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Exclusion processes and applications. Part I

1. The M/M/1 queue, graphical construction, dynamic reversibility and Burke's theorem.

2. The totally asymmetric simple exclusion process. Invariant measures. Multiclass stationary processes. Multiline processes and dual points.

3. Second class particles as representative of shocks and characteristics. Multiclass processes in the rarefaction fan.

4. Oriented percolation. Rost theorem. Competition interface.

5. Extension of Burke theorem to the ASEP.

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1 The M/M/1 queue

Continuous time Markov process Q_t on $\mathbb{N}$ ($\{0,1,2,\dots\}$)

Non-null Transition rates:

$q(x, x+1) = \lambda$, $q(x+1, x) = 1$, $x \geq 0$

If $\lambda < 1$, invariant measure is geometric: $\pi(x) = \lambda^x(1-\lambda)$

Proof: reversibility: $\pi(x)q(x, x+1) = \pi(x+1)q(x+1, x)$.

Arrival process: $A_t = \sum_{s=0}^{t} \mathbf{1}\{Q_s = Q_{s-} + 1\}$

Departure process: $D_t = \sum_{s=0}^{t} \mathbf{1}\{Q_s = Q_{s-} - 1\}$

Arrival process is Poisson process: has independent increments and

$\mathbb{P}(A_{t+h} - A_t = 1) = \lambda h + o(h)$

$\mathbb{P}(A_{t+h} - A_t > 1) = o(h)$

Burke Theorem

Theorem 1 *If Q_0 has law π then D_t is a Poisson process of rate λ.*

Proof. Fix T and consider the reverse process $Q_t^* := Q_{T-t-}$. Then by reversibility $(Q_s^*, 0 \leq x \leq t)$ has the same law as $(Q_s, 0 \leq x \leq t)$.

But defining A_s^* and S_s^* for the reverse process we have:

$A_s^* = D_T - D_{T-s}$

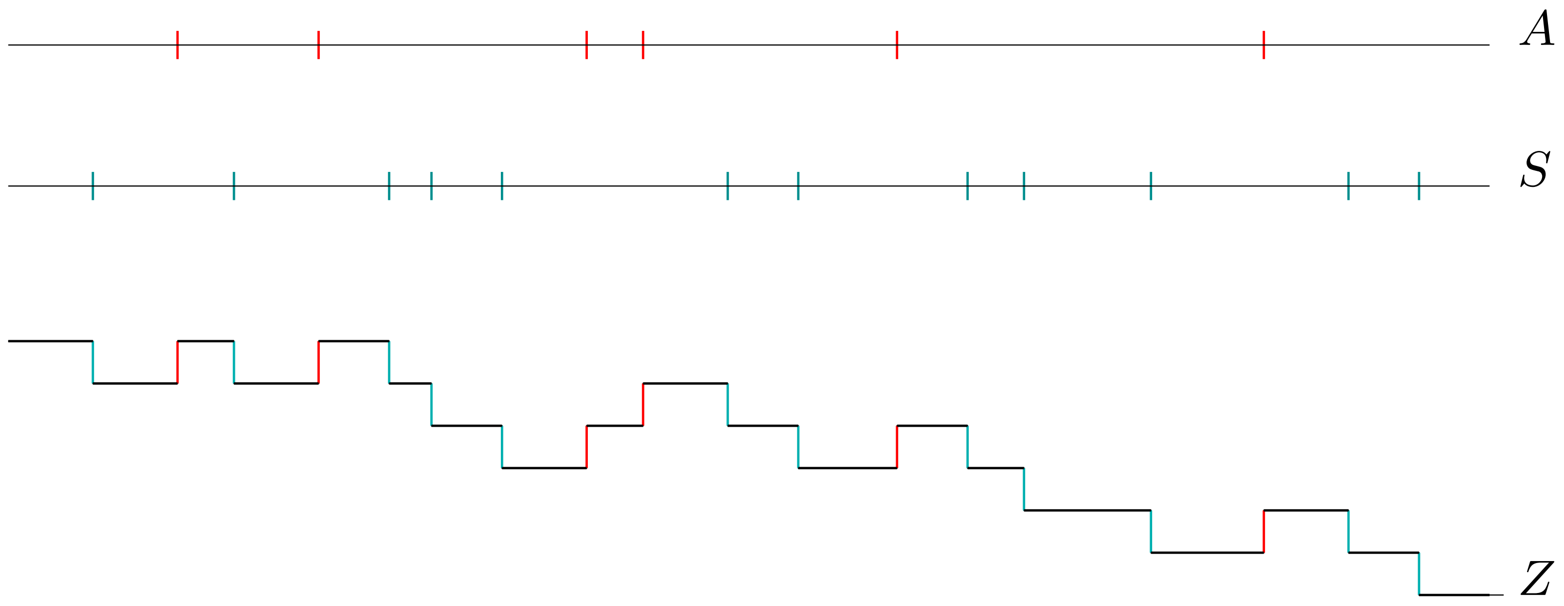
On the other hand, $(A_s^*, s \in [0, T])$ has the same law as $(A_s, s \in [0, T])$, hence it is a Poisson process of rate λ.

This implies $(D_T - D_{T-s}, s \in [0, T])$ is Poisson process , and so is D_s which is just a time reflected process. $\square$

Graphic construction of a stationary version of queue

Now A and S are Poisson processes of parameters $\lambda_1 < \lambda_2$ in $\mathbb{R}$

$|A \cap [s,t]|$ = number of points in $[s,t]$ (change of notation, this was $A_t - A_s$).



$Z(t) - Z(s) := |A \cap [s,t]| - |S \cap [s,t]|$

Asymmetric continuous time random walk; stationary increments.

Define $Q(t) = Z(t) - \inf_{s \le t} Z(s)$ for $t \in \mathbb{R}$.

Exercise: Show that $(Q(t),\, t \in \mathbb{R})$ is a Stationary $M/M/1$ queue. That is, show that

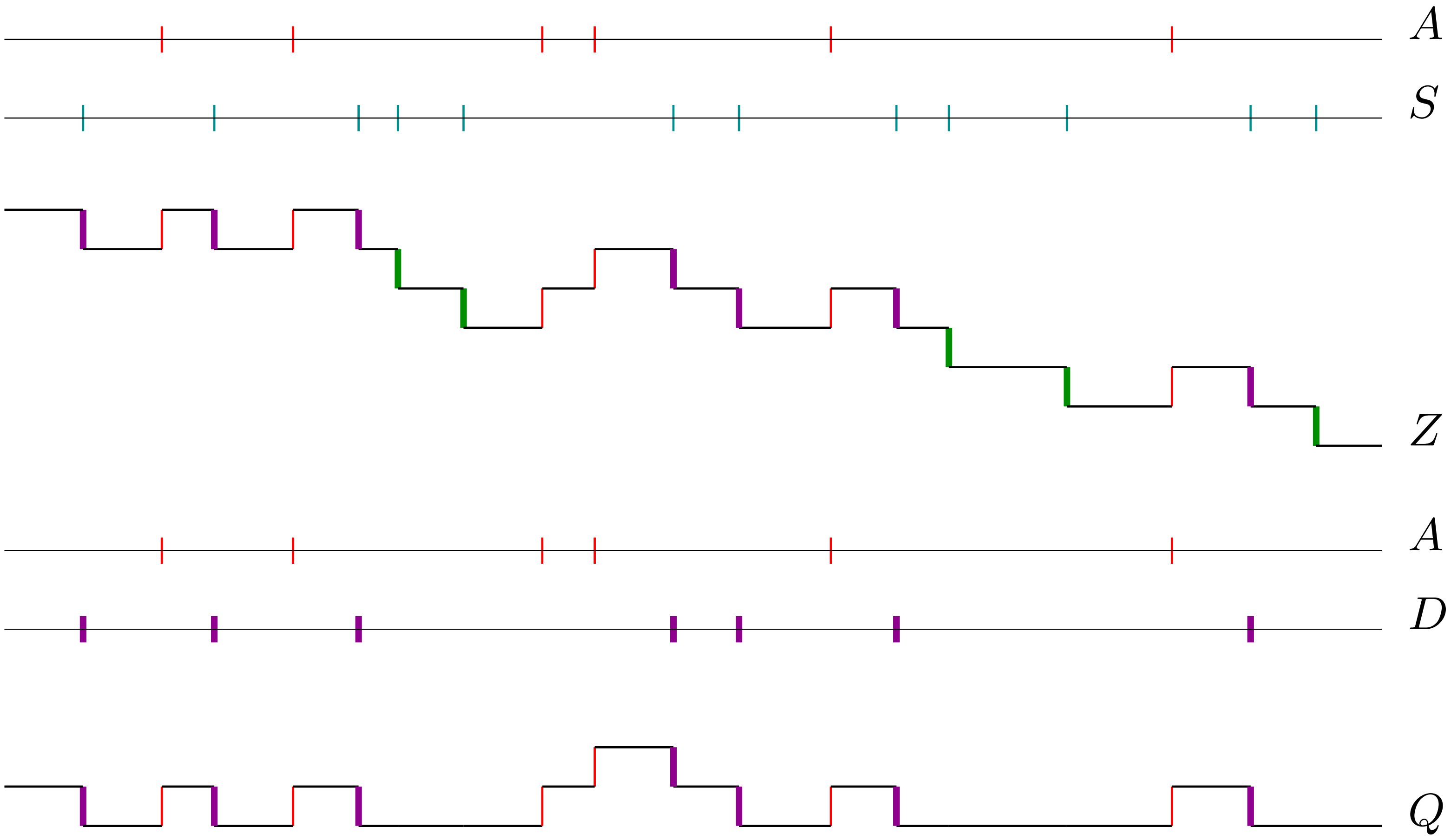
1) $(Q(s),\, s \in [a,b])$ has the same law as $(Q(s),\, s \in [a+t, b+t])$.

2) $\mathbb{P}(Q(t+h) = y | Q(t) = x) = hq(x,y) + o(h)$

Notice that $Q = (Q(t),\, t \in \mathbb{R})$ is a function of A and S. In this case we say that A and S *govern* Q and write $Q = Q(A,S)$.

Let $D = \{t \in S,\, Q(t) < Q(t-)\}$ be the set of times where there are *departures* of the queue system.

The times in $U := S \setminus D$ *cannot* be recovered from the knowledge of Q. These are service times that have not been used by a customer to be served. We call them *unused* service times.


A
S
Z
A
D
Q

Non Markov queues $G/G/1$

The graphic construction does not require neither A nor S to be Poisson. Any ergodic processes A and S with density of A strictly smaller than the density of S would provide the construction of a queue $Q(A,S)$. Of course the resulting queue Q is not Markovian.

To do that it is sufficient that the random walk $Z = Z(A,S)$ has asymptotic strictly negative and strictly positive drift to the left and right respectively.

A graphical Burke' s theorem

We want to construct A^* and S^* *as a function of* A and S in such a way that the reverse process $(Q^*(t), t \in \mathbb{R})$ is governed by A^* and S^*, where $Q^*(t) := Q(-t)$.

Consider an auxiliary process $X(t)$ which flips at the unused service times as follows. Define

$\tilde{Q}(t) = (Q(t), X(t)) \in \mathbb{N} \times \{0, 1\}$

with rates

$\tilde{q}((x, \ell), (x + 1, \ell)) = q(x, x + 1)$

$\tilde{q}((x + 1, \ell), (x, \ell)) = q(x + 1, x)$

$\tilde{q}((0, \ell), (0, 1 - \ell)) = 1$

The marginal process Q has the same law as before (hence we can use the same name).

The process $X(t)$ flips at service times when the queue is empty.

Exercise: The measure $\tilde{\pi}$ on $\mathbb{N} \times \{0,1\}$ defined by $\tilde{\pi}(x,\ell) = \frac{1}{2}\pi(x)$ is reversible for the process $(Q(t), X(t))$.

The arrival and service processes can be recovered from the chain (Q, X) as follows

$A = \{t : Q(t) > Q(t-)\}$

$S = \{t : Q(t) < Q(t-) \text{ or } X(t) \neq X(t-)\}$

The process $((Q^*(t), X^*(t)),\ t \in \mathbb{R})$ defined by

$(Q^*(t), X^*(t)) := (Q(t-), X(t-))$

is the reverse process of (Q, X), hence has the same law.

The processes A^*, S^* defined by (notice reflection)

$S^* = \{-t : Q^*(t) < Q^*(t-) \text{ or } X(t) \neq X(t-)\}$

$A^* = \{-t : Q^*(t) > Q^*(t-)\}$

are the Poisson processes governing (Q^*, X^*). Furthermore $A^* = D$ and $S^* = A \cup U$.

Factorizing property

Lemma 2 *Let* Q *be the stationary queue constructed above. Then* $Q(t)$ *is independent of* $A \cap (t, \infty)$ *and* $D \cap (-\infty, t]$ *for all* t*.*

Proof. The Markov property (and our construction of Q) implies that $Q(t)$ is independent of the future arrivals $A \cap (t, \infty)$ and $Q(t-)$ is independent of the future arrivals $A \cap [t, \infty)$. For the very same reason $Q^*(-t)$ is independent of $A^* \cap (-t, \infty)$. But $Q^*(-t) = Q(t-)$ and $A^* \cap [-t, \infty) = D \cap (-\infty, t]$). $\square$

Arrival-departure identification The construction of D and Q as functions of A and S induces naturally a bijection from A to D. The bijection associates to each arrival time the corresponding departure time.

Order the events of A such that $A = (a_i,\, i \in \mathbb{Z})$ with $a_i < a_{i+1}$. The FIFO (first-in-first-out) schedule is a function $\mathrm{FIFO}(a) = d$ which satisfies

$$\mathrm{FIFO}(a_i) = \min(D \cap (\mathrm{FIFO}(a_{i-1}), \infty))$$

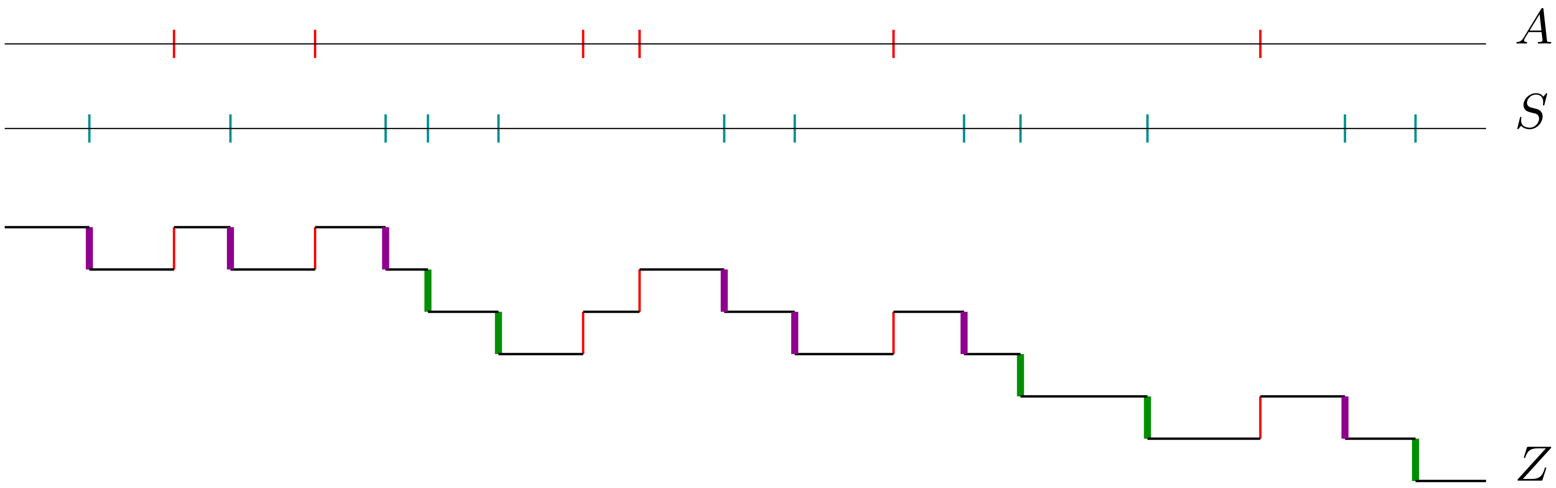
This is well defined because there are infinitely many unused services and the map can be determined in the intervals (u_i, u_{i+1}). In particular:

$$[a, \mathrm{FIFO}(a)] \cap U = \emptyset$$

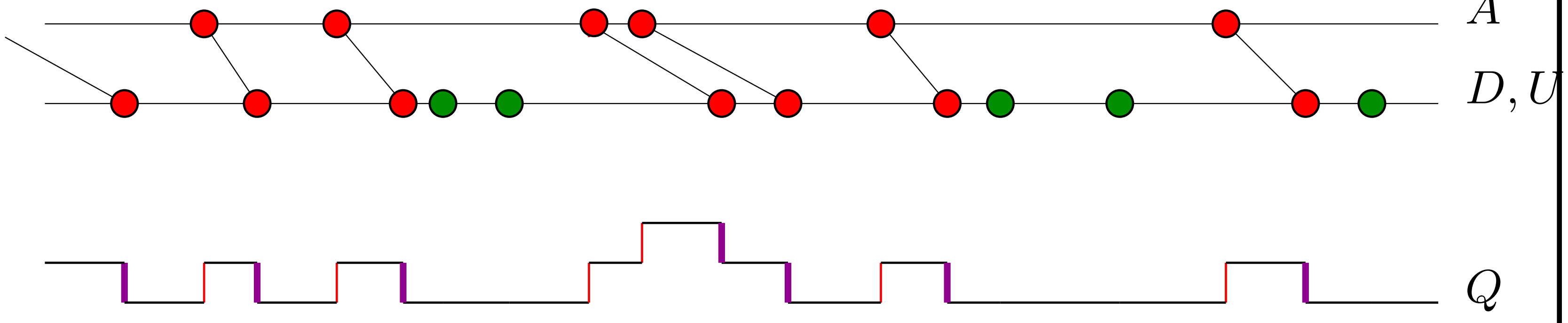
The queue Q can then be recovered from this bijection:

$$Q(t) = \sum_{a \in A} \mathbf{1}\{[a, \mathrm{FIFO}(a)) \ni t\}$$

There are other possible bijections, LIFO, for instance.



FIFO:



Queues in tandem

Consider a process $Q_1(t), \ldots, Q_n(t)$ of n queues with service rates 1 and arrival rate $\lambda < 1$ to the first queue. When a customer departs from queue i, it jumps to queue $i+1$, where it has to wait, etc. Customers departing from queue n just leave the system.

The nonzero transition rates for $x_i \geq 0$ are given by:

$q((x_1, \ldots, x_n), (x_1 + 1, \ldots, x_n)) = \lambda$ (arrivals to system)

$q((x_1, \ldots, x_j + 1, x_{j+1}, \ldots, x_n), (x_1, \ldots, x_j, x_{j+1} + 1, \ldots, x_n)) = 1$

$q((x_1, \ldots, x_n + 1), (x_1, \ldots, x_n)) = 1$ (departures from system)

Lemma 3 *The invariant measure π for this process is product of geometrics:* $\pi(x_1, \ldots, x_n) = (1-\lambda)^n \prod_{i=1}^n \lambda^{x_i}$.

Proofs.

