to the Edmonds–Giles conjecture, we can limit ourselves to the counterexamples that, in some sense, do not “include” other counterexamples. We say that such counterexamples are minimal.

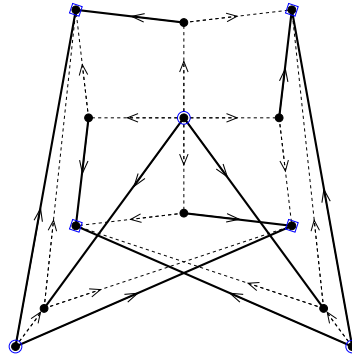


Figure 10: The counterexample (D_3, u_3) of Cornuéjols and Guenin. The capacity vector u_3 has values in $\{0, 1\}$. The null arcs are indicated by dashed lines; the others by solid lines. The digraph is a DAG.

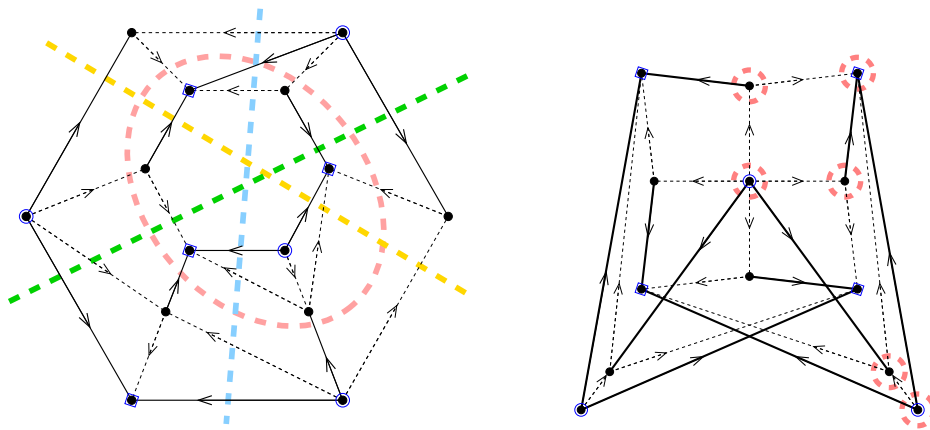


Figure 11: The first drawing represents the four critical cuts of (D_2, u_2) . The second drawing marks the vertices on the positive margin of one of the critical cuts of (D_3, u_3) ; the other three critical cuts are defined by symmetry.

To define “include”, we begin by introducing an order relation between capacity vectors. Given two capacity vectors u and u' for a digraph, we say that $u' < u$ if $u'_a \leq u_a$ for every arc a and $u'_a < u_a$ for some arc a . Clearly the relation is transitive (i.e., if $u'' < u'$ and $u' < u$ then $u'' < u$) and antisymmetric (i.e., if $u' < u$ then $u \not< u'$).

We also need some auxiliary notation: for any capacitated digraph (D, u) , we will denote by $I(D, u)$ and $N(D, u)$ the set of infinite arcs and the set of null arcs of the digraph.

We can now define an inclusion relation. We say that a capacitated digraph (D, u) *includes* a capacitated digraph (D', u') if either

- i. $V' \subseteq V$ and $A' \subset A$ or
- ii. $V' = V$ and $A' = A$ and $I' \supset I$ or
- iii. $V' = V$ and $A' = A$ and $I' = I$ and $N' \supset N$ or
- iv. $V' = V$ and $A' = A$ and $I' = I$ and $N' = N$ and $u' < u$,

where V', A', I' and N' are abbreviations for $V(D'), A(D'), I(D', u')$ and $N(D', u')$ respectively and V, A, I and N are the analogous abbreviations for (D, u) . This inclusion relation is transitive and antisymmetric. Here are some examples to illustrate the concept:

1. Williams [Wil04, WG05] noted that the counterexample (D_2, u_2) of Cornuéjols–Guenin includes another counterexample. Let v and w be the vertices 14 and 8 of Figure 9 and let D'_2 be the digraph $D_2 - vw$. Let u'_2 the restriction of u_2 to the set $A(D'_2)$. Then (D'_2, u'_2) is included in (D_2, u_2) . Moreover, (D'_2, u'_2) is a counterexample, since $\nu(D'_2, u'_2) = \nu(D_2, u_2) < \tau(D_2, u_2) = \tau(D'_2, u'_2)$.
2. The counterexample (D'_1, u'_1) in Figure 8 includes the capacitated digraph (D''_1, u''_1) in Figure 12. The latter is a counterexample because it is “equivalente” to the counterexample (D_1, u_1) in Figure 6.
3. Schrijver's counterexample (D_1, u_1) (see Figure 6) includes no other counterexample, although this is not obvious.

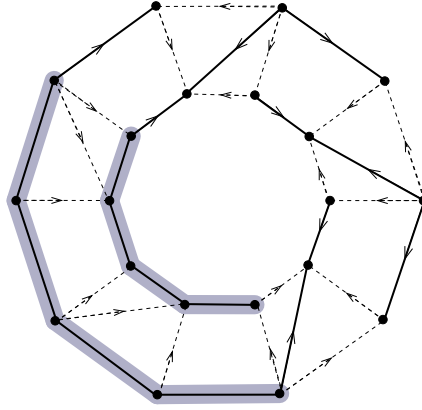


Figure 12: Capacitated digraph (D''_1, u''_1) . (Compare with (D'_1, u'_1) in Figure 8.) The gray bands indicate the infinite arcs. The orientation of these arcs has been omitted because they can be traversed in any direction.

We can now define the concept of a minimal counterexample: a counterexample (D, u) is *minimal* if it does not include another counterexample.

If $A(D) = \emptyset$ or $A(D) = I(D, u) \cup N(D, u)$ then (D, u) is not a counterexample. It follows from this observation and from the transitivity and antisymmetry of the inclusion relation that every counterexample includes a minimal counterexample.

Every minimal counterexample is, of course, connected. We will examine next some other properties of minimal counterexamples. The counterexamples (D_1, u_1) , (D_2, u_2) , and (D_3, u_3) in figures 6, 9, and 10 have these properties, although the latter two are not minimal.

(Williams [Wil04, WG05], a student of Guenin, proposed a much stronger definition of minimality and conjectures that, for this definition, every minimal counterexample belongs to a small set of digraphs derived from (D_1, u_1) , (D_2, u_2) and (D_3, u_3) .)

9.1 There are no null transitive arcs

An arc vw is *transitive* in (D, u) if there exists a directed path from v to w in $(D - vw, u')$, where u' is the restriction of u to the set of arcs of $D - vw$. (According to the definitions in Section 5, the path may have backward infinite arcs.)

Proposition 9.1 *Minimal counterexamples do not have null transitive arcs.*

PROOF: We will show that the removal of a null transitive arc does not create new cuts and does not change the values of the parameters ν and τ .

Let (D, u) be a capacitated digraph and b be a null transitive arc. Let D' be the digraph $D - b$ and u' the restriction of u to the set of arcs of $D - b$. Let v be the positive end and w the negative end of b . Let B the set of arcs of a directed path from v to w in (D', u') .

A cut of D intersects B if and only if it contains b . Furthermore, every cut of D contains at most one arc of B . It follows that D and D' have the same set of sources and therefore also (D, u) and (D', u') have the same set of sources. Furthermore, for each source F , we have $u'(C') = u(C)$, where C' and C are the cuts associated to F in D' and D respectively. Therefore,

$$\tau(D', u') = \tau(D, u). \quad (6)$$

Now, let \mathcal{P} be a packing of joins of (D, u) and \mathcal{P}' a packing of joins of (D', u') . Since (D, u) and (D', u') have the same set of sources, every join of (D', u') is also a join of (D, u) . Therefore, \mathcal{P}' is a packing in (D, u) . Conversely, every join of (D, u) that does not contain b is a join of (D', u') . Since b is null, none of the joins in \mathcal{P} contains b , and therefore \mathcal{P} is a packing in (D', u') . It follows that

$$\nu(D', u') = \nu(D, u). \quad (7)$$

By virtue of (6) and (7), if (D, u) is a counterexample then (D', u') is also a counterexample. Since $V(D') = V(D)$ and $A(D') \subset A(D)$, the counterexample (D, u) is not minimal. ■

The counterexample (D''_1, u''_1) in Figure 12 has several null transitive arcs and therefore is not minimal.