1. Classical. Balance equations:

$\sum_{x \neq y} \pi(x) q(x,y) = \pi(y) \sum_{z \neq y} q(y,z)$

just check that the above π satisfies this equations.

2. Dynamic reversibility

Easy lemma (prove it as an exercise): *Given rates q and probability measure π, if one finds rates q^* satisfying*

(a) $\pi(x)q(x,y) = \pi(y)q^*(y,x)$

(b) $\sum_{x \neq y} \pi(x)q(x,y) = \sum_{x \neq y} \pi(x)q^*(x,y)$

then π is invariant for q and q^* are the rates of the reverse *process $Q^*(t)$ which in equilibrium is defined by*

$Q(0)$ has law π and for $t \in [0,T]$, $Q^*(t) = Q(T-t-)$.

To show that π is invariant for q, check conditions (a) and (b) for q, π and q^* defined by

$q^*((x_1,\ldots,x_j,x_{j+1}+1,\ldots,x_n),(x_1,\ldots,x_j+1,x_{j+1},\ldots,x_n)) = 1$

$q^*((x_1,\ldots,x_n),(x_1,\ldots,x_n+1)) = \lambda$

$q^*((x_1+1,\ldots,x_n),(x_1,\ldots,x_n)) = 1$

That is, the reverse process is just a system of queues in tandem going backwards.

This proof is useful when one does not know the invariant measure but has a good guess of it. *And* is lucky enough to have a nice reverse process.

3. Use graphical construction, Burke's theorem and induction on n.

Let A be the arrival process to queue 1 (Poisson rate λ).

$S_1, \ldots, S_n$ service processes for queues 1 to n (Poisson rate 1).

Construct the stationary queuing process $Q_1 = Q(A, S_1)$ and its departure process $D_1 = D(A, S_1)$. Then inductively,

$Q_k := Q(D_{k-1}, S_k)$ and $D_k := D(D_{k-1}, S_k)$.

By construction, the process $((Q_1(t), \ldots, Q_n(t)),\ t \in \mathbb{R})$ is stationary.

The marginal process $((Q_1(t), \ldots, Q_k(t)),\ t \in \mathbb{R})$ is Markovian for each $k \leq n$.

In particular, $(Q_1(t),\ t \in \mathbb{R})$ is just a stationary $M/M/1$ queue, so its marginal time distribution is π for each t.

$Q_1(t)$ is independent of $D_1 \cap (-\infty, t]$

$Q_2(t)$ is a function of $D_1 \cap (-\infty, t]$ and $S_2 \cap (-\infty, t]$.

Hence $Q_2(t)$ is independent of $Q_1(t)$.

By Burke, D_1 is Poisson(λ), hence the marginal distribution of $Q_2(t)$ is also π.

On the other hand $Q_k(t)$ depends on $D_{k-1} \cap (-\infty, t]$ and $S_k \cap (-\infty, t]$, both processes are independent of $(Q_1(t), \ldots, Q_{k-1}(t))$ by induction, hence $Q_k(t)$ is independent of $(Q_1(t), \ldots, Q_{k-1}(t))$ and with marginal law π.

This proves the product measure π^n is invariant for the queues in tandem.

Graphic dynamical reversibility for queues in tandem The unused services at each queue are given by

$U_k = S_k \setminus D_k$

Define the arrivals and service times of the reverse process:

$A^* = D_n$, $S_1^* = D_{n-1} \cup U_n$

$S_k^* = D_{n-k} \cup U_{n-k+1}$

Exercise: show (a) A^* and $S_1^*, \ldots, S_n^*$ are mutually independent Poisson processes of rates λ and 1, respectively.

(b) (The reflections around the origin of) A^* and $S_1^*, \ldots, S_n^*$ govern the process $(Q_1^*(t), \ldots, Q_n^*(t)$, $t \in \mathbb{R})$ defined by

$Q_k^*(t) = Q_{n-k}(-t-)$

A dynamics of point processes The map $\Phi : (A, S) \mapsto (D, U)$ can be extended to general ergodic point processes satisfying

"$\lambda =$ (density of A) $< 1 =$ (density of S) "

where "density of A" $= \lim_{x\to\infty} \frac{1}{2x} |A \cap [-x, x]|$.

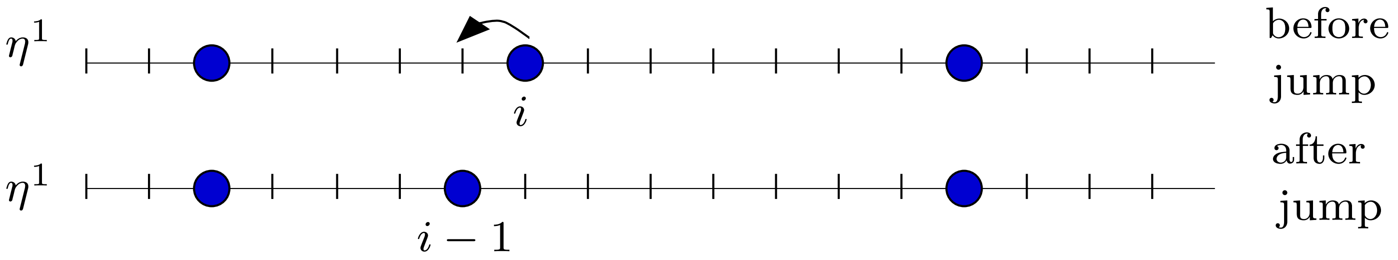
If $S_1, S_2, \dots$ are independent, then (D_n, U_n) defined by

$(D_n, U_n) = \Phi(D_{n-1}, S_n)$ is a Markov process. Mountford proposes to call it "Loynes dynamics".

When S_n are Poisson(1), we have a succession of queues with non Markovian arrival process. Mountford and Prabhakar prove that D_n converges to Poisson(λ). In fact Poisson(λ) is invariant for this dynamics.

2 TASEP

The totally asymmetric simple exclusion process.



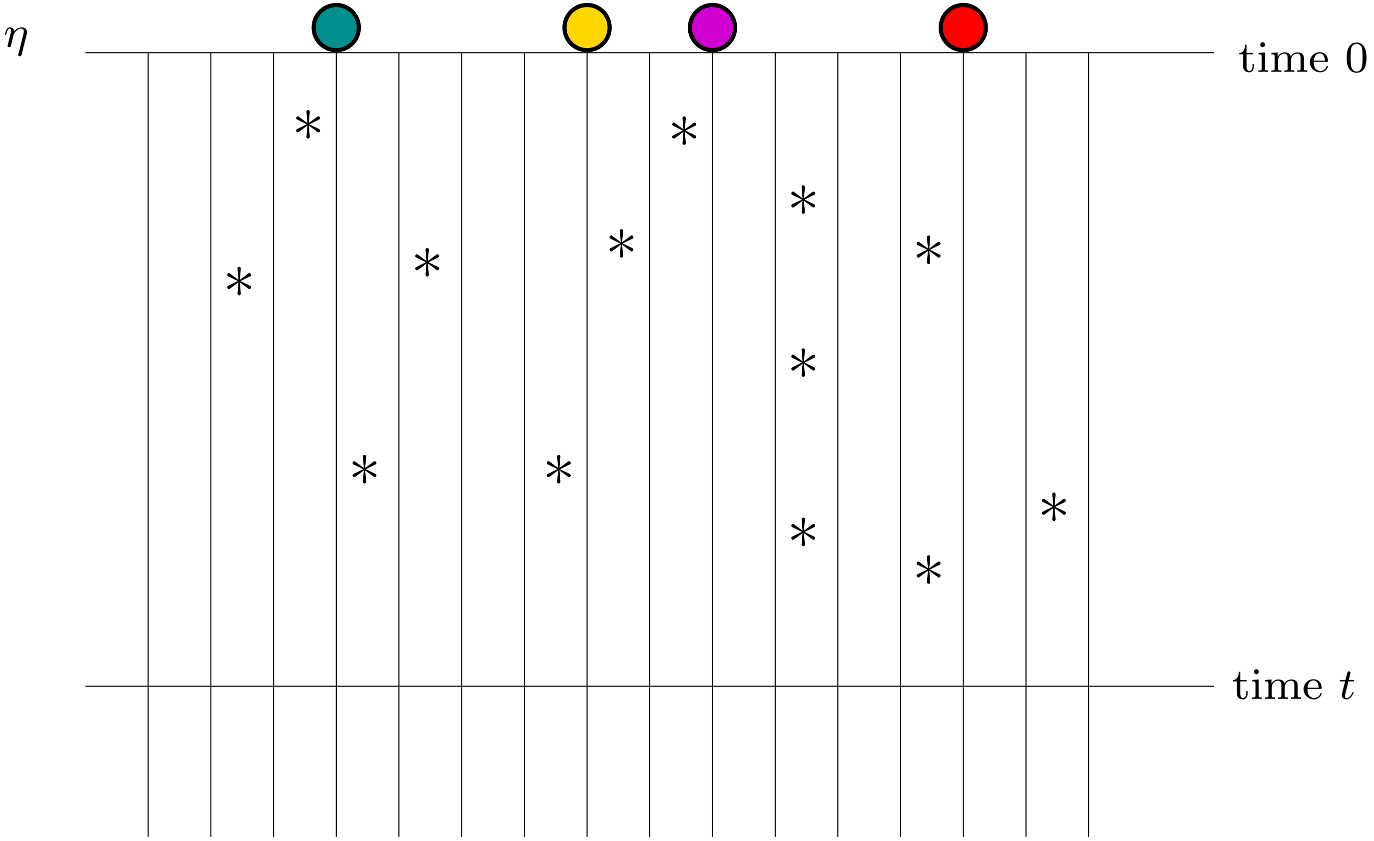
Generator:

$$Lf(\eta) = \sum_j \eta(j)(1-\eta(j-1))[f(\eta-\delta_j+\delta_{j-1})-f(\eta)]$$

where δ_j is the configuration defined by $\delta_j(i) = \mathbf{1}\{j=i\}$.

Graphical construction

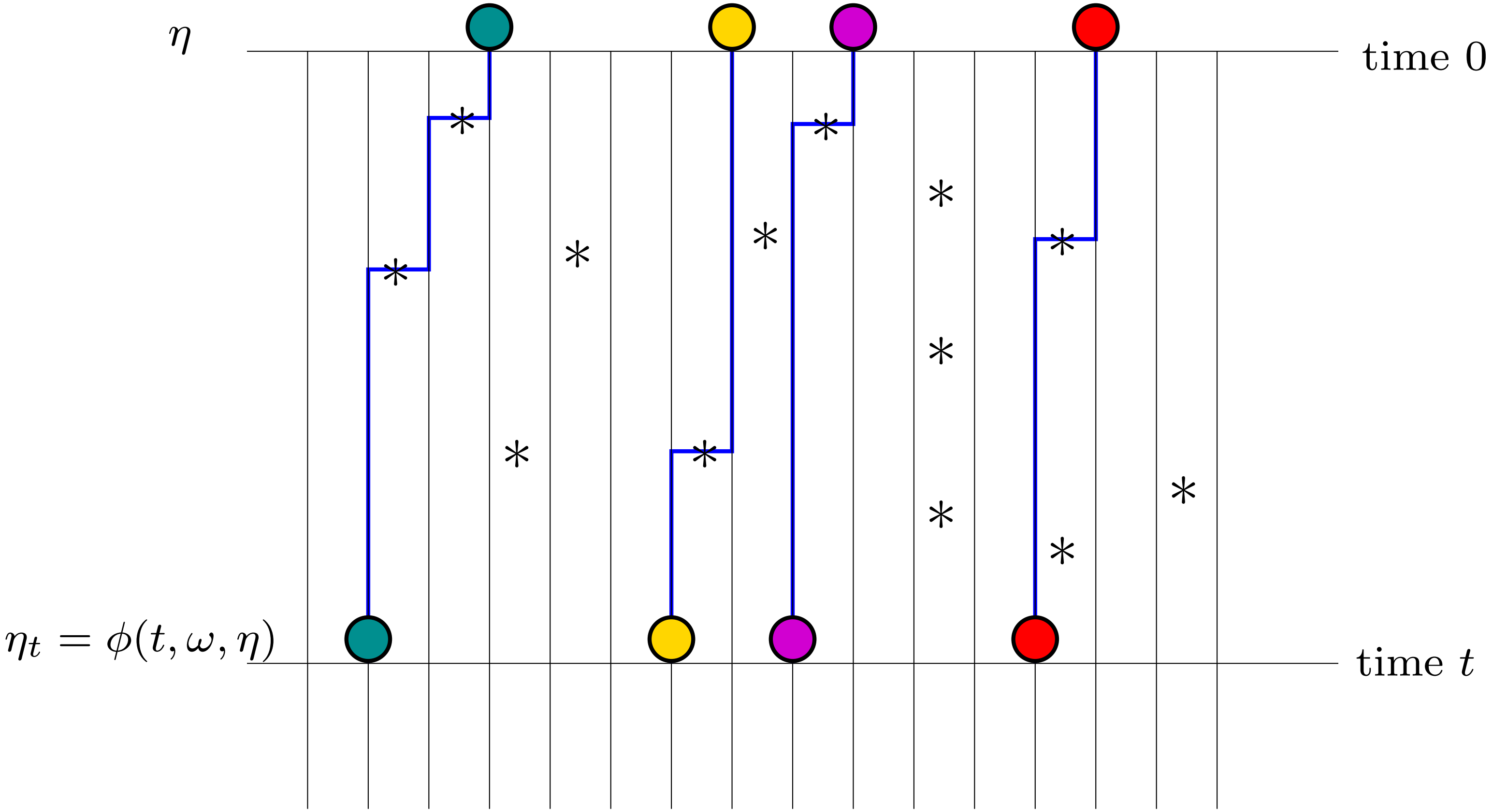
Initial η and Poisson points or marks ω on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$.



The * represent the events of the Poisson process ω.

Graphical construction

Poisson points or marks, on ω on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$.



$\eta_t = \phi(t, \omega, \eta)$ *governed* by ω with initial distribution η.

Lemma 4 *The Bernoulli measures ν^ρ (and mixtures of them) are invariant for the TASEP. (Also* blocking measures*).*

Proof. 1. Direct proof: $\nu^\rho L f = 0$.

2. Guess reverse process:

$L^* f(\eta) = \sum_j \eta(j-1)(1-\eta(j))[f(\eta - \delta_j + \delta_{j-1}) - f(\eta)]$

Check $\nu^\rho(fLg) = \nu^\rho(gL^*f)$ for indicators of cylinders. This is in general easier than to check $\nu^\rho Lf = 0$ because gLf is non null only when the cylinders associated to g and f differ at two points. □

Stationary version governed by ω with marginal law ν^ρ

$$\eta_t = \phi(t - s, \tau_s\omega, \eta_s) \tag{1}$$

for all $0 \leq s < t < \infty$.

Coupled TASEP

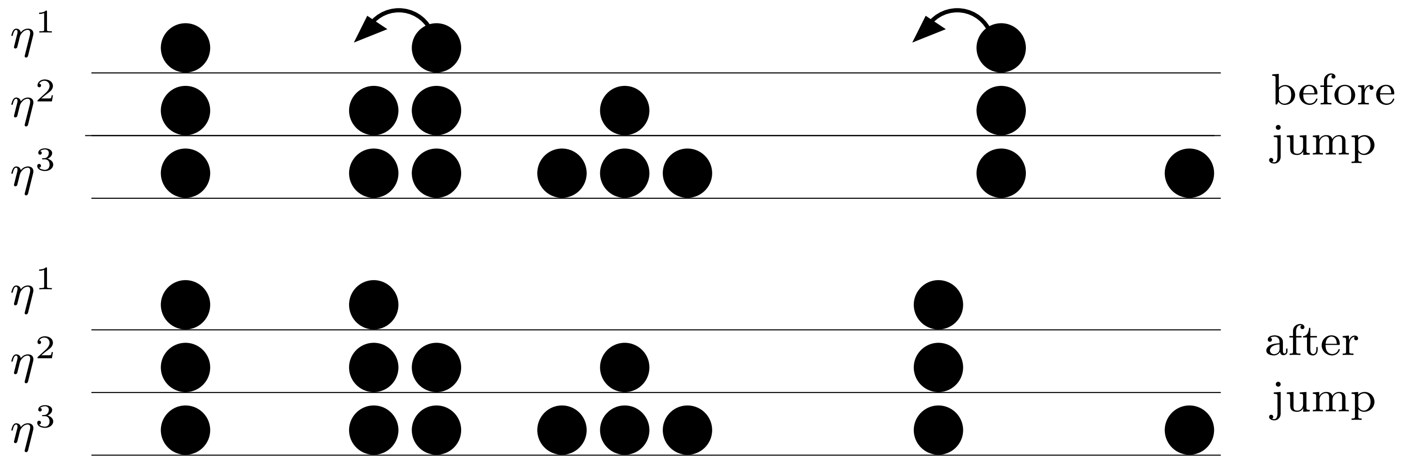


Figure 1: Coupling in TASEP

The basic coupling with initial configurations $\eta = (\eta_0^1, \ldots, \eta_0^n)$:

$\eta_t = (\eta_t^1, \ldots, \eta_t^n) = \phi^{(n)}(t, \omega, \eta_0)$, where

$(\phi^{(n)}(t, \omega, \eta_0))^k = \phi(t, \omega, \eta_0^k)$.

Multiclass process $\xi_t = R\eta_t$.

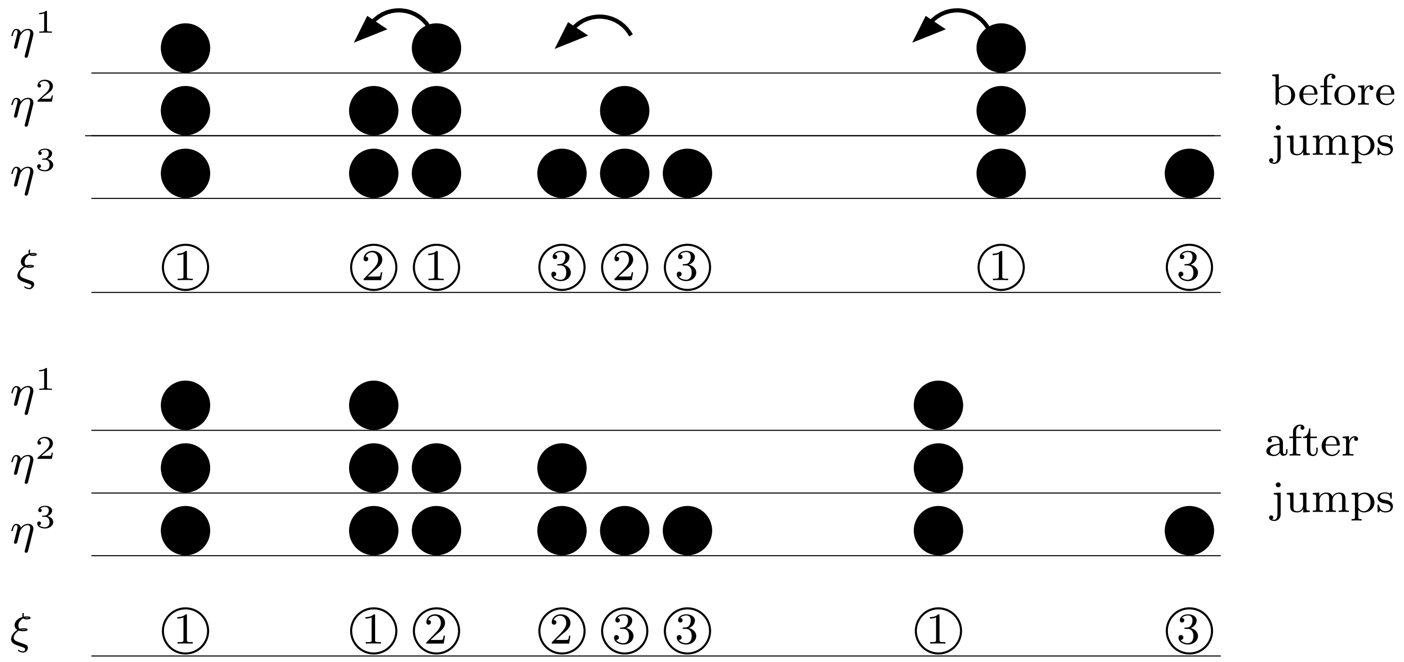


Figure 2: Coupled and multiclass TASEP

Discrete time stationary $M/M/1$ queue

A Bernoulli law parameter λ_1 (arrivals)

S Bernoulli law parameter λ_2, $\lambda_1 < \lambda_2$ (services)

$Z(t) - Z(s) = |A \cap [s,t]| - |S \cap [s,t]|$ stationary increments asymmetric random walk.