9.2 The capacity vector is critical

An arc of a capacitated digraph (D, u) is *active* if it is neither null nor infinite. The capacity vector u is *critical* if every active arc belongs to a minimum cut.

Proposition 9.2 *In every minimal counterexample, the capacity vector is critical.*

PROOF: Let (D, u) be a counterexample such that u is not critical. Then some active arc a does not belong to a minimum cut. Let u' be the capacity vector defined by

$$u'_a := u_a - 1 \quad \text{and} \quad u'_e := u_e \quad \text{for each } e \neq a.$$

Clearly $I(D, u') = I(D, u)$ and therefore (D, u') and (D, u) have the same set of cuts. It is also clear that $u'(C) = u(C) - 1$ for every cut C that contains a and $u'(C) = u(C)$ for all the remaining cuts. Since minimum cuts of (D, u) do not contain a , we have

$$\tau(D, u') = \tau(D, u).$$

Now consider the joins. Let \mathcal{P}' be a maximum packing of joins of (D, u') . Since \mathcal{P}' is also a packing in (D, u) , we have

$$\nu(D, u') = |\mathcal{P}'| \leq \nu(D, u).$$

But $\nu(D, u) < \tau(D, u)$, whence $\nu(D, u') < \tau(D, u')$, and therefore (D, u') is a counterexample. As $N(D', u') \supseteq N(D, u)$ and $u' < u$, the counterexample (D, u) is not minimal. ■

9.3 All directed circuits are infinite

Assigning ∞ to the arcs of a directed circuit does not change the set of cuts of the digraph.

Proposition 9.3 *In a minimal counterexample, the arcs of every directed circuit are infinite.*

PROOF: Let (D, u) be a capacitated digraph and O a directed circuit in (D, u) . (According to the definitions in Section 5, O may have backward infinite arcs.) Suppose that $u_a < \infty$ for some forward arc a of O . Define a new capacity vector u' as follows:

$$u'_a := \infty \quad \text{and} \quad u'_e := u_e \quad \text{for each } e \neq a.$$

Since O is directed in (D, u) , no cut of (D, u) contains arcs of O . Therefore, the set of cuts of (D, u') is identical to the set of cuts of (D, u) . Thus,

$$\tau(D, u') = \tau(D, u).$$

No minimal join of (D, u) contains a since no cut of (D, u) contains a . Thus, (D, u') and (D, u) have the same minimal joins. Therefore, every packing of minimal joins of (D, u) is also a packing in (D, u') , and vice versa. It follows that

$$\nu(D, u') = \nu(D, u).$$

Now suppose that (D, u) is a counterexample. Then $\nu(D, u) < \tau(D, u)$ and therefore $\nu(D, u') < \tau(D, u')$, that is, (D, u') is a counterexample. Since $I(D, u') \supset I(D, u)$, and therefore the counterexample (D, u) is not minimal. ■

It follows from this proposition that every minimal counterexample is essentially a DAG.

9.4 All minimum cuts are peripheral

A cut C is *peripheral* in (D, u) if, for one of the two margins of C , every arc that has both ends on that margin is infinite. For example, C is peripheral if either the positive margin or the negative margin of C has a single vertex.

Proposition 9.4 *In a minimal counterexample, every minimum cut is peripheral.*

PROOF: We will show that any capacitated digraph can be divided, along a minimum non-peripheral cut, into two "independent" capacitated digraphs.

Let C be a minimum cut of a capacitated digraph (D, u) . Let u' be the capacity vector defined as follows:

$$u'_a := \begin{cases} \infty & \text{if } a \text{ has both ends on the negative margin of } C, \\ u_a & \text{otherwise.} \end{cases}$$

(Informally, u' describes the contraction of the negative margin of C to a vertex.) Let u'' be the capacity vector defined as follows:

$$u''_a := \begin{cases} \infty & \text{if } a \text{ has both ends on the positive margin of } C, \\ u_a & \text{otherwise.} \end{cases}$$