$Q(t) = Z(t) - \inf\{Z(s),\ s \le t\}$ stationary discrete time queue.

$U = \{t \in S : Z(t) < Z(s),\ \text{for all } s < t\}$. Records, unused services.

$D = S \setminus U$

Since there may be arrivals and departures at the same time, $Q(t)$ does not identify all arrivals and departures, but the formula

$Q(t) - Q(s) = A(t) - A(s) - (D(t) - D(s))$

still holds.

Departures and Unused services

Burke theorem still holds: D is Bernoulli (λ_1).

We have introduced a transformation:

Departures: $D = D(A, S)$.

Unused services: $U = U(A, S)$.

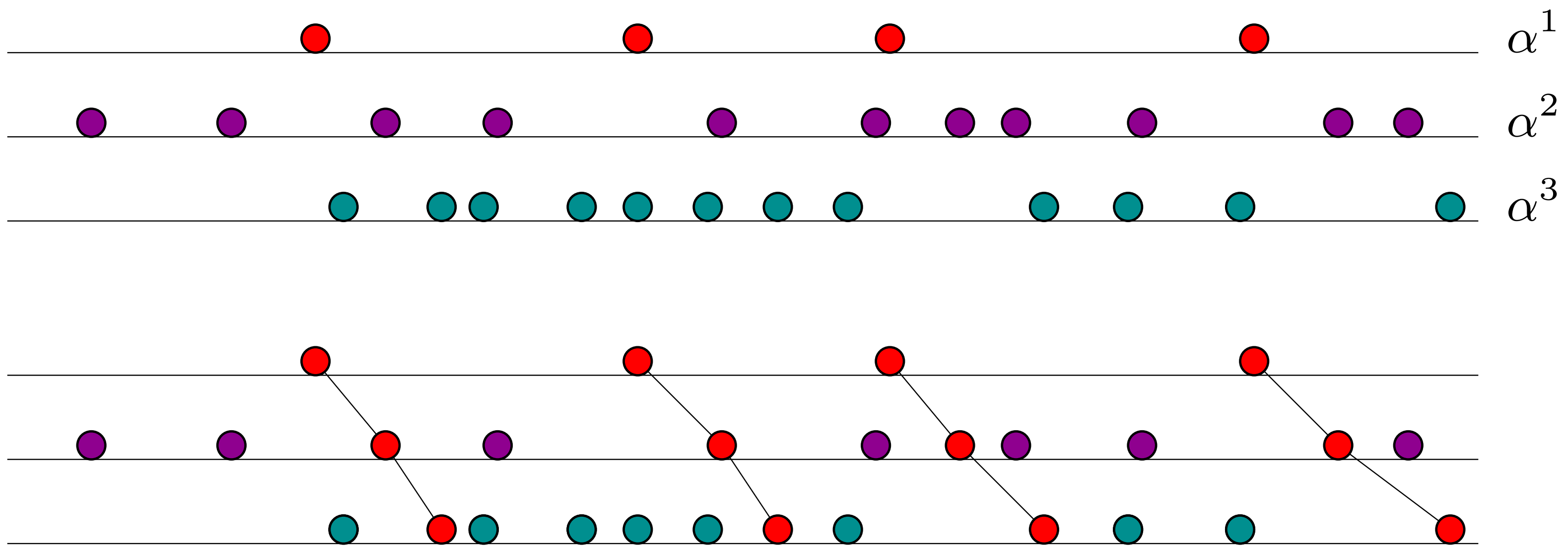
$D \cup A = S$

Multiclass queue

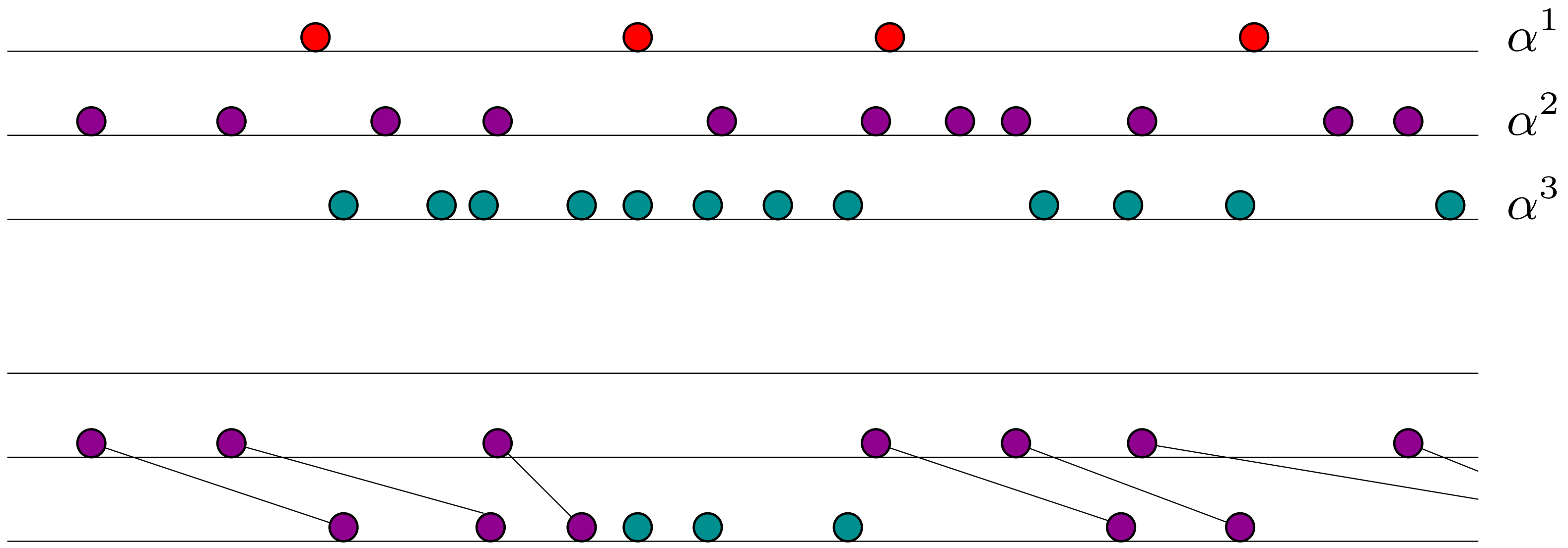
Take $(\alpha^1, \ldots, \alpha^n)$ iid Bernoulli with $0 < \rho^1 < \ldots < \rho^n < 1$.

α^1 are arrivals,

$\alpha^2, \ldots, \alpha^n$ are services at queues $1, \ldots, n-1$.



Define $\xi^1 = D(D(\alpha^1, \alpha^2), \alpha^3)$, departures at queue 2 of the customers arrived to queue 1 at times α^1.



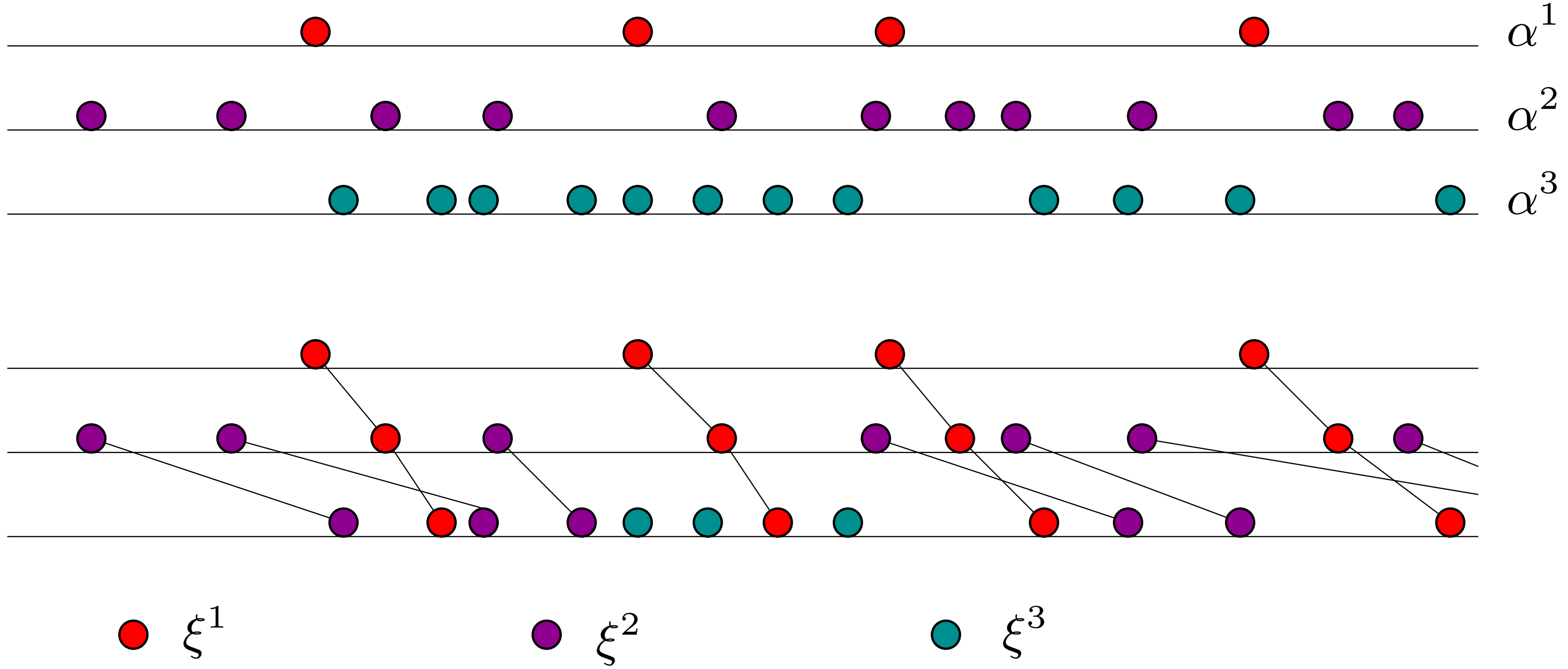
Consider the unused services of queue 1 as arrivals of customers of second class to queue 2 and define its departures from queue 2 as follows:

$\xi^2 = D(U(\alpha^1, \alpha^2), \alpha^3 \setminus \xi^1)$

The unused services of queue 2 are considered arrivals of customers of third class:

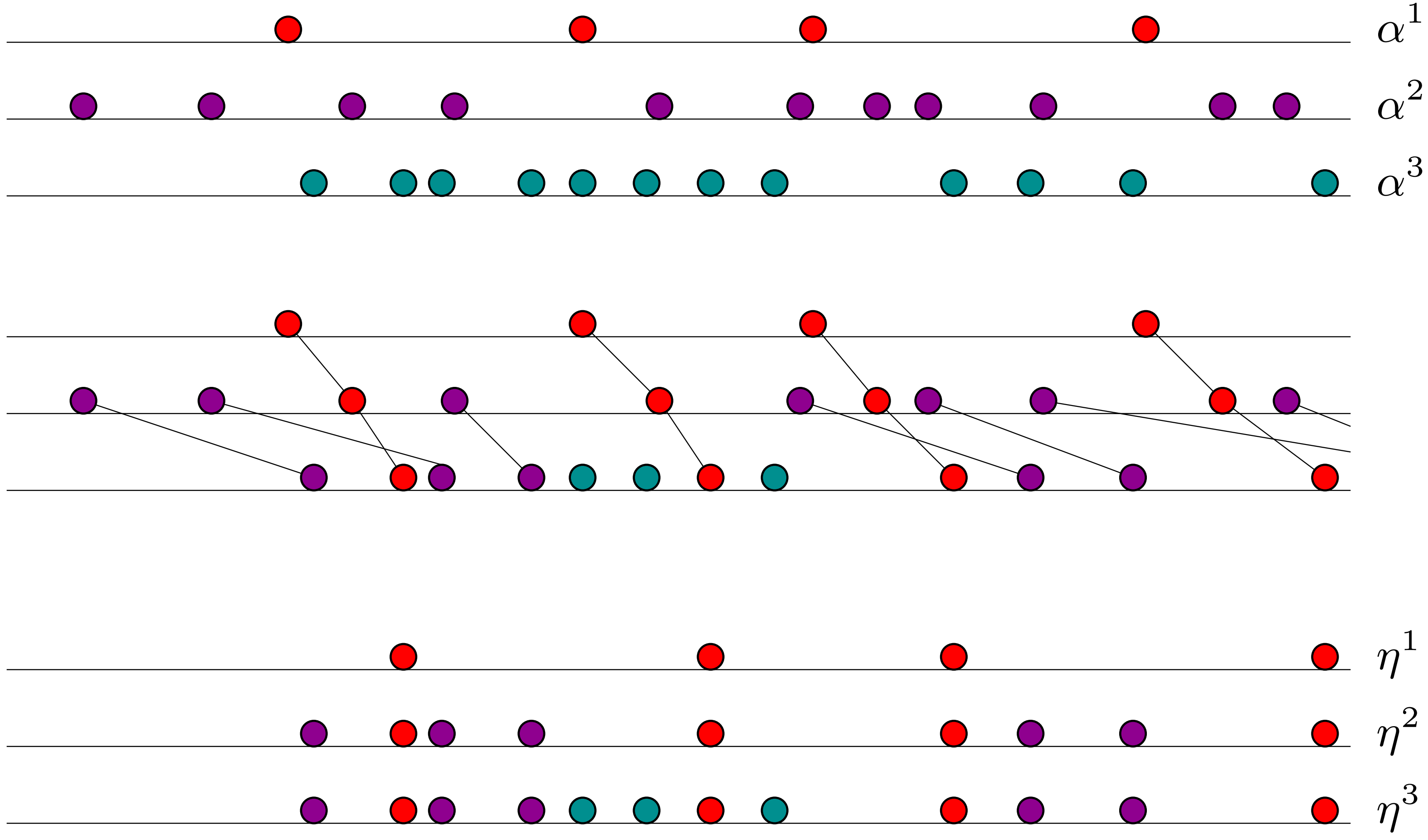
$\xi^3 = U(U(\alpha^1, \alpha^2), \alpha^3 \setminus \xi^1)$

The final picture is:



- We have defined the map $M\alpha = \xi$
- Under this map we have $\xi^1 \cup \xi^2 \cup \xi^3 = \alpha^3$.

From the multiclass queue to the coupled measure



Coupled measure

- Start with α independent Bernoulli of increasing densities.
- Construct a multiclass queue $\xi = M\alpha$
- Define $\eta^k := \xi^1 \cup \ldots \cup \xi^k$.
- It holds $\eta^1 \subset \ldots \subset \eta^n$.
- for each k, η^k has marginal distribution ν^{ρ^k} (Burke).
- We have constructed the map $T\alpha = \eta$.
- Define $\pi = T\nu$ the induced distribution of η.

Invariance of μ

Theorem 5

Let $\alpha = (\alpha^1, \ldots, \alpha^n)$ *have law* ν*, product of Bernoulli with densities* $\rho^1 < \ldots < \rho^n$*. Then*

- π*, law of* $T\alpha$*, is invariant for coupled* TASEP (η_t)
- μ*, law of* $M\alpha$*, is invariant for multiclass* TASEP (ξ_t).

The case $n = 2$

• Computed by Derrida, Janowsky, Lebowitz and Speer (1993) using **matrix method**.

• F. Fontes Kohayakawa (1994), probabilistic construction exploiting **renewal structure**, Speer (1994) **uniqueness**

• Duchi and Schaeffer (2005). Angel (2005) **introduce the map** T without the queuing interpretation.

• **Queueing interpretation** F. and Martin (2005).

General case

• F. and Martin (2005) introduce the maps T and M for the **multiclass** process.

• Evans, F., Mallik (2008) compute the weight of a given multiclass configuration using *tensors* which count number of compatible queues.

Sketch of proof Since $M = RT$, the statements are equivalent.

We first introduce a new dynamics $\alpha_t = (\alpha_t^1, \ldots, \alpha_t^n)$ *multi-line process*, and then show:

1) The product measure ν is invariant for the multi-line process α_t.

2) $T\alpha_t$ is the coupled process η_t. $\square$

Dual points in TASEP

Density ρ fixed

ω be a Poisson process on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$.

Let $(\eta_t, t \in \mathbb{R})$ be the TASEP trajectory governed by ω

Define $\Delta_\rho(\omega)$ as the dual points for the TASEP trajectory with density ρ governed by ω:

$$\begin{aligned} \Delta_\rho(\omega) &= \{(x,t) \in \omega : \eta_{t-}(x + \frac{1}{2}) = 1\} \\ &\qquad \cup \{(x+1,t) : (x,t) \in \omega \text{ and } \eta_{t-}(x + \frac{1}{2}) = 0\}. \end{aligned}$$

See Figure 3.