According to Lemma 9.1 below, if (D, u) is a counterexample then either (D, u') or (D, u'') is a counterexample. Moreover, if C is not peripheral then $I(D, u') \supset I(D, u)$ (since some non-infinite arc has both ends on the negative margin of C) and, similarly, $I(D, u'') \supset I(D, u)$. Therefore, if the counterexample (D, u) is minimal, the cut C must be peripheral. ■

To conclude the proof of the proposition, we must establish the following lemma:

Lemma 9.1 *Let C be a minimum cut of a capacitated digraph (D, u) and let u' and u'' be the capacity vectors defined at the beginning of the proof of Proposition 9.4. If (D, u') and (D, u'') are not counterexamples then (D, u) is also not a counterexample.*

PROOF: On the one hand, C is a cut of (D, u') (since C does not contain infinite arcs) and $u'(C) = u(C)$, whence $\tau(D, u') \leq u'(C) = u(C) = \tau(D, u)$. On the other hand, $\tau(D, u') \geq \tau(D, u)$ since the set of cuts of (D, u') is part of the set of cuts of (D, u) and the capacity of a cut of (D, u') is equal to the capacity of that cut in (D, u) . Hence $\tau(D, u') = \tau(D, u)$ and therefore C is a minimum cut of (D, u') . An analogous reasoning shows that $\tau(D, u'') = \tau(D, u)$ and C is a minimum cut of (D, u'') .

1. Suppose that (D, u') is not a counterexample, i.e., that $\nu(D, u') = \tau(D, u')$. Let \mathcal{P}' be a maximum packing of joins of (D, u') . Of course $|\mathcal{P}'| = \nu(D, u') = \tau(D, u')$. Since $\tau(D, u') = u'(C)$, we have $|\mathcal{P}'| = u'(C)$. Lemma 5.1 (see Section 5) guarantees that

$$|\mathcal{P}'(a)| = u_a \quad \text{for each } a \text{ in } C \text{ and} \quad (8)$$

$$|J' \cap C| = 1 \quad \text{for each } J' \text{ in } \mathcal{P}'. \quad (9)$$

Now suppose that (D, u'') is not a counterexample and let \mathcal{P}'' be a maximum packing of joins of (D, u'') . A reasoning similar to the previous paragraph shows that

$$|\mathcal{P}''(a)| = u_a \quad \text{for each } a \text{ in } C \text{ and} \quad (10)$$

$$|J'' \cap C| = 1 \quad \text{for each } J'' \text{ in } \mathcal{P}''. \quad (11)$$

2. By virtue of (8) and (9), for each nonnull arc a of C , there are elements $J'_{a,1}, \dots, J'_{a,u_a}$ of \mathcal{P}' such that

$$J'_{a,i} \cap C = \{a\} \quad (12)$$

for $i = 1, \dots, u_a$. By virtue of (10) and (11), there are elements $J''_{a,1}, \dots, J''_{a,u_a}$ of \mathcal{P}'' such that $J''_{a,i} \cap C = \{a\}$ for $i = 1, \dots, u_a$. Let

$$J_{a,i} := J'_{a,i} \cup J''_{a,i} \quad (13)$$

for each $a \in C$ and each $i \in \{1, \dots, u_a\}$. Given any pair (a, i) , let J' , J'' and J be abbreviations for $J'_{a,i}$, $J''_{a,i}$ and $J_{a,i}$ respectively. Our next task is to show that J is a join of (D, u) . Since J' , J'' and J have no null and no infinite arcs, we only need to show that $J \cap B \neq \emptyset$ for every cut B of (D, u) .

3. Let B be a cut of (D, u) and X be the positive margin of B . Let Y be the positive margin of C . If $X \cap Y = \emptyset$ or $X \supseteq Y$ then B is a cut of (D, u'') , whence $J'' \cap B \neq \emptyset$. If $X \cup Y = V$ or $X \subseteq Y$ then B is a cut of (D, u') , whence $J' \cap B \neq \emptyset$. In both cases we have $J \cap B \neq \emptyset$. In the other cases, thanks to (9), (11), (12) and (13), Lemma 9.2 below guarantees that $J \cap B \neq \emptyset$. Hence, J is a join of (D, u) .