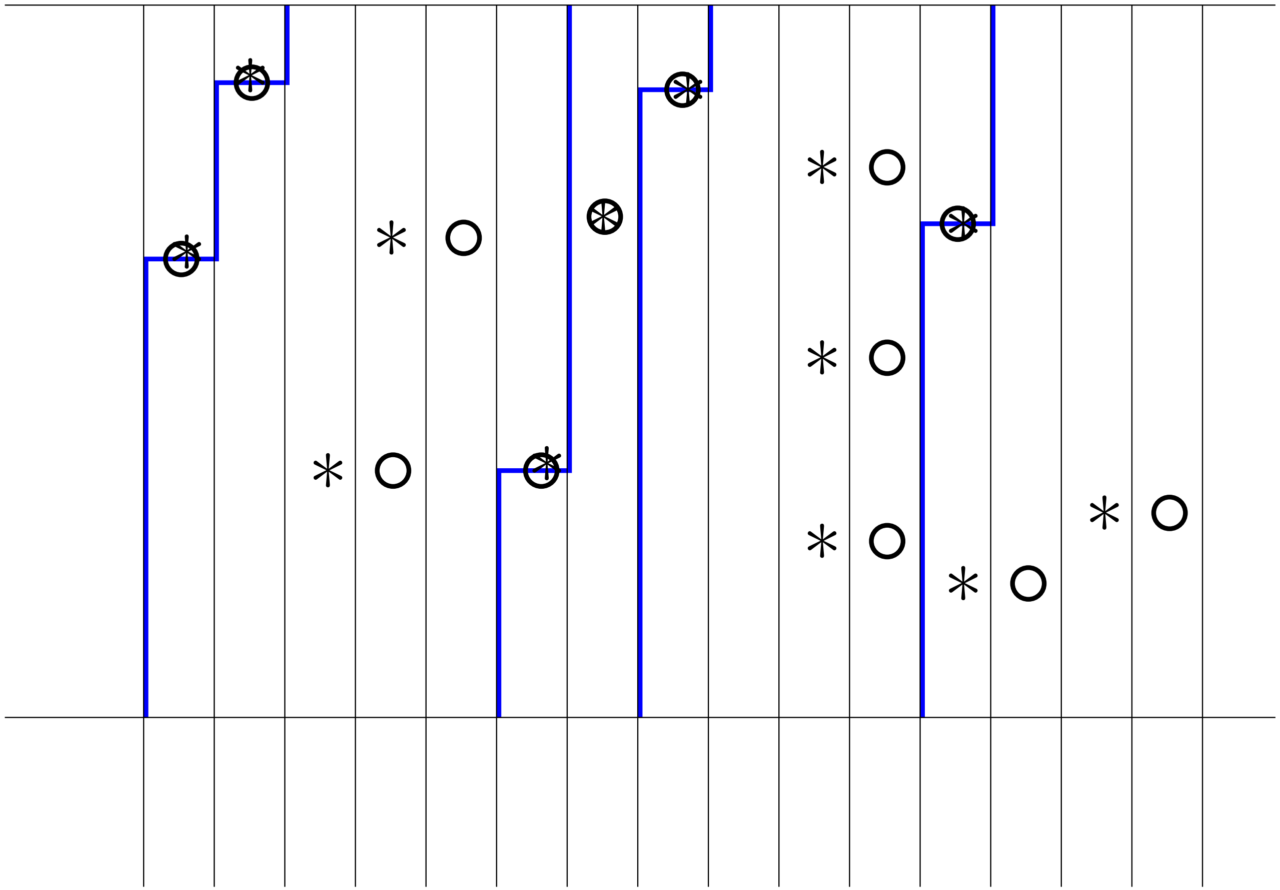


Figure 3: Circles represent the dual points of the TASEP trajectory. Turning the picture upside-down exchanges the roles of circles and stars.

Properties of dual points

- Dual points $\Delta_\rho(\omega)$ govern the time-reversal of the trajectory.

- Time reversed trajectory is TASEP with jumps to right.

- The law of the dual points $\Delta_\rho(\omega)$ is also Poisson in $\mathbb{R} \times \mathbb{Z}$.

- $\{(x, s) \in \Delta_\rho(\omega) : \, s < t\}$, the set of dual points in the past of t, is independent of the configuration η_t.

Proof: Along Reich proof of Burke, Cator and Groeneboom (2005). Need two spin flip dynamics to mark dual points missed by the trajectory. TASEP plus spin-flips identify all points in ω. Their reverse process identify the totality of the dual points. Since same law, dual points must be Poisson. Dual points are in the future of the reverse process $\eta^*_{-t} = \eta_t$, hence independence by Markov property.

Multi-line TASEP process

$\alpha_t = (\alpha_t^1, \ldots, \alpha_t^n)$ in $\mathcal{X}^n$ governed by ω.

Let $\rho^1 < \ldots < \rho^n$.

Let $\omega^n = \omega$, and ω^k dual points of ω with respect to ρ^{k+1}:

$$\omega^k = \Delta_{\rho^{k+1}}(\omega^{k+1})$$

is Poisson on $\mathbb{R}^2$.

The "kth line" of $(\alpha_t^k, t \in \mathbb{R})$ is TASEP trajectory with density ρ^k governed by ω^k

Each line is a TASEP trajectory governed by the dual points produced from the line below.

Product of Bernoulli is invariant for multiline process

Proposition 6 *The multi-line TASEP process* $(\alpha_t, t \in \mathbb{R})$ *is stationary, and the distribution of* α_t *for each* t *is the product measure* $\nu = \nu^{\rho^1} \times \ldots \times \nu^{\rho^n}$.

Proof By construction, process is stationary and the marginal distribution of α_t^k is ν^{ρ^k} for any k and t.

Need to show $\alpha_t^1, \alpha_t^2, \ldots, \alpha_t^n$ are independent.

Let $2 \leq k \leq n$.

$\alpha_t^k, \ldots, \alpha_t^n$ is independent of the set of dual points

$$\{(x, s) \text{ in } \Delta_{\rho^k}(\omega^k) \text{ such that } s < t\}.$$

because they govern the future of the dual process.

But $(\alpha_s^{k-1}, s \leq t)$ is a function of precisely this set of dual points. $\square$

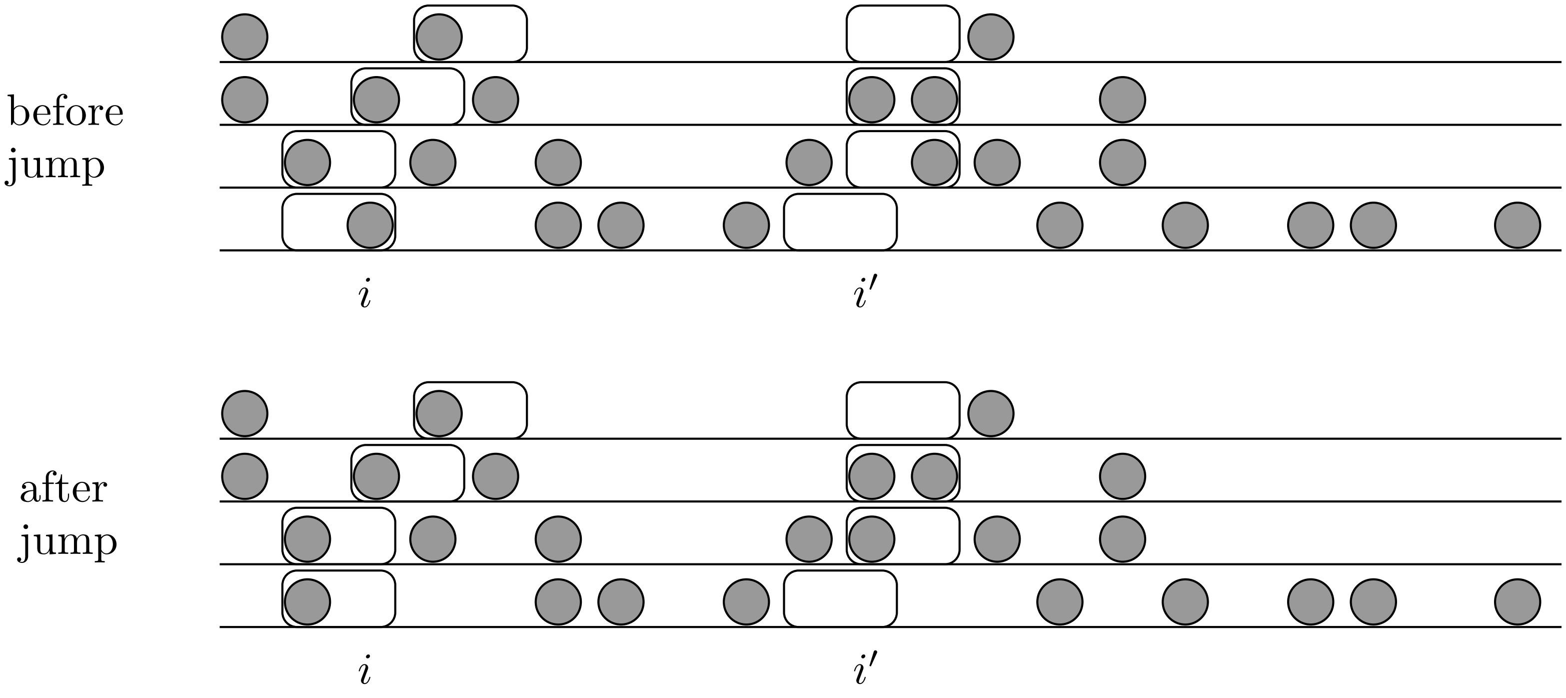


Figure 4: **Multi-line process and time reversal.** The effect of two possible bells, at i and i'. For i, the bells ring at $b_4 = b_3 = i$, $b_2 = i + 1$ and $b_1 = i + 2$, and $b_0 = i + 3$. The only particle which jumps is on the bottom line, due to the bell at i. For i' the bells ring at $b'_4 = i'$ and $b'_3 = b'_2 = b'_1 = i' + 1$, and $b'_0 = i' + 2$. A particle in the third line jumps due to this bell.

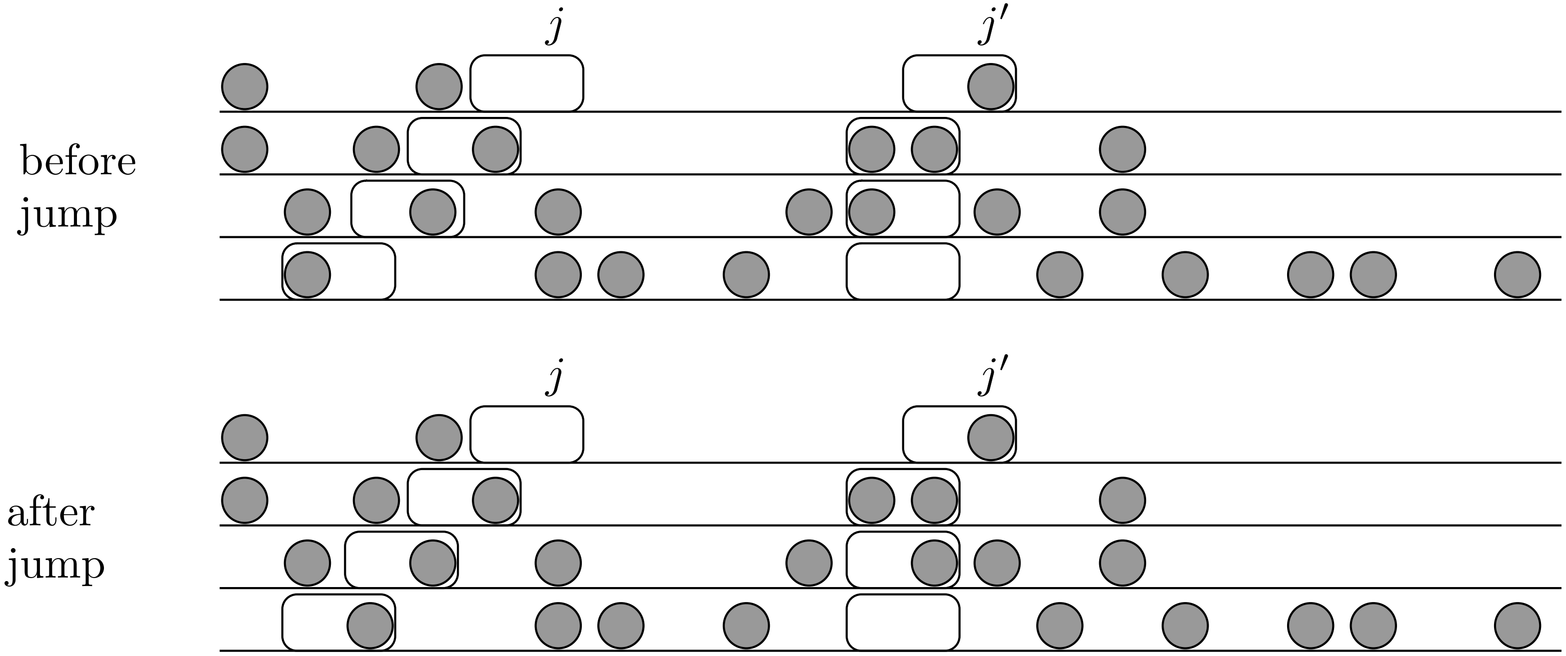


Figure 5: **Reverse multi-line process.** The two reverse jumps corresponding to the forward jumps in Figure 2. The corresponding reversed bells ring at sites j and j'. For j, the bells ring at $c_1 = j$, $c_2 = j - 1$, $c_3 = j - 2$ and $c_4 = j - 3$; also $c_5 = j - 3$. Again, only one particle, on the bottom line, jumps due to the bell at j. For j' the bells ring at $c'_1 = j'$ and $c'_2 = c'_3 = c'_4 = j' - 1$, with $c'_5 = j' - 2$. Again, a particle in the third line jumps due to this bell.

Image of multi-line process under map T is coupled process

Proposition 7

Let $0 < \rho^1 < \ldots < \rho^n < 1$, *and*

let $(\alpha_t, t \in \mathbb{R})$ *multiline TASEP governed by* ω

Let $\eta_t = T\alpha_t \in \mathcal{X}^{n\uparrow}$.

Then for each k, $(\eta_t^k, t \in \mathbb{R})$ *is the TASEP trajectory governed by* ω.

Sketch of Proof From the definition of T:

$$\eta_t^k = D^{(n-k+1)}\left(\alpha_t^k, \dots, \alpha_t^n\right). \tag{2}$$

We know that η_t^k has distribution ν^{ρ^k}.

Need to show that RHS of (2) is a TASEP governed by ω.

Since $(\alpha^k, \dots, \alpha^n)$ is multi-line governed by ω, enough to show that $D^{(n)}(\alpha_t^1, \dots, \alpha_t^n)$ is TASEP governed by ω.

$D^{(n)}$:= departures of first class customers at nth queue. Induction:

$$D^{(n)}(\alpha_t^1, \dots, \alpha_t^n) = D^{(2)}\left(D^{(n-1)}(\alpha_t^1, \dots, \alpha_t^{n-1}), \alpha_t^n\right)$$

and the fact that $(\alpha_t^1, \dots, \alpha_t^{n-1})$ is an $(n-1)$-line multiline process governed by ω^{n-1}.

This concludes the recursion argument.

$n = 2$:

need to show $D(J(\alpha^1, y), J(\alpha^2, x)) = J(D(\alpha^1, \alpha^2), x)$, where the configuration $J(\alpha^1, y)$ is the configuration obtained when a ω mark appear between $y - 1$ and y defined by

$J(\alpha^1, y)(y-1) = 1 - J(\alpha^1, y-1)(x) = 1$ if
$\alpha^1(y) = 1 - \alpha^1(y-1) = 1$,

$J(\alpha^1, y)(x) = \alpha^1(x)$ otherwise

Check a small number of cases. $\square$

Invariant measure is fix point of priority queue

Fix n and $\rho^1, \ldots, \rho^n$, and let $m < n$.

Let $\xi^{(n)}$ be distributed according to $\mu^{(n)}_{\rho^1,\ldots,\rho^n}$ (multiclass with n classes) and $\xi^{(m)}$ according to $\mu^{(m)}_{\rho^1,\ldots,\rho^m}$.

Let the truncated configuration $[\xi^{(n)}]^m$ be defined by

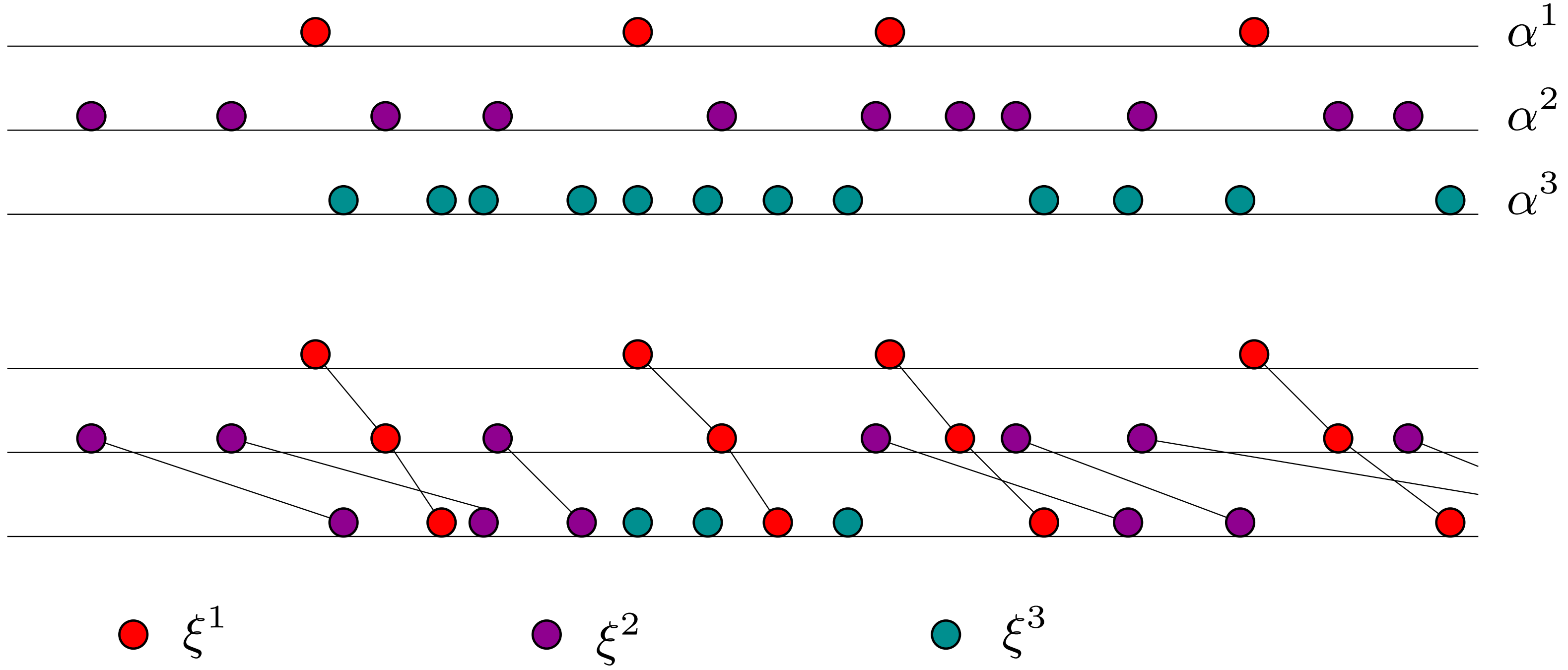
$$[\xi^{(n)}]^m(i) = \min\left\{\xi^{(n)}(i), m+1\right\}.$$

Theorem 8 (F. Martin) *The m-class input process to queue m has the same law as the m-class departure process from the same queue:*

$$\xi^{(m)} \text{ has the same distribution as } [\xi^{(n)}]^m, \tag{3}$$

Proof: follows from uniqueness of the multiclass invariant measure with fixed marginals. Martin and Prabhakar prove this using a result of queuing theory.

Multiclass Burke.



Law of (ξ^1, ξ^2) in second line is the same as law of (ξ^1, ξ^2) in third line.

Other properties of invariant measure Take $n = 2$

Let multiclass measure $\mu' = \mu(\cdot|\xi^2(0) = 1)$ the measure conditioned to have a second class particle at the origin.

Then under μ', the second class particles are regeneration points: calling x_i the positions of the second class particles,

$\xi^1 \cap (x_i, x_{i+1}$ are iid. The weight of $\xi^1 \cap (x_i, x_{i+1}$ is giving by "pushing procedure" $W(001) = 1$, $W(1100) = 5$, number of configurations of zeros and ones that can be obtained by pushing ones to the right.

The positions of the second class particles are the records of rw.

The distribution of first class particles to the left of x_0 is product ρ^1.

The distribution of holes to the right of x_0 is product $(1 - \rho^2$

Coupling between μ' and μ:

Q invariant Q' conditioned to have an unused service at the origin.

3 Hydrodynamics and second class particles

This lecture relates hydrodynamics with the microscopic properties of the TASEP.

I will give a short description of the hydrodynamic theorems. Ellen Saada will prove these theorems in the second part of the course.

I describe some important properties of the limiting Burgers equation that can be found in the paper of Lax.

The goal is to show how the second class particles represent shocks and characteristics of the Burgers equation.

Non viscous Burgers equation

$u(r,t) \in [0,1]$, $r \in \mathbb{R}$, $t \geq 0$.

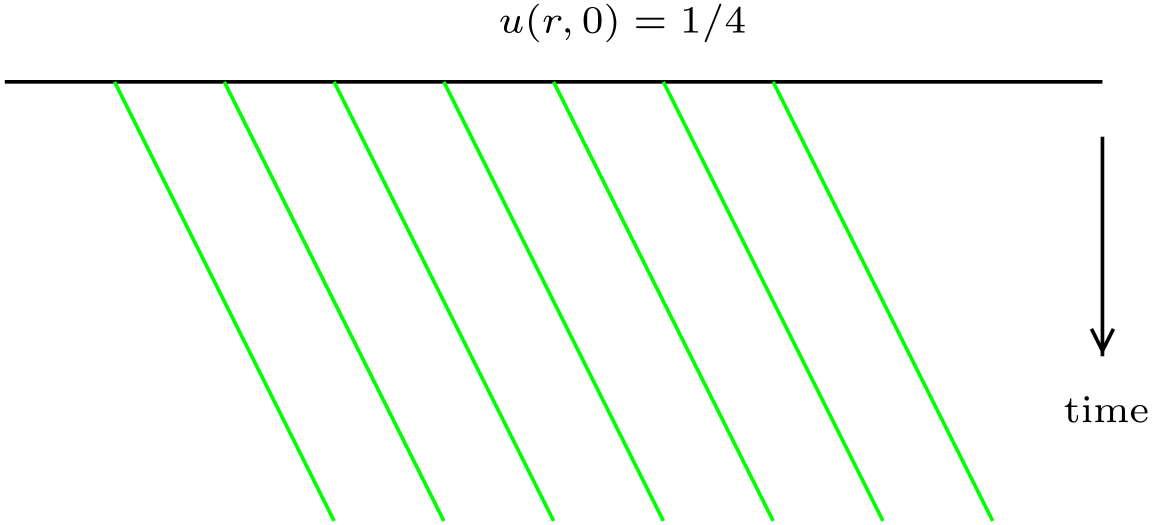
$$\frac{\partial u}{\partial t} = -\frac{\partial(u(1-u))}{\partial r} = -(1-2u)\frac{\partial u}{\partial r}$$

Characteristics

$$\begin{cases} \dfrac{dW_a(t)}{dt} = 1 - 2u(W_a(t), t) \\ W_a(0) = a \end{cases}$$

$u(r,t)$ is constant along the characteristics.

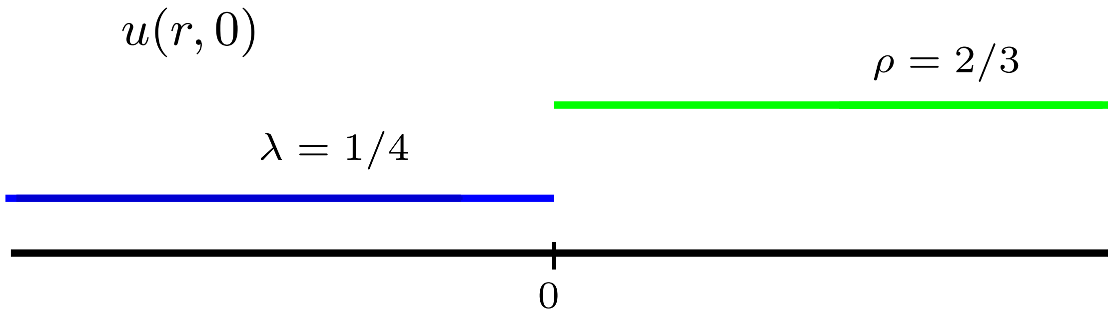
Characteristics



For constant solutions $u(r,t) = \rho$, velocity $1 - 2\rho$:

$$W_a(t) = a + (1 - 2\rho)t$$

Initial condition



$$u(r,0) = \begin{cases} \lambda & r > 0 \\ \rho & r \leq 0 \end{cases}$$

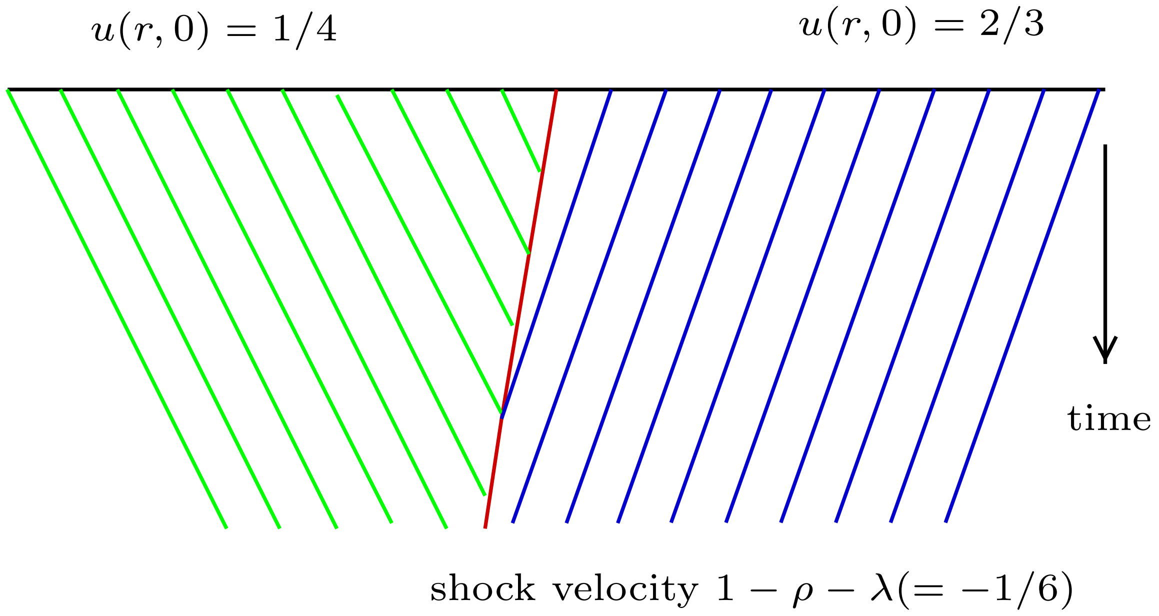
Two cases: 1) Stationary shocks $\lambda < \rho$

2) rarefaction fan $\lambda > \rho$

1): When characteristics conflict: **shock**

$$\textbf{Solution:} \quad u(r,t) = u(r - vt, 0)$$

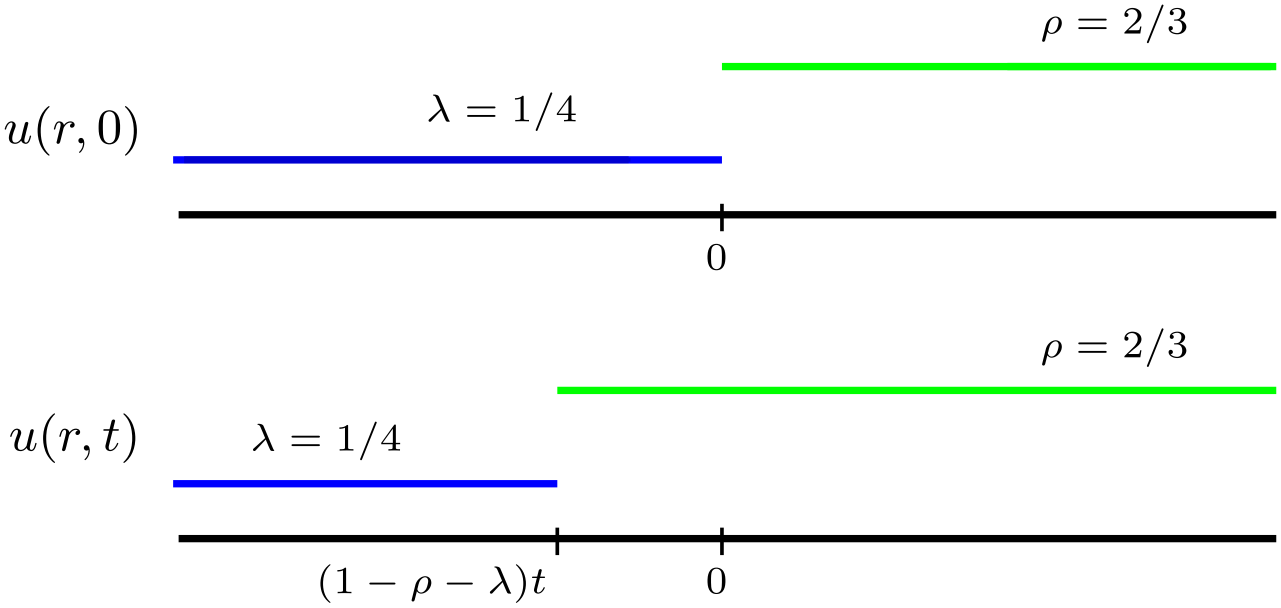
where $v = 1 - \lambda - \rho$.



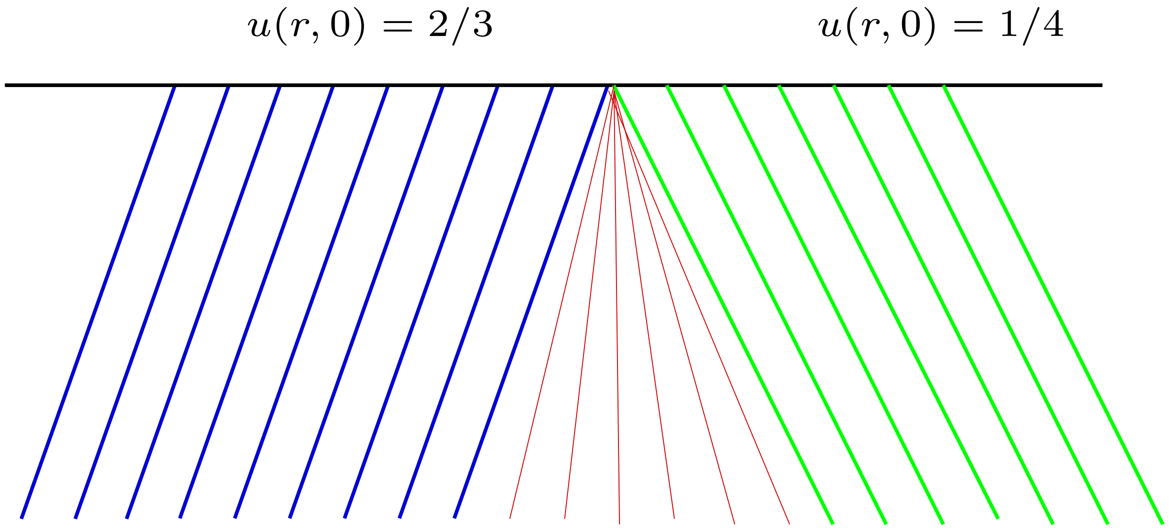
Shock solution

$$u(r,t) = u(r - vt, 0)$$

where $v = 1 - \lambda - \rho$.



2) Rarefaction fan $\lambda > \rho$



rarefaction fan

Solution

$$u(r,t) = \begin{cases} \lambda & r < 1 - 2\lambda t \\ \text{interpolation} & r \text{ in the middle} \\ \rho & r > 1 - 2\rho t \end{cases}$$

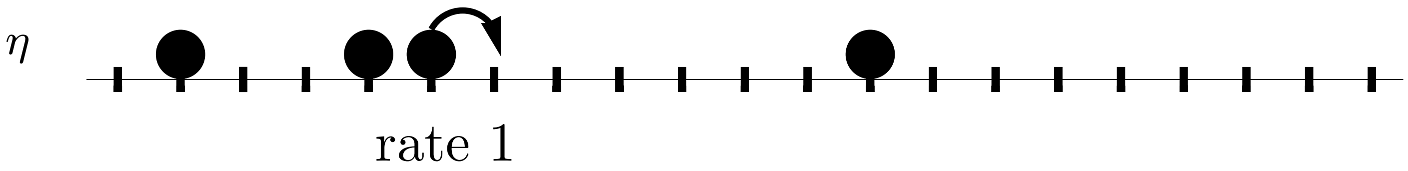
2) Rarefaction fan $\lambda > \rho$

$$u(r,t) = \begin{cases} \lambda & r < 1 - 2\lambda t \\ \text{interpolation} & r \text{ in the middle} \\ \rho & r > 1 - 2\rho t \end{cases}$$

(Non uniqueness, this is the entropic solution).

TASEP with jumps to right

$\eta_t(x) \in \{0,1\}$, t time (real), x integer.



Invariant measures: **Product** measures ν_ρ density $\rho \in [0,1]$

Blocking measures ν_n concentrating mass in

$$\underset{}{\dots,0,0,0,}\underset{n}{1},1,1,\dots$$

(and convex combination).

Hydrodynamics.
Connection between micro and macro

Scale parameter ε (to go to zero)

$u^\varepsilon(r,t) := \mathbb{E}\eta_{t/\varepsilon}(r/\varepsilon)$

Heuristics:

$$\begin{aligned}\frac{\partial u^\varepsilon(r,t)}{\partial t} \quad &= \quad \frac{1}{\varepsilon}\mathbb{E}\Big[-\eta_{t/\varepsilon}(\frac{r}{\varepsilon})(1-\eta_{t/\varepsilon}(\frac{r+\varepsilon}{\varepsilon})) \\ &\qquad + \eta_{t/\varepsilon}(\frac{r-\varepsilon}{\varepsilon})(1-\eta_{t/\varepsilon}(\frac{r}{\varepsilon}))\Big]\end{aligned}$$

converges to

$$\frac{\partial u}{\partial t} = -\frac{\partial(u(1-u))}{\partial r}$$

assuming independence of $\eta_t(x)$ and $\eta_t(x+1)$!!!

Hydrodynamic theorem (Law of large numbers)

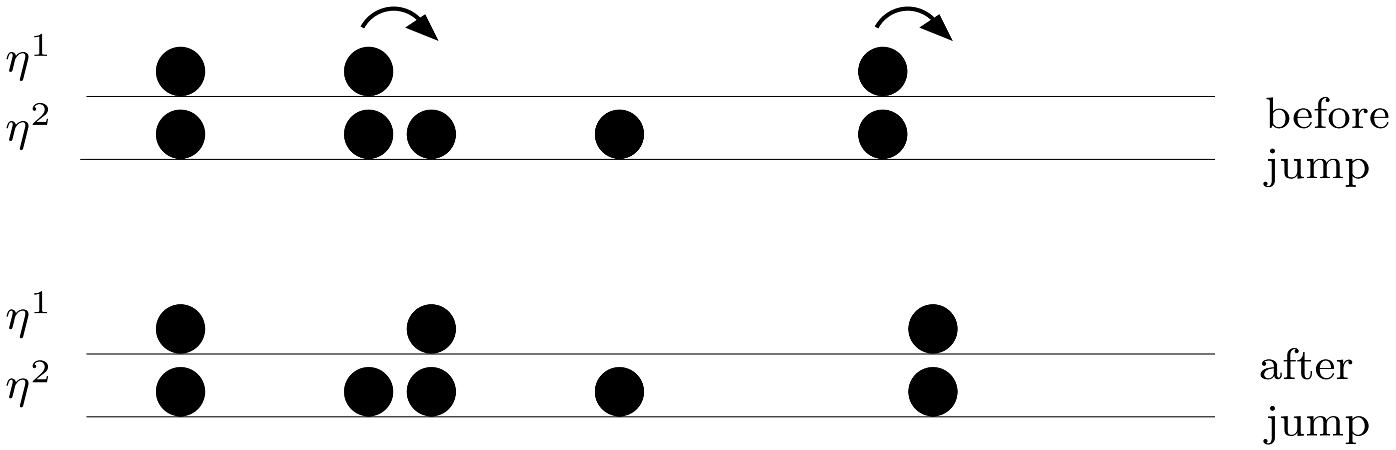
$$\varepsilon \sum_{k=r/\varepsilon}^{s/\varepsilon} \eta_{t/\varepsilon}(k) \to \int_r^s u(z,t)dz$$

for a family of initial product distributions with profile $u(\cdot,0)$.

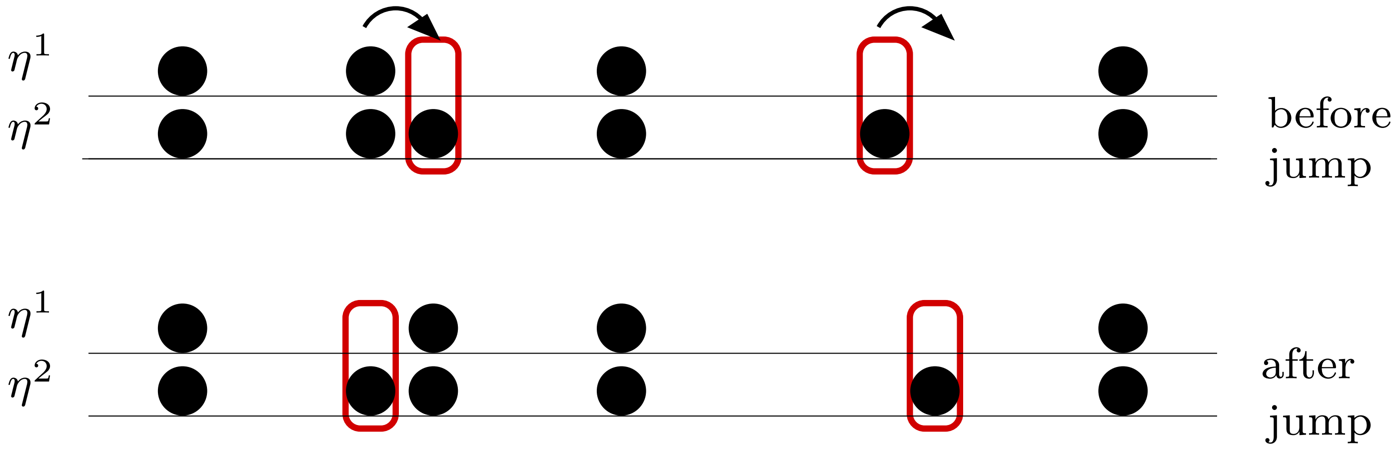
Local equilibrium Convergence in distribution:

$$\tau_{r/\varepsilon}\eta_{t/\varepsilon} \longrightarrow \nu_{u(r,t)}$$

Recall coupling Realization of two process with different initial configurations with the same Poisson bells: (η_t^1, η_t^2)



If at time zero there is **only one discrepancy**:



then at future times $t > 0$ there will also be one discrepancy at a position called $X(t)$:

$$X(t) := \sum_x x\mathbf{1}\{\eta_t^1(x) \neq \eta_t^2(x)\}$$

We call

$X(t)$ Position of the **second class particle**

Second class particles

- Identifies shock: The process as seen from a second class particle has *shock invariant measures* with densities λ and ρ for all $\lambda < \rho$.
- **Follows characteristics and shocks:**

$$\lim_{t\to\infty} \frac{X(t)}{t} = \begin{cases} 1-\rho-\lambda & \text{if } \lambda \le \rho \\ U & \text{if } \lambda > \rho \end{cases} \quad \text{almost surely,} \tag{4}$$

where $U \sim$ uniform in $[1-2\lambda, 1-2\rho]$.