4. Let \mathcal{P} be the collection of all joins $J_{a,i}$ such that a is a nonnull arc of C and i belongs to $\{1, \dots, u_a\}$. For every arc e of D , if e has positive end on the positive margin of C then

$$|\mathcal{P}(e)| \leq u_e$$

since \mathcal{P}' is a packing in (D, u') and $u'_e = u_e$. Similarly, if e has negative end on the negative margin of C then $|\mathcal{P}(e)| \leq u_e$. Therefore, \mathcal{P} is a packing in (D, u) .

5. It follows from the previous paragraph that $\nu(D, u) \geq |\mathcal{P}|$. But $|\mathcal{P}| = |\mathcal{P}'| = |\mathcal{P}''| = \tau(D, u)$, and therefore $\nu(D, u) \geq \tau(D, u)$. Hence, (D, u) is not a counterexample. ■

To conclude the proof of the lemma, we must establish the following consequence of the modularity of ∂ :

Lemma 9.2 (modularity) *Let Y be a nontrivial source of a digraph D . Let J be a set of arcs that intersects all cuts ∂X of D for which X is a source such that*

$$X \cup Y = V \quad \text{or} \quad X \cap Y = \emptyset \quad \text{or} \quad X \supseteq Y \quad \text{or} \quad X \subseteq Y.$$

If $|J \cap \partial Y| = 1$ then J is a join of D .

PROOF: Let X be a nontrivial source of D such that $X \cup Y \neq V$ and $X \cap Y \neq \emptyset$. To prove that J is a join of D , it suffices to show that $J \cap \partial X \neq \emptyset$.

It is clear that $X \cup Y$ and $X \cap Y$ are nontrivial sources of D . Therefore, $\partial(X \cup Y)$ and $\partial(X \cap Y)$ are cuts of D . The union of $\partial(X \cup Y)$ with $\partial(X \cap Y)$ is equal to the union of ∂X with ∂Y and the intersection of $\partial(X \cup Y)$ with $\partial(X \cap Y)$ is equal to the intersection of ∂X with ∂Y . Therefore, the sum $|\partial(X \cup Y)| + |\partial(X \cap Y)|$ is equal to the sum $|\partial X| + |\partial Y|$. Similarly,

$$|J \cap \partial(X \cup Y)| + |J \cap \partial(X \cap Y)| = |J \cap \partial X| + |J \cap \partial Y|. \quad (14)$$

Since $X \cup Y \supseteq Y$ and $X \cap Y \subseteq Y$, the assumptions of the lemma guarantee that each term on the left-hand side of (14) is at least 1. Since the second term on the right-hand side of (14) is exactly 1, the first term on the right-hand side must be at least 1. Therefore, $J \cap \partial X \neq \emptyset$, as claimed. ■

9.5 There are no active circuits

An arc is *active* in (D, u) if it is neither null nor infinite. Williams [Wil04] showed that in a minimal counterexample the subdigraph induced by the set of active arcs is a forest:

Proposition 9.5 *No minimal counterexample has a circuit of active arcs.*

PROOF: Let (D, u) be a counterexample that has a circuit O whose arcs are active. We will show that the counterexample is not minimal.

Let e be a minimum capacity arc in O and $k := u_e$. Adjust the notation so that e is forward in O . Let u' be the capacity vector defined as follows:

$$u'_a := \begin{cases} u_a - k & \text{if } a \text{ is a forward arc of } O, \\ u_a + k & \text{if } a \text{ is a backward arc of } O, \\ u_a & \text{otherwise.} \end{cases}$$

It is clear that $u'_e = 0$ and therefore $N(D, u') \supset N(D, u)$. Thus, if (D, u') is a counterexample then the counterexample (D, u) is not minimal. We turn now to the case in which (D, u') is not a counterexample.