Central limit Theorem In the shock case: $\lambda < \rho$:

$$\frac{X_t - vt}{\sqrt{t}} \to \text{Normal}(0, \sigma^2)$$

$$\sigma^2 = \frac{\rho(1-\rho) + \lambda(1-\lambda)}{\lambda - \rho}$$

Dependence on the initial configuration Define $N_t(\eta) =$ number of holes of η in $[0, \rho - \lambda]$ minus number of particles of η in $[-(\rho - \lambda), 0]$.

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(X_t - N_t/(\lambda - \rho))^2 = 0$$

This result follows from the fluctuations of the flux.

Dynamic phase transition

Start with product measure with densities λ, ρ

$$\tau_{r/\varepsilon}\nu_{\lambda,\rho}S(t/\varepsilon) \to \begin{cases} \nu_\lambda, & r < vt \\ \nu_\rho, & r > vt \end{cases}$$

At $r = vt$ the system converges to a convex combination of both densities.

Fluctuations For $\lambda < \rho$,

$$\frac{\mathbb{E}(X_t - vt)^2}{t} \to \sigma^2$$

When $\lambda = \rho$ the variance is infinity. In this case, $\mathbb{E}X_t = 1 - 2\rho$ and

$$\frac{\mathbb{E}(X_t - (1-2\rho)t)^2}{t^{4/3}} \sim \text{constant}$$

Johanson; Patrick Ferrari and Spohn, Balazs-Cator-Seppalainen (EJP 11, 46).

Tracy-Widom distribution.

Correspondence between variance of flux and first absolute moment of second class particle (Ferrari Fontes, see generalization in Balazs-Seppalainen)

$$VJ_{(1-2\rho)t}(t) = \rho(1-\rho)\mathbb{E}|X_t - (1-2\rho)t|$$

Initial measure product ν^ρ; $J_r(t)$ is the flux: number of particles to the left of the origin at time zero and to the right of r at time t.

$1-2\rho$ is the speed of the characteristic for density ρ.

$$VJ_{(1-2\rho)t}(t) = \rho(1-\rho)\mathbb{E}|X_t - (1-2\rho)t|$$

Variance of flux (or current) Initial measure ν^ρ, $r \neq 1-2\rho$.

$$VJ_{rt}(t) = \rho(1-\rho)|r-(1-2\rho|$$

(zero when $r = 1-2\rho$). Scales as $t^{2/3}$ when $r = 1-2\rho$.

Second class particle identifies shock Let η' be a TASEP configuration starting with a second class particle at the origin. Let $X(t)$ be the position of the second class particle; $X(0) = 0$.

Theorem 9 *Let $\lambda < \rho$. Then there exists a distribution μ' of η'_0 with the following properties:*

1) Under μ' there is only a second class particle and it is at the origin.

2) μ' is invariant for the process $\eta'_t = \tau_{X(t)}\eta_t$.

3) μ' approaches the product distributions ν_ρ and ν_λ exponentially fast to the right and left of the origin.

Construction of μ' Let (ξ^1, ξ^2) be a two-class configuration chosen with the two-class invariant measure μ_2 for densities $\rho^1 = \lambda$ and $\rho^2 = \rho$ (stress sub-index 2 in the notation, this measure was called just μ in previous lectures, as $n = 2$ was sub-understood):

ξ^1 first class particles

ξ^2 second class particles

Let $\mu_2' = \mu_2(\cdot|\xi^2(0) = 1)$, the measure conditioned to have a second class particle at the origin.

Let $S_2(t)$ the semigroup describing the law of the two class process at time t

$S_2'(t)$ the semigroup of the process as seen from the second class particle.

Theorem 10 (Harris) *If μ_2 is translation invariant, then*

$$(\mu_2 S_2(t))' = \mu_2' S_2'(t)$$

For each t, the translation invariant process conditioned to have a second class particle at the origin has the same law as the process as seen from a second class particle.

Let $X(t)$ be the position of the tagged second class particle initially at the origin for the process $(\xi_t^1, \xi_t^2)'$ with initial distribution μ_2',

Harris theorem implies $(\xi_t^1, \xi_t^2)' = \tau_{X(t)}(\xi_t^1, \xi_t^2)$ has law μ_2' for all t.

Consider the operator $T(\xi_t^1, \xi_t^2) \mapsto \eta$, which erases the second class particles to the left of the origin and identifies the first and second class particles to the right of the origin:

$[T(\xi_t^1, \xi_t^2)](x) = \xi^1(x) + \xi^2(x)\mathbf{1}\{x > 0\} + 2\mathbf{1}\{x = 0\}$

$\eta_0' = [T(\xi_0^1, \xi_0^2)]$ implies $\tau_{X(t)}\eta_t' = T(\xi_t^1, \xi_t^2)'$

That is, if process starts with $T\mu_2'$ then η_t' has law $T\mu_2'$ for all t.

Proposition 11 *$T\mu_2'$ approaches exponentially fast $T\nu_2$:*

$$|T\mu_2'\tau_x f - T\nu_2\tau_x f| < C' e^{-|x|C}$$

where τ_x is translation by x.

Proof. We exhibit a coupling η', η with marginals $T\mu_2'$ and $T\nu_2$ such that

$$\mathbb{P}(\eta(y) \neq \eta(y), \text{ for some } y \text{ with } |y| > |x|) < e^{-|x|}$$

In terms of the queue:

$T\mu_2'$ is the law of $(D \cap (-\infty, 0)) \cup (S \cap (0, \infty)$ given $0 \in U$.

$T\nu_2'$ is the law of $(D \cap (-\infty, 0)) \cup (S \cap (0, \infty)$.

To couple we fix the queue in 0 for the first case and in a random geometric position in the second one. Then run both processes backwards and forwards with the same marks.

4 Last passage percolation – Growth model

In this section we prove Rost Theorem using law of large numbers for the TASEP.

Then we prove the convergence in distribution of the second class particle in the rarefaction front to a uniform random variable (F. Kipnis)

Finally, we establish the relation between the second class particle and a competition interface and indicate how to show almost sure convergence of the second class particle in the rarefaction front (F., Martin, Pimentel).

Law of large numbers for tagged particle

We consider TASEP with jumps to the right with initial product measure ν^ρ (this is an invariant measure).

Add a particle at the origin if there is no one and tag the particle at the origin. Let Y_t position of tagged particle at time t.

Correspondence with system of queues: Label the particles: let $y_t(i)$ be the position of the ith particle at time t, $y_t(i) < y_t(i+1)$, $y_t(0) = Y_t$.

Let $\zeta_t(i) = y_t(i+1) - y_t(i) - 1$, the number of empty sites between particles i and $i+1$.

ζ_t is a system of queues in tandem, also known as a *zero range process* (discussed in the first lecture). When particle i jumps, a customer (hole) jumps from queue i to queue $i-1$.

The position of the tagged particle Y_t is just the departure process of queue 0.

Since the initial measure is product ρ, the initial measure for ζ_t is product of geometrics, that is, the invariant measure for the zero range process.

Burke theorem implies the departure process of queue 0 is a Poisson process with rate $1-\rho$ and so is the position of the tagged particle $Y(t)$.

This implies the following law of large numbers for the tagged particle:

$$\lim_{t\to\infty}\frac{Y_t}{t}=1-\rho \qquad \nu^\rho\mathbb{P} \text{ a.s.}$$

LLN for flux of particles

Let $J_{rt,t}(\eta)$ be the number of particles of η to the left of the origin at time zero and to the right of rt at time t. This is the *flux* through the space-time line containing the points $(0,0)$ and (rt,t).

Then

$$\lim_{t\to\infty} \frac{J_{rt,t}}{t} = \rho(1-\rho) - r\rho, \qquad \nu^\rho \mathbb{P} \text{ a.s.}$$

Proof: Since no particles can cross $Y(t)$,

$$J_{rt,t} = \sum_{x=rt}^{Y(t)} \eta_t(x)$$

Now use the law of large numbers for $Y(t)$ and the one for η_t which is a product measure ν^ρ to get

$$\sum_{x=rt}^{Y(t)} \eta_t(x) \sim \rho[(1-\rho) - r]$$

LLN for the flux of second class particles

Consider $\lambda < \rho$ and (η^1, η^2) the coupled process with *any* initial distribution ν with marginals ν^λ and ν^ρ.

Consider the two-class process (ξ^1, ξ^2) given by

$\xi^1 = \eta^1$ and $\xi^2 = \eta^2 - \eta^1$

Since the flux of second class particles can be expressed as the difference of fluxes of first class particles we have the following LLN:

$$\begin{aligned} \lim_{t\to\infty} \frac{J_{rt,t}(\xi^2)}{t} &= \lim_{t\to\infty} \frac{J_{rt,t}(\eta^2)}{t} - \lim_{t\to\infty} \frac{J_{rt,t}(\eta^1)}{t}, \qquad \nu\mathbb{P} \text{ a.s.} \\ &= \rho(1-\rho) - \lambda(1-\lambda) - r(\rho - \lambda) \end{aligned}$$

Notation: $J_{rt,t}(\xi^2)$ is the flux of particles for the process ξ^2, etc.

LLN for tagged second class particle Consider $\lambda < \rho$ and the same system (ξ^1_t, ξ^2_t) of two-class particles with η-marginals ν^λ and ν^ρ. Put a ξ^2 particle at the origin and call $X(t)$ its position. Since

$$J_{0,t}(\xi^2) = \sum_{x=0}^{X(t)} \xi^2_t(x)$$

we have

$$\rho(1-\rho) - \lambda(1-\lambda) \sim (\rho - \lambda) \lim_{t\to\infty} \frac{X(t)}{t}$$

from where

$$\lim_{t\to\infty} \frac{X(t)}{t} = 1 - \rho - \lambda, \qquad \nu\mathbb{P} \text{ a.s.}$$

Consider $\lambda < \rho$ and the product measures ν^λ, $\nu^{\lambda,\rho}$, ν^ρ. Couple the initial configurations such that $\eta^1 \le \eta^2$. Let

$X^{\lambda,\rho}(t)$ a tagged second class particle in (ξ_t^1, ξ_t^2),

$X^\rho(t)$ an isolated second class particle in ν^ρ first class.

$X^\lambda(t)$ an isolated second class particle in ν^λ first class.

Then almost surely:

$$X^\rho(t) \le X^{\lambda,\rho}(t) \le X^\lambda(t)$$

$$\limsup_{t\to\infty} X^\rho(t) \le \lim_{t\to\infty} X^{\lambda,\rho}(t) \le \liminf_{t\to\infty} X^\lambda(t)$$

$$\limsup_{t\to\infty} X^\rho(t) \le 1 - \lambda - \rho \le \liminf_{t\to\infty} X^\lambda(t)$$

Taking $|\lambda - \rho| \to 0$, we get

$$\lim_{t\to\infty} \frac{X(t)}{t} = 1 - 2\rho \qquad \nu^\rho \mathbb{P} \text{ a.s.}$$

Rost Theorem TASEP to the right

Let η^0_t be the process starting with Heaviside configuration $\eta^0(x) = \mathbf{1}\{x \le 0\}$ with rightmost one at origin.

Theorem 12 (Rost) *Law of large numbers:*

$$\lim_{t\to\infty} \sum_{x \ge rt} \eta^0_t(x) \;=\; \int_r^\infty u(z,1)dz \;=\; \frac{(1-r)^2}{4} \qquad a.s.$$

Local equilibrium:

$$\lim_{t\to\infty} \prod_{x\in A} \eta_t(x+rt) \;=\; u(r,1)^{|A|}$$

Rost's proof of this theorem uses the sub-additive ergodic theorem to show the existence of a limit and then properties of the solution to identify it. We give an alternative proof using law of large numbers for the second class particles with initial measure ν^ρ.

Then use an idea of Landim to deduce local equilibrium from LLN.

Lemma 13 *Coupling between η_t^0 and η_t with initial measure ν^a such that*

$$\lim_{t\to\infty} \sum_{(1-2a)t<x\leq(1-2b)t} \eta_t^0(x) \leq \lim_{t\to\infty} \sum_{at<x\leq bt} \eta_t(x) \qquad a.s.$$

or, equivalently:

$$\lim_{t\to\infty} S((1-2a)t,t) - S((1-bt),t) \leq \lim_{t\to\infty} \sum_{at<x\leq bt} \eta_t(x)$$

Proof: Take η_0 with law ν^a and consider the two-class process with initial condition

$\xi_0^1 = \eta_0$

$\xi_0^2(x) = (1-\eta_0(x))\mathbf{1}\{x \leq 0\}$

Put a ξ^2 particle, at the origin. This is a tagged second class particle starting with a non-equilibrium distribution. Call $X(t)$ its position at time t.

The position of $X(t)$ does not depend on the way we have chosen ξ^2, because it does not feel the second class particles behind it.

Since there are no second class particles to the right of $X(t)$,

$\eta_t^0(x) \le \eta_t(x)$ for all $x \ge X(t)$

By LLN for $X(t)$ under ν^a,

$$\lim_{t\to\infty} \frac{X(t)}{t} = 1 - 2a$$

$\square$

Partitioning the interval $[0,1]$ in n identical parts, and taking $a_k = k/n$, $b_k = a_{k+1}$,

$$\lim_{t\to\infty} S((1-2a_\ell)t, t) \le \sum_{k\ge\ell} \lim_{t\to\infty} \sum_{x=(1-2a_k)t}^{(1-2a_{k+1})t} \eta_t^k(x)$$

This limits equals

$$\sum_{k \geq \ell} [(1 - 2a_k) - (1 - 2a_{k+1})] a_k \to \frac{(1-r)^2}{4}$$

if $n \to \infty$ and $\ell = \ell(n) \to r$ as $n \to \infty$.

To show a limit from below a similar argument is used. This shows the law of large numbers in Rost theorem.

Local equilibrium

Lemma 14 (Landim) *If* η_t *is attractive, then the law of large numbers implies local equilibrium. See [C. Landim, Conservation of Local Equilibrium for Attractive Particle Systems on* Z^d*, Ann. Probab. Volume 21, Number 4 (1993), 1782-1808]*

Definition η_t is *attractive* if there exists a coupling between two processes η_t and η'_t in such a way that $\eta \le \eta'$ implies $\eta_t \le \eta'_t$, a.s.. TASEP is clearly attractive.

Proof: Let f be a local increasing function. Then, in the set $\{X(t) < (1-2a)t\}$, almost surely:

$$f(\tau_{(1-2a)t}\eta_t^0) \le f(\tau_{(1-2a)t}\eta_t^a)$$

where η_t^a is the stationary process with initial distribution ν^a.

Hence, for all $r > (1-2a)$, $\lim_{t\to\infty} \mathbb{E} f(\tau_{rt}\eta_t^0) \le \nu^a f$ and in particular, as $r \to (1-2a)$ we get $u(r,1) = a$ and $\nu^a f$

Second class particles in the rarefaction fan

Theorem 15 (Ferrari, Kipnis) *The second class particle in the rarefaction fan chooses one of the characteristics uniformly at random. That is, if the initial configuration is chosen with* $\nu^{\lambda,\rho}$*, then*

$$\lim_{t\to\infty} \mathbb{P}(X_t > rt) = \frac{(1-2\rho)-r}{2(\lambda-\rho)}.$$

for $r \in [(1-2\lambda),(1-2\rho)]$.

To simplify notation, we show the theorem for $\lambda = 1$ and $\rho = 0$, that is, the initial configuration η is the Heaviside configuration. Let

$J_{rt,t}(\eta)$ = number of particles of η to the left of the origin (including it) that end up at time t strictly to the right of rt minus the number of particles of η strictly to the right of the origin that end up at time t to the left of rt (including it).

Write $\nu = \nu^{1,0}$ We compute in two different ways

$$\int d\nu(\eta)\mathbb{E}J_{rt,t}(\eta) - \int d(\tau_{-1}\nu)(\eta)\mathbb{E}J_{rt,t}(\eta).$$

where τ_x is the translation by x operator: $(\tau_x\eta)(z) = \eta(z-x)$. For *any* coupling $\bar{\mu}$ of ν and $\tau_{-1}\nu$ and *any* coupling $\bar{\mathbb{P}}$ of the two processes the previous quantity is also equal to

$$\int d\bar{\mu}(\eta^0,\eta^1)\bar{\mathbb{E}}(J_{rt,t}(\eta^0) - J_{rt,t}(\eta^1)).$$

In the sequel we write $\mathbb{E}$ for the expectation with respect to the coupled process. We first couple ν and $\tau_{-1}\nu$ in such a way that if η^0 and η^1 are two configurations with those distributions respectively, then $\eta^0(x) = \eta^1(x)$ for all $x \neq 0$ and there is a particle in the origin for the first marginal and no particle for the second marginal: $\eta^0(0) = 1 - \eta^1(0)$. The discrepancy in the origin under the basic coupling behaves like a second class particle. If we label the particles at the other sites and call them first class particles, then under this coupling the positions of these particles are exactly the same for both marginals. This implies that the current produced by the first class particles are identical for both marginals and that the only difference can arise from the second class particle. It is then easy to see that the currents through rt at time t for the two marginals differ if and only if at time t the second class particle is beyond rt. Hence, taking expectations, from this coupling we see that it is equal to

$$\mathbb{P}(X_t > rt).$$

We now couple ν and $\tau_{-1}\nu$ in such a way that $\eta^1(x) = \eta^0(x-1)$ for all x. Then we use the particle to particle coupling: $\eta_t^1(x) = \eta_t^0(x-1)$ for all x. The currents through rt for the two marginals differ by one if and only if for the first marginal there is a particle at $[rt]+1$ at time t.

Taking expectations and noting that the above event depend only on the first marginal, it is also equal to

$$\mathbb{P}(\eta_t([rt] + 1) = 1$$

Letting t tending to infinity we obtain by convergence to local equilibrium that

$$\lim_t \mathbb{P}(\eta_t([rt] + 1) = 1) = u(r, 1),$$

where $u(r, t)$ is solution of Burgers. Putting all together we get the proof.

Growth model, last passage percolation

Rost constructed the TASEP in the same space as a last-passage percolation problem in the positive integer quadrant.

The trajectory of a second class particle in TASEP can be almost surely mapped to the *competition interface* of spatial growth process in the last passage percolation model.

As a consequence we can show a strong law of large numbers for the second class particle in the rarefaction fan for the exclusion process and describe the distribution of the asymptotic angle of the competition spatial growth interface.

Correspondence also shows that **fluctuations** of second class particle must be the same as those of competition interface.

Growth model Let the *initial growth interface* γ_0 be a right-down sequence of neighboring points in $\mathbb{Z}^2$ such that

1) $\gamma_0(j) - \gamma_0(j+1) \in \{(0,1),(1,0)\}$

2) $\gamma_0(0) = (1,1)$, $\gamma_1 = (-1,1)$, $\gamma_{-1} = (1,-1)$.

Let $\mathcal{W} = (w(z), z \in \mathbb{Z}^2)$ be a sequence of iid exp(1) in some probability space; denote $\mathbb{P}$, $\mathbb{E}$ the respective probability and expectation.

Let $\Pi(z,z')$ be the set of up/right paths from z to z'.

Let $G(z,z')$ *maximal length* between z and z':

$$G(z,z') := \max_{\pi\in\Pi(z,z')} \Big\{ \sum_{z''\in\pi} w(z'') \Big\}. \tag{5}$$

It satisfies the following local rule:

$$G(z,z') = w(z') + \max(G(z,z'-e_1), G(z,z'-e_2)) \tag{6}$$

In words, the value of G in a site z' is the maximum of the values of G in the sites to the left and below it plus the exponential $w(z')$.

$$G(z') = \max_{z \in \gamma_0} G(z, z') \tag{7}$$

Infected region at time t:

$$\Gamma_t := \{z \in \mathbb{Z}^2 \, : \, G(z) < t\}$$

describe a growing Young cluster.

Let $R(z)$ be the square of area 1 with NE vertex z $(R(z_1, z_2) = \{(a, b) : z_1 - 1 < a \leq z_1, \; z_2 - 1 < b \leq z_2\})$. We abuse notation writing: $\Gamma_t = \{R(z) \in \mathbb{Z}^2 \, : \, G(z) < t\}$, a connected region of the positive quadrant of $\mathbb{R}^2$.

The *growth interface* at time t, $\gamma_t = (\gamma_t(j))_{\mathbb{Z}}$, is the NE boundary of Γ_t.

Growth process and TASEP

TASEP and growth process can be realized in the same space in such a way that for a TASEP configuration η, we define $\gamma = \gamma(\eta)$ by

$\gamma(0) := (-1, -1)$ and

$$\gamma(j) - \gamma(j-1) := \begin{cases} (1-\eta(j), -\eta(j)) & \text{if } j < -1 \\ (1-\eta(j-1), -\eta(j-1)) & \text{if } j > 1 \end{cases}$$

If $\eta_0 = \bar{\eta}_0$, then Rost theorem implies:

$$\lim_{t\to\infty} \frac{1}{t}\Gamma_t = \{(r_1, r_2) :\, 0 \le \sqrt{r_1} + \sqrt{r_2} \le 1\}$$

in the Hausdorff metric defined by
$d(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} |x - y|, \sup_{y\in B} \inf_{x\in A} |x - y|\}$.

Competition growth

Let color the initial interface:

$\gamma^{\text{Blue}} = (\gamma_0(j))_{j>0}$ *blue* boundary.

$\gamma^{\text{Red}} = (\gamma_0(j))_{j<0}$ *red* boundary.

Blue: $\Gamma_t^1 := \{z' \in \Gamma_t : G(z') = G(z, z') \text{ for some } z \in \gamma^{\text{Blue}}\}$

Red: $\Gamma_t^2 := \{z' \in \Gamma_t : G(z') = G(z, z') \text{ for some } z \in \gamma^{\text{Red}}\}$

Heuristically, each site $z \in C_0$ is acquired by the cluster which spend more time to reach its neighbor.

Locally, when a site $z \in C_0$ has a left neighbor blue and a neighbor below red, it takes the color of the site that was occupied later.

The infected region is the union of the blue and red regions:
$\Gamma_t = \Gamma_t^1 \cup \Gamma_t^2 \cup \{(1, 1)\}$.

Competition interface $\varphi = (\varphi_n)_{\mathbb{N}}$: $\varphi_1 := (1,1)$ and for $n \geq 1$,

$$\varphi_{n+1} = \begin{cases} \varphi_n + (1,0) & \text{if } \varphi_n + (1,1) \in \Gamma^1_\infty \\ \varphi_n + (0,1) & \text{if } \varphi_n + (1,1) \in \Gamma^2_\infty . \end{cases} \tag{8}$$

separates the blue and red regions.

In words: The competition interface starts at $(1,1)$ and when sees a blue site in the direction NE, then it goes to the right (East), if it sees a red site, it goes up (North).

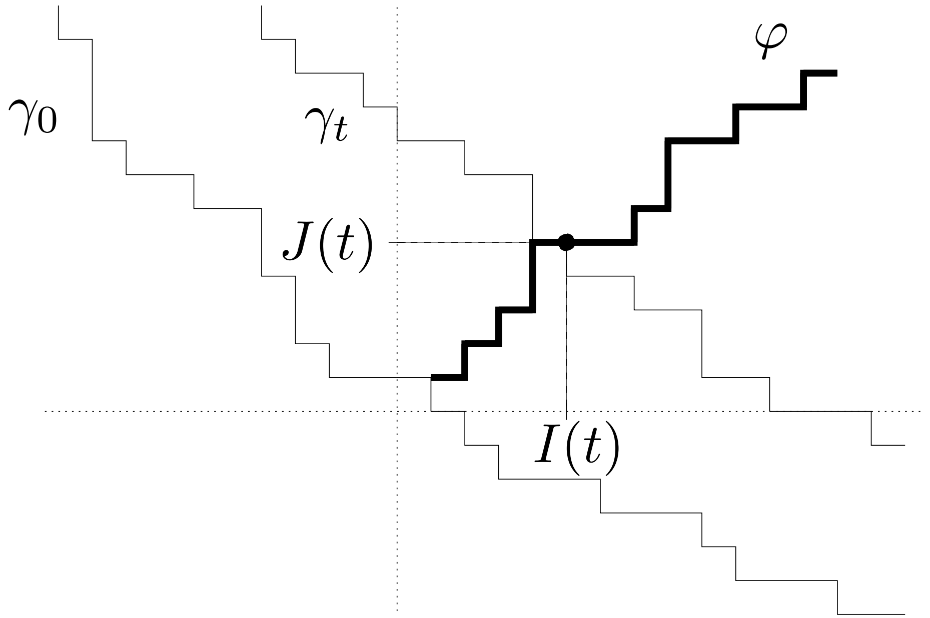


Figure 6: Illustration of the growth interface at times 0 and t and the competition interface. The position of the competition interface at time t is in the intersection of both interfaces. $I(t)$ and $J(t)$ are the coordinates of the competition interface.

Theorem 16 *It holds almost surely,*

$$\lim_{n\to\infty} \frac{\varphi_n}{|\varphi_n|} = e^{i\theta} \tag{9}$$

where $\theta \in [0, 90^o]$ *is given by*

$$\tan\theta \ = \ \begin{cases} \frac{\lambda\rho}{(1-\lambda)(1-\rho)} & \textit{if } \rho \geq \lambda \\ \left(\frac{U-1}{U+1}\right)^2 & \textit{if } \rho < \lambda \end{cases} \tag{10}$$

and U *is a random variable uniformly distributed in* $[1-2\lambda, 1-2\rho]$.

Growth model and simple exclusion Rost representation:
$\gamma_0(0) = (1,1)$ and for $j \neq 0$,

$\gamma_0(j) - \gamma_0(j-1) = (1-\eta_0(j), -\eta_0(j))$.

Particles labeled from right to left and holes from left to right.

There is a particle at site 1 labeled 1 and a hole at site 0 labeled 1.

$P_0(j)$ and $H_0(j)$, $j \in \mathbb{Z}$ positions of particles and holes respectively at time zero.

$P_t(j)$ and $H_t(i)$, as function of $\{G(z) \,:\, z \in C_0 \backslash \gamma_0\}$.

Rule: *at time $G((i,j))$ the jth particle and the ith hole interchange positions.*

Resulting process is exclusion: $\{\zeta_t \,:\, t \geq 0\}$ defined by $\zeta_t(P_t(j)) = 1$, $\zeta_t(H_t(j)) = 0$, $j \in \mathbb{Z}$ has the same law as the process $\{\eta_t \,:\, t \geq 0\}$.

Second class particle and competition interface

(φ_n) competition interface.

$\tau_n := G(\varphi_n)$

$\{\psi(t) = (I(t), J(t)) \,:\, t \geq 0\}$ by interpolation:

$$\psi(t) := \varphi_n \qquad \text{if } t \in [\tau_n, \tau_{n+1}) \tag{11}$$

$\psi(t) \in \gamma_t$ for all $t \geq 0$

$I(t)$ and $J(t)$ evolution of labels of *pair (*hole–*particle).

Position of *pair at time t is $(H_t(I(t)), P_t(J(t)))$.

Initially $I(0) = J(0) = 1$.

$$\varphi_{n+1} = \arg\min\{G(\varphi_n + (1,0)), G(\varphi_n + (0,1))\}$$

Position of *pair:

$$(H_{I(t)}(t), P_{J(t)}(t)) = (I(t) - J(t), I(t) - J(t) + 1),$$

Hence $\{I(t) - J(t)) \, : \, t \geq 0\}$ is a second class particle.

Consequence: almost surely:

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = \begin{cases} ((1-\rho)(1-\lambda), \lambda\rho) & \text{if } \lambda \leq \rho \\ \frac{1}{4}((U+1)^2, (U-1)^2) & \text{if } \lambda > \rho \end{cases} \tag{12}$$

where U random variable uniformly distributed in $[1-2\lambda, 1-2\rho]$.

Proof of (12): $X(t) := I(t) - J(t)$.

$P_t(1)$ 1st tagged particle at time t.

$J(t) :=$ particles crossing $X(t)$

$$\frac{J(t)}{t} \;=\; \frac{1}{t} \sum_{j=X(t)}^{P_t(1)} \eta_t(j) \longrightarrow \int_U^{1-\rho} u(r,1)dr = \frac{1}{4}(1-U)^2 \tag{13}$$

$I(t) :=$ holes crossing $X(t)$

$$\frac{I(t)}{t} = \frac{1}{t}\sum_{x=H_t(1)}^{X(t)} (1-\eta_t(x)) \longrightarrow \int_{-\lambda}^{U} (1-u(r,1))dr = \frac{1}{4}(1+U)^2 \tag{14}$$

This show (12) for $\lambda > \rho$. For $\lambda \le \rho$ the same argument works by substituting U above by $1-\lambda-\rho$, the limit position of the second class particle in this case, and taking the solution $u(r,1)$.

Theorem 16 is an immediate consequence of (12) and the definition of $\psi(t) = (I(t), J(t)) \in \varphi$.

Geodesics and growing interfaces Alternative to Guiol-Mountford. Geodesics plus ergodicity of last-passage percolation:

Proposition 17 *For $\lambda = 1$, $\rho = 0$:*

$$\lim_{n\to\infty} \frac{\varphi_n}{|\varphi_n|} = e^{i\theta} \qquad \mathbb{P} - a.s. \tag{15}$$

where $\theta = \theta(w)$ is a (so far unidentified) random angle in $[0, \pi/2]$.

π is a **geodesic** from z to z' if

$$G(z, z') = \sum_{z''\in\pi} w(z''). \tag{16}$$

$\pi(z, z')$ is the *unique* geodesic from z to z'.

Alternative to Guiol-Mountford.

Studied by Newman-Licea, James Martin and ....

Sufficient to prove proposition: Existence of geodesics in fixed direction and

Lemma 18 (Coalescence of Geodesics) *For all* $\alpha \in (0, \frac{\pi}{2})$, $\mathbb{P}$*-a.s. there exist* $c_\alpha = c_\alpha(w)$ *such that for all* $r > 0$ *sufficiently large,* $\pi((1,2), re^{i\alpha}) = \pi((1,2), c_\alpha) \cup \pi(c_\alpha, re^{i\alpha})$ *and* $\pi((2,1), re^{i\alpha}) = \pi((2,1), c_\alpha) \cup \pi(c_\alpha, re^{i\alpha})$.

Alternatively (James Martin): Proof follows from the following (known) h-straight property:

$\pi(z, z') \subset Q(z, z')$ almost surely for all but a finite number of z',

where $Q(z, z')$ is a rectangle of height $z' - z$ and basis $(z' - z)^{3/4+\epsilon}$.

5 Poisson approximation of output process in systems of queues. Applications to PASEP

- $S = \{-1, 0, 1, \ldots\}$, $\eta \in \mathbb{N}^S$
- $\eta(x)$ = number of customers at queue $x \in S$
- service time of queue $x \geq 0$ is exponential with rate $\mu(x)$,
- served customers at x jump to queue y with probability $p(x, y)$.
- $\eta(-1) \equiv \infty$: "Queue" -1 is out of the system and it is considered as a queue only for notational convenience.
- customers enter system to queue $y \geq 0$ at rate $\mu(-1)p(-1, y)$.
- zero range process with an external source/sink of customers.

- generator:

$$\mathbf{L}f(\eta) = \sum_{x,y\geq -1} 1\{\eta(x) > 0\}\mu(x)p(x,y)[f(\eta^{x,y}) - f(\eta)], \tag{17}$$

- $\eta^{x,y}$ is defined by

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y \\ \eta(x) - 1, & \text{if } z = x \\ \eta(y) + 1, & \text{if } z = y, \end{cases}$$

Notation: $q(x,y) = \mu(x)p(x,y),\ x \neq y;\quad q(x,x) = -\mu(x)$

Assume total arrival and departure rates are uniformly bounded:

$$\sup_y \sum_x q(x,y) < \infty,\ \sup_y \sum_x q(y,x) < \infty; \tag{18}$$

η_t resulting process in $\mathbb{N}^{\mathbb{N}}$

$\eta_t(x)$ is the number of customers in queue x by time t.

- Want positive net flux of customers out of the system:

Assume exists unique finite measure m on S satisfying

$$\textstyle\sum_{x\geq -1} m(x)q(x,y) = 0, \quad y \geq -1, \tag{19}$$

$$m(-1) = 1 > m(x), \; x \geq 0. \tag{20}$$

(q is positive recurrent and m invariant for q.)

- Assume that for each $\lambda > 0$ there exists a sigma finite measure $\rho = \rho_\lambda$ on S satisfying

$$\mu(y)\rho(y) = \sum_{x\neq y} \rho(x)q(x,y), \quad y \geq 0, \tag{21}$$

$$\rho(-1) = 1 > \rho(x) > m(x), \; x \geq 0; \tag{22}$$

$$\lambda = \sum_{x\geq 0} (\rho(x)q(x,-1) - q(-1,x)) \tag{23}$$

λ is the mean current of customers out to -1

Also “rate of entrance of customers at infinity”.

Let ν^ρ product measure:

$$\nu^\rho(\eta(x) = k(x), x \in A) = \prod_{x \in A} \rho(x)^{k(x)}(1 - \rho(x)) \qquad (24)$$

ν^ρ is invariant for the system Jackson (1963), Andjel (1982).

$\rho(x)$ = probability there is at least one customer at queue x

- Departure process D_t: # customers leaving the system in $[0, t]$
- Arrival process A_t: # customers entering the system in $[0, t]$.
- Net output process: $X_t = D_t - A_t$.

Burke (1956): $S = \{0\}$, $q(-1,0) = \lambda < 1$; $q(0,-1) = 1$

One queue in equilibrium with Poisson arrivals and exponential service times:

Departure process D_t is a Poisson (λ).

Finite number of queues: Kelly (1979).

Infinite case:

Theorem 19 *Let* $q(x,y)$ *satisfying previous conditions.* η_t *starting with stationary* ν^ρ*. Then the departure process* D_t *is a Poisson process of parameter* $\sum_{x\geq 0} \rho(x) q(x,-1)$.

Sketch proof. Follow Reich (1957), Kelly (1979):

η_t^*: reverse process of η_t with respect to ν^ρ:

Generator

$$\mathbf{L}^* f(\eta) = \sum_{x,y\geq -1} 1\{\eta(x) > 0\} q^*(x,y)[f(\eta^{xy}) - f(\eta)], \tag{25}$$

with

$$q^*(x,y) = \frac{\rho(y)q(y,x)}{\rho(x)}$$

Verify that $\{D_t, t \geq 0\} = \{A_t^*, t \geq 0\}$ have the same distribution, where A_t^* is the number of customers entering the reverse process.

By construction A_t^* is Poisson with rate

$$\sum_{x\geq 0} q^*(-1,x) \;=\; \sum_{x\geq 0} \rho(x) q(x,-1)$$

□

Theorem 20 *Let $q(x,y)$ satisfying (18) to (23) and*

$$\sum_{x\geq 0}\frac{m(x)}{1-\rho(x)+m(x)}<\infty; \tag{26}$$

$$\sup_y m(y)\sum_{x\geq -1}\frac{q(y,x)}{m(x)}<\infty, \tag{27}$$

$$\sup_y(\rho(y)-m(y))\sum_{x\geq -1}\frac{q(y,x)}{\rho(x)-m(x)}<\infty. \tag{28}$$

Let η_t with generator (17) and initial invariant measure ν^ρ.

Then

$$X_t = R_t - B_t + B_0, \tag{29}$$

where R_t is Poisson(λ); λ given in (23)

B_t is a nonnegative stationary process on $\mathbb{N}$ with all exponential moments finite: $\mathbb{E}e^{\epsilon B_t}<\infty$, for all $\epsilon>0$.

Remark: B_t is neither Markov nor independent of X_t.

The exact distribution of B_t is given in the proof of the theorem.

Sketch proof • Distinguish between two type of customers: those that enter the system from -1 (black customers) and the others (red customers) "coming from infinity".

- Black and red customers behave in the same way in the queues $x \geq 0$: When the server in queue x finishes a service —this happens at rate $\mu(x)$— it chooses uniformly one customer among the ones in its queue.

- The chosen customer jumps to queue $y \neq x$ with probability $p(x,y) = q(x,y)/\mu(x)$.

- Only different at the boundary: only black customers enter into the system from -1 and they do so to queue y at rate $q(-1,y)$.

We call $(\sigma_t, \xi_t) \in \mathbb{N}^S \times \mathbb{N}^S$ the resulting system, where

$\sigma_t(x)$:= # black customers in x at time t

$\xi_t(x)$:= # red customers in x at time t

By Lemma 21 below the process $\sigma_t + \xi_t$ has same law as η_t.

(If we disregard colors we recover the original system η_t.)

R_t := departure process of red customers and

B_t := number of black customers in the system at time t.

Since no red customers enter the system, we have

$$X_t = R_t - B_t + B_0$$

Next step: find invariant measure ν_2 for (σ_t, ξ_t) and the reverse process $(\sigma_t, \xi_t)^* = (\sigma_t^*, \xi_t^*)$ with respect to ν_2.

The two species system.