The set of cuts of (D, u') is identical to the set of cuts of (D, u) . It follows that the sets of joins of (D, u') and (D, u) are identical. Thus, we can say “cut” and “join” without adding “of (D, u) ” or “of (D, u') ”. Note that every cut contains the same number of backward and forward arcs of O . Therefore,

$$u'(C) = u(C) \quad (15)$$

for every cut C , and so

$$\tau(D, u') = \tau(D, u). \quad (16)$$

Let \mathcal{P}' be a maximum packing of joins of (D, u') . Since (D, u') is not a counterexample, $|\mathcal{P}'| = \tau(D, u')$. Let J_0 be an element of \mathcal{P}' . Lemma 9.3 below shows that $u'(C) - |J_0 \cap C| \geq |\mathcal{P}'| - 1$ for every cut C . Therefore,

$$u'(C) - |J_0 \cap C| \geq \tau(D, u') - 1$$

for every cut C . By virtue of (15) and (16), everything holds with u in place of u' , that is,

$$u(C) - |J_0 \cap C| \geq \tau(D, u) - 1 \quad (17)$$

for every cut C . Moreover, it is clear that J_0 has no null arcs of (D, u') and therefore no null arcs of (D, u) .

Now that we have a join J_0 that satisfies (17), we can discard u' and \mathcal{P}' . Let u'' be the vector defined as follows: for each arc a ,

$$u''_a := \begin{cases} u_a - 1 & \text{if } a \in J_0 \text{ and} \\ u_a & \text{otherwise.} \end{cases} \quad (18)$$

Since J_0 has no null arcs, u'' is a capacity vector. The sets of cuts of (D, u'') and of (D, u) are identical and therefore the sets of joins of (D, u'') and (D, u) are identical. Thus, we can say “cut” and “join” without adding “of (D, u) ” or “of (D, u'') ”. For every cut C , we have $u''(C) = u(C) - |J_0 \cap C|$, whence $u''(C) \geq \tau(D, u) - 1$ by virtue of (17). Therefore,

$$\tau(D, u'') \geq \tau(D, u) - 1.$$

Let \mathcal{P}'' be a maximum packing of joins of (D, u'') . Suppose for a moment that (D, u'') is not a counterexample. Then $|\mathcal{P}''| = \nu(D, u'') = \tau(D, u'')$. Consider now the collection $\mathcal{P} := \mathcal{P}'' \cup \{J_0\}$ and observe that

$$|\mathcal{P}| = |\mathcal{P}''| + 1 = \tau(D, u'') + 1 \geq \tau(D, u) - 1 + 1 = \tau(D, u).$$

Also observe that \mathcal{P} is a packing in (D, u) , since $|\mathcal{P}(a)| = |\mathcal{P}''(a)| + 1 \leq u''_a + 1 = u_a$ for each a in J_0 and $|\mathcal{P}(a)| = |\mathcal{P}''(a)| \leq u''_a = u_a$ for each a outside J_0 . Therefore, $\nu(D, u) \geq |\mathcal{P}| \geq \tau(D, u)$ and so (D, u) is not a counterexample. This contradicts the way (D, u) was chosen at the beginning of the proof. Therefore, contrary to what we had supposed for a moment, (D, u'') is a counterexample. Since $I(D'', u'') = I(D, u)$ and $N(D'', u'') \supseteq N(D, u)$ and $u'' < u$, the counterexample (D, u) is not minimal. ■

To conclude the proof of the proposition, we must establish the following lemma:

Lemma 9.3 *For any packing \mathcal{P} of joins of (D, u) , any element J_0 of \mathcal{P} , and any cut C , the inequality $u(C) - |J_0 \cap C| \geq |\mathcal{P}| - 1$ holds.*

PROOF: Let \mathcal{P} be a packing of joins of (D, u) . Then $|\{J \in \mathcal{P} : J \ni a\}| \leq u_a$ for each a in C and therefore

$$\begin{aligned} u(C) &= \sum_{a \in C} u_a \\ &\geq \sum_{a \in C} |\{J \in \mathcal{P} : J \ni a\}| \\ &= \sum_{J \in \mathcal{P}} |\{a \in C : a \in J\}| \\ &= \sum_{J \in \mathcal{P}} |J \cap C| \\ &= |J_0 \cap C| + \sum_{J \in \mathcal{P} \setminus \{J_0\}} |J \cap C|. \end{aligned}$$

Since $|J \cap C| \geq 1$ for each J , we have $u(C) \geq |J_0 \cap C| + |\mathcal{P} \setminus \{J_0\}|$. Hence, $u(C) - |J_0 \cap C| \geq |\mathcal{P}| - 1$. ■

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