Generator of (σ_t, ξ_t):

$$\mathbf{L}_2 f(\sigma,\xi) = \sum_{x,y\geq -1} q(x,y) \quad \left\{ \frac{\sigma(x)}{\sigma(x)+\xi(x)} [f(\sigma^{xy},\xi) - f(\sigma,\xi)] \right. \quad (30)$$

$$+ \quad \left. \frac{\xi(x)}{\sigma(x)+\xi(x)} [f(\sigma,\xi^{xy}) - f(\sigma,\xi)] \right\}, (31)$$

with $\sigma(-1) = +\infty$, $\xi(-1) = 0$, $\infty/\infty = 1$ and $0/0 = 0$.

Lemma 21 *$\{\sigma_t + \xi_t\}$ has the same distribution as $\{\eta_t\}$ with initial configuration $\sigma + \xi$. I.e., for cylinder $f : \mathbb{N}^S \to \mathbb{R}$,*

$$\mathbf{L}_2 f(\sigma + \xi) = \mathbf{L} f(\sigma + \xi)$$

The two species invariant measure

Define ν_2 on $\mathbb{N}^S \times \mathbb{N}^S$ as follows.

η chosen from ν^ρ.

Let $\alpha(x) = m(x)/\rho(x)$.

Each η customer in x is black, with probability $\alpha(x)$ or red, with probability $1 - \alpha(x)$, independently of "everything".

For each x let $\sigma(x)$ and $\xi(x)$ be the number of black and red customers respectively in queue x.

The distribution of (σ, ξ) so obtained is called ν_2.

ν_2 is product measure with marginals

$$\nu_2(\sigma(x) = k, \xi(x) = j - k) = \rho(x)^j (1 - \rho(x)) \binom{j}{k} \alpha(x)^k (1 - \alpha(x))^{j-k} \tag{32}$$

$j \geq k \geq 0$

Using the probability generating function of $\sigma(x)$, one sees that under ν_2, $\sigma(x)$ is geometric:

$$\nu_2(\sigma(x) \geq k) = \left(\frac{\alpha(x)\rho(x)}{1 - \rho(x) + \alpha(x)\rho(x)} \right)^k . \tag{33}$$

Also $(\sigma(x),\, x \in S)$ is a family of independent random variables.

From Lemma 1 and the construction of ν_2 we have

$$\int d\nu_2(\sigma, \xi) \mathbf{L}_2 f(\sigma + \xi) = \int d\nu^\rho(\eta) \mathbf{L} f(\eta) \tag{34}$$

The two species reverse process

Recall rates $q^*(x, y)$ for reverse process η_t^* :

$$q^*(x, y) = \frac{q(y, x)\rho(y)}{\rho(x)} \tag{35}$$

Service rates $\mu^*(x) = \sum_y q^*(x,y)$. (Notice $\mu^*(x) = \mu(x)$)

$\alpha(x)$: probability that walk on S with rates q^* starting at x hits -1.

Define

$$\begin{aligned} q_b^*(x,y) &= q^*(x,y)\alpha(y)/\alpha(x), \quad x \geq -1 \\ q_r^*(x,y) &= q^*(x,y)(1-\alpha(y))/(1-\alpha(x)), \quad x \geq 0 \\ q_r^*(-1,y) &= q^*(-1,y)(1-\alpha(y)). \end{aligned}$$

q_b^* are rates of walk q^* conditioned to eventual absorption at -1

q_r^* are rates of walk q^* conditioned to non-absorption at -1

Notice $\sum_y q_b^*(x,y) = \sum_y q_r^*(x,y) = \mu^*(x) = \mu(x)$.

Define a process (σ_t^*, ξ_t^*) as follows. Here $\sigma_t^*(x)$ and $\xi_t^*(x)$ respectively count the number of black* and red* customers in queue x at time t.

At rate $\mu^*(x)$, a customer is selected in queue x uniformly among the $\sigma_t^*(x) + \xi_t^*(x)$ customers.

If it is a black* one, then jumps to y with probability $q_b^*(x, y)/\mu^*(x)$

if it is a red* one, then jumps with probability $q_r^*(x, y)/\mu^*(x)$.

The black* customers enter the system to queue y at rate $q_b^*(-1, y)$ and red* ones do it at rate $q_r^*(-1, y)$.

red* customers do not leave the system (go to infinity).

The generator of the process is:

$$\mathbf{L}_2^* f(\sigma,\xi) = \sum_{x,y\geq -1} \left\{ q_b^*(x,y)\frac{\sigma(x)}{\sigma(x)+\xi(x)}[f(\sigma^{xy},\xi)-f(\sigma,\xi)] \right. \tag{36}$$

$$\left. + \quad q_r^*(x,y)\frac{\xi(x)}{\sigma(x)+\xi(x)}[f(\sigma,\xi^{xy})-f(\sigma,\xi)] \right\}, \tag{37}$$

where we used the convention

$$\frac{\sigma(-1)}{\sigma(-1)+\xi(-1)} = \frac{\xi(-1)}{\sigma(-1)+\xi(-1)} = 1.$$

Condition (28) insures the existence of this process.

Main ingredient *The processes* (σ_t, ξ_t) *and* (σ_t^*, ξ_s^*) *are the reverse of one another with respect to* ν_2*.*

Proof. We want to show that for local f and g

$$\int f\mathbf{L}_2 g d\nu_2 = \int g\mathbf{L}_2^* f d\nu_2, \tag{38}$$

Suffice to show: black jump:

$$\begin{aligned} & q(x,y)\int \frac{\sigma(x)}{\sigma(x)+\xi(x)} f(\sigma^{xy},\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi) \\ &= q_b^*(y,x)\int \frac{\sigma(y)}{\sigma(y)+\xi(y)} f(\sigma,\xi)g(\sigma^{yx},\xi)d\nu_2(\sigma,\xi), \end{aligned} \tag{39}$$

red jump:

$$\begin{aligned} & q(x,y)\int \frac{\xi(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi^{xy})g(\sigma,\xi)d\nu_2(\sigma,\xi) \\ &= q_r^*(y,x)\int \frac{\xi(y)}{\sigma(y)+\xi(y)} f(\sigma,\xi)g(\sigma,\xi^{yx})d\nu_2(\sigma,\xi) \end{aligned} \tag{40}$$

and “rate of jump for direct = rate of jump for reverse”:

$$
\begin{aligned}
&\textstyle\sum_{x,y} q(x,y)\displaystyle\int 1\{\sigma(x)+\xi(x)>0\}f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi)\\
&\quad= \textstyle\sum_{x,y} q_b^*(x,y)\displaystyle\int \frac{\sigma(x)}{\sigma(x)+\xi(x)}f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi)\\
&\quad+ \textstyle\sum_{x,y} q_r^*(x,y)\displaystyle\int \frac{\xi(x)}{\sigma(x)+\xi(x)}f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi), \qquad (41)
\end{aligned}
$$

sums are done in $\{x,y : x \text{ or } y \in [-1,\dots,N]\} \supset \mathrm{supp}(f)\cup\mathrm{supp}(g)$.

To prove (39) and (40), we can take f, g of the type

$$
\prod_{v\in A} 1\{\sigma(v)=k_v\}\prod_{w\in B} 1\{\xi(w)=l_w\},
$$

where A and B are arbitrary finite subsets of $\mathbb{N}$ and k_v and l_w are arbitrary nonnegative integers. Since ν_2 is a product measure, it is then sufficient to consider the cases

$$
f(\sigma,\xi)=1\{\sigma(x)=k_x+1,\sigma(y)=k_y,\xi(x)=l_x,\xi(y)=l_y\} \quad (42)
$$

$$g(\sigma,\xi) = 1\{\sigma(x) = k_x, \sigma(y) = k_y + 1, \xi(x) = l_x, \xi(y) = l_y\} \quad (43)$$

and

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x, \sigma(y) = k_y, \xi(x) = l_x + 1, \xi(y) = l_y\} \quad (44)$$
$$g(\sigma,\xi) = 1\{\sigma(x) = k_x, \sigma(y) = k_y, \xi(x) = l_x, \xi(y) = l_y + 1\} \quad (45)$$

respectively for (39) and (40), when $x, y \in \mathbb{N}$. When $y = -1$, take

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x + 1, \xi(x) = l_x\} \quad (46)$$
$$g(\sigma,\xi) = 1\{\sigma(x) = k_x, \xi(x) = l_x\} \quad (47)$$

and

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x, \xi(x) = l_x + 1\} \quad (48)$$
$$g(\sigma,\xi) = 1\{\sigma(x) = k_x, \xi(x) = l_x\} \quad (49)$$

respectively for (39) and (40). When $x = -1$, (40) is trivially satisfied (both sides vanish), for (39) one should take

$$f(\sigma,\xi)=1\{\sigma(y)=k_y,\xi(y)=l_y\} \tag{50}$$

$$g(\sigma,\xi)=1\{\sigma(y)=k_y+1,\xi(y)=l_y\}. \tag{51}$$

For all these cases, it is a simple calculation to verify (39) and (40). The equality (41) is easily verified for the sums taken in the set $\{0\le x\le N,\ y\ge -1\}$ from the equalities

$$\sum_y q(x,y)=\sum_y q_b^*(x,y)=\sum_y q_r^*(x,y),\ x\ge 0.$$

So it suffices to show that

$$\int f(\sigma,\xi)g(\sigma,\xi)\sum_y q(-1,y)d\nu_2(\sigma,\xi)$$

$$+\textstyle\sum_{x>N}\sum_{y\le N}q(x,y)\displaystyle\int 1\{\sigma(x)+\xi(x)>0\}f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi)$$

$$= \int fg \sum_y q_b^*(-1,y)d\nu_2 + \int fg \sum_y q_r^*(-1,y)d\nu_2$$

$$+ \textstyle\sum_{x>N} \sum_{y\le N} q_b^*(x,y) \displaystyle\int \frac{\sigma(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi)$$

$$+ \textstyle\sum_{x>N} \sum_{y\le N} q_r^*(x,y) \displaystyle\int \frac{\xi(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi).$$

The second term in the l.h.s. and the third and fourth terms in the r.h.s. equal respectively

$$\int fg \sum_{x>N} \sum_{y\le N} \rho(x)q(x,y)d\nu_2, \tag{52}$$

$$\int fg \sum_{x>N} \sum_{y\le N} \alpha(x)\rho(x)q_b^*(x,y)d\nu_2, \tag{53}$$

$$\int fg \sum_{x>N} \sum_{y\le N} (1-\alpha(x))\rho(x)q_r^*(x,y)d\nu_2, \tag{54}$$

so (52) can be rewritten (when $\int fg d\nu_2 \neq 0$, otherwise ok) as

$$\begin{aligned} & \textstyle\sum_y q(-1,y) + \sum_{x>N} \sum_{y\leq N} \rho(x) q(x,y) && (55) \\ = \quad & \textstyle\sum_y q_b^*(-1,y) + \sum_y q_r^*(-1,y) + \sum_{x>N} \sum_{y\leq N} \rho(y) q(y,x). && (56) \end{aligned}$$

So, it suffices to prove the equality

$$\textstyle\sum_{x>N} \sum_{y\leq N} \rho(x) q(x,y) - \sum_{x>N} \sum_{y\leq N} \rho(y) q(y,x)$$
$$= \textstyle\sum_y \rho(y) q(y,-1) - \sum_y q(-1,y),$$

which we do by rewriting the l.h.s. as

$$\sum_{x\geq -1} \sum_{y\leq N} \rho(x) q(x,y) - \sum_{x\geq -1} \sum_{y\leq N} \rho(y) q(y,x).$$

This equals the r.h.s. of (57) plus

$$\sum_{x\geq -1} \sum_{0\leq y\leq N} \rho(x) q(x,y) - \sum_{x\geq -1} \sum_{0\leq y\leq N} \rho(y) q(y,x)$$

and the equality of these two terms follows from (21). $\square$

Proof of Theorem 2: ν^ρ is invariant for the process η_t.

(σ_t, ξ_t) under initial invariant distribution ν_2

$\eta_t = \sigma_t + \xi_t$ and

$$X_t = R_t - B_t + B_0 \tag{57}$$

B_t is the number of black customers in the system at time t

R_t: number of red customers leaving system in $[0, t]$

Proposition 1 implies

$$\{(R_t, B_t)\} = \{(R_t^*, B_t^*)\} \quad \text{as processes in distribution,} \tag{58}$$

B_t^*: number of customers at time t

R_t^*: number of red* customers entering the system in $[0, t]$ for the reverse process.

By definition R^*_t is a Poisson process of rate

$$\sum_{y\geq 0} p^*_r(-1,y) = \sum_{x\geq 0}[\rho(x) - m(x)]q(x,-1) \tag{59}$$

But this is just λ given in (23) because

$$\sum_{x\geq 0} m(x)q(x,-1) = \sum_{x\geq 0} q(-1,x)$$

is the balance equation that m must satisfy for state -1.

Under ν_2 B^*_t and B_t have all the same distribution which is also independent of t: it is the distribution of

$$B_0 = \sum_{x\geq 0}\sigma(x), \tag{60}$$

where $\{\sigma(x)\}_x$ is a collection of independent random variables with geometric distribution given by (33). Hence

$$\sum_{x\geq 0}\nu_2(\sigma(x)\geq 1)=\sum_{x\geq 0}\frac{\alpha(x)\rho(x)}{1-\rho(x)+\alpha(x)\rho(x)}<\infty, \tag{61}$$

by (26). This suffices to get the required exponential decay. □

Example: the nearest neighbor asymmetric random walk

$$q(x,y)=\begin{cases} b, & \text{if } x\geq -1,\ y=x+1\\ a, & \text{if } x\geq 0,\ y=x-1\\ 0, & \text{otherwise,}\end{cases}$$

where $a>b\geq 0$. In this case

$$m(x)=(b/a)^{x+1},$$

and since $q(x,y)$ satisfies

$$\sum_{y:y\neq x} q(x,y)=\sum_{x:x\neq y} q(x,y)=a+b$$

we can take

$$\rho(x)=m(x)+(1-m(x))\rho,$$

with $\rho\in(0,1)$. Here $\lambda=(a-b)(1-\rho)$.

For $b=0$ Theorem 1 applies and X_t is a Poisson process. Kesten (1970), Kipnis (1986).

Application to asymmetric simple exclusion process

Y_t tagged particle

nearest neighbors asymmetric simple exclusion process (ASEP)

jumps: b to left; a to right; $a > b$.

Initial distribution: μ'_β, product measure density β conditioned to have a particle at the origin (Palm).

Theorem [F. & Fontes] *There exist a Poisson process N_t with rate $(a-b)(1-\beta)$ and a stationary processes A_t in $\mathbb{Z}$ with all exponential moments bounded such that for all $t \geq 0$,*

$$Y_t = N_t - A_t + A_0.$$

Proof Coupling introduced by Ferrari, Kipnis and Saada (1991): decomposes η_t into a first and second class particles:

Let η_0 be the initial configuration chosen from μ'_λ.

$\ldots < x_{-2} < x_{-1} < x_0 = 0 < x_1 < x_2 < \ldots$ particle positions of η_0

Label x_i "first class" with probability

$$(a/b)^i/(1 + (a/b)^i)$$

Label x_i "second class" with the complementary probability.

(γ, ζ) configurations of first and second class particles

Call $\bar{\nu}$ the law of (γ, ζ)

Motion: Particles move like ordinary particles except that when a first class particle attempts a jump over a second class one, they exchange positions.

Call γ_t and ζ_t configurations of first and second class particles.

Define $\eta_t = \gamma_t + \zeta_t$ coordinatewise.

Disregarding labels: η_t doubly infinite asep system.

Y_t: position of the η tagged particle initially at the origin.

Label of Y_t may change with time.

At time zero, Y_0 is labeled γ (or first class) with probability $1/2$.

$\bar{\nu}$ is invariant for the coupled process as seen from the η tagged particle $\tau_{Y_t}(\gamma_t, \zeta_t)$.

Key point: $\bar{\nu}$ is reversible for exchanges between first and second class particles.

Leftmost first class particle under $\bar{\nu}$; denote by X_t its (absolute) position at time t.

Under $\bar{\nu}$, the distribution of

$$C_t = Y_t - X_t \tag{62}$$

is independent of t

C_t has bounded exponential moments.

It suffices to study X_t

γ_t and $\tau_{X_t}\gamma_t$ are Markovian

γ marginal of $\bar{\nu}$ as seen from the leftmost γ particle is invariant for $\tau_{X_t}\gamma_t$

Define $\xi_t(i) =$ number of sites between the ith and $(i+1)$-th γ_t particles. ξ_t is zero range process in $\{-1, 0, \ldots\}$

The queue at -1 has infinite number of customers.

$Z_t = X_t - X_0$ is the net output of customers from queue 0 to -1.

Let $\bar{\nu}_\lambda$ the measure induced by the γ marginal on the semi-infinite zero range process, then $\bar{\nu}_\lambda$ is invariant for ξ_t.

$\bar{\nu}_\lambda$ is product of geometrics with parameter

$$\rho(x) = m(x) + (1 - m(x))(1 - \beta)$$

with

$$m(x) = (b/a)^{x+1}$$

We apply the queue theorem for the case
$q(x,y) = a\mathbf{1}\{y = x-1\} + b\mathbf{1}\{y = x+1\}$, $x \geq 0$,
$q(-1,0) = q$ and $\phi = (a-b)(1-\beta)$.

Recalling that $C_t = Y_t - X_t$, $Y_0 = 0$ and $Z_t = X_t - X_0$, we can write

$$Y_t = Z_t + C_t - C_0$$

By the queue approximation:

$$Z_t = N_t - B_t + B_0$$

C_t between Y_y and X_t is independent of time and with a finite exponential moment.

Conclude the proof by taking

$$A_t = B_t - C_t$$

Open problems

- Possible to extend the result to other zero range processes? $g(k)$ general. Mysanthropes?
- Not nearest neighbors simple exclusion?
- Other systems with same Poisson approximation?
- Tagged particle in TAZRP

Application to a double infinite system It is possible to prove Theorem 2 for the net flux of customers between two consecutive queues in a doubly infinite system.

let ζ_t be the process on $\mathbb{Z}^{\mathbb{N}}$ with rates a and b for customer jumps to the right (respectively left) nearest neighbor.

Let ν^ρ be the product measure of geometrics of parameter ρ.

Let Y_t be the net flux of customers between queues 0 and -1.

Then Y_t is a Poisson process of parameter $(b-a)\rho$ plus a perturbation of order one as in Theorem 2.

$a = 0$ is immediate.

The case $a > b > 0$ is based in a coupling between the semi infinite process η_t and the double infinite process ζ_t.

Lectures 1 and 2 are based on

[1] Pablo A. Ferrari, James B. Martin Multiclass processes, dual points and M/M/1 queues Markov Process. Related Fields (2006) vol 12, 175-201. math.ph/0509045

[2] Pablo A. Ferrari, James B. Martin Stationary distributions of multi-type totally asymmetric exclusion processes Ann. Probab. Volume 35, Number 3 (2007), 807-832, math.PR/0501291

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Lecture 3 is based on

[1] P.A. Ferrari Shocks in one-dimensional processes with drift in G. Grimmett (ed.), Probability and Phase Transition NATO ASI Series C: Mathematical and Physical Sciences Vol 420. 35-48. Kluwer Academic Publishers. Dordrecht (1994).

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