SHOCKS IN THE BURGERS EQUATION AND THE ASYMMETRIC SIMPLE EXCLUSION PROCESS

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We review recent results related to the approximation of the Burgers equation using the totally asymmetric simple exclusion process. The goal is to show the existence of shocks at a microscopic level. The approach aims at simplicity rather than explicit technical details. First we discuss the graphical representation of the process and coupling, techniques which are used to describe the set of invariant measures for the process. We then obtain law of large numbers and central limit theorems for the shock. Some related cellular automata are reviewed.

1. Introduction

The Burgers equation is used as a model of transport in highways. The function $u(r,t)$ represents the density of cars at $r \in I\!R$ at time t. We assume that, in the absense of other cars, the velocity of a single car is θ. Due to the presence of other cars this velocity can be lower. The variation of density at r in an infinitesimal time interval dt is given by the number of cars entering in the infinitesimal interval $(r, r+dr)$ that is $\theta u(r-dr,t)(1-u(r,t))$ minus the number of cars exiting that interval: $\theta u(r,t)(1-u(r+dr,t))$ (note that $1-u$ is the density of free space in the interval). Hence the density must satisfy the equation

$$\frac{\partial u}{\partial t} = -\theta \frac{\partial [u(1-u)]}{\partial r} \qquad (1.1.a)$$

This is called the (unviscous) Burgers equation. We think of the highway as having only one lane. One of the most interesting phenomena in highways is the formation of shock waves. They form when for some reason one car lowers its velocity or stops. Then, after a small but positive time, the next car must lower its velocity and gets closer to the previous one. Then the third car lowers its velocity and so on. In this way the cars are divided into two well differentiated regions: one of high density where the cars are packed and run slowly and the other of fast cars and low density. This phenomena has been observed in actual highways by Walker [wa]. The shock at a given time is given by the car that is applying brakes at that time. To model the highway at a microscopic level –*i.e.* taking into account the interaction between single cars– a stochastic model called the asymmetric simple exclusion process has been used. The assumptions that we used to derive the Burgers equation (1.1.a) are rigorously proven for this model when the initial conditions are non decreasing [bfsv]. We study the case of non decreasing initial conditions that present only one shock: the initial condition will be constant to the right and left of the origin and present a discontinuity at the origin.

In the simple exclusion process, cars are represented by particles that sit at the integers. At most one particle is allowed at each site. Each particle waits an exponentially distributed random time of parameter θ and attempts to jump to the nearest right neighbor. The jump is actually realized only if the site is empty.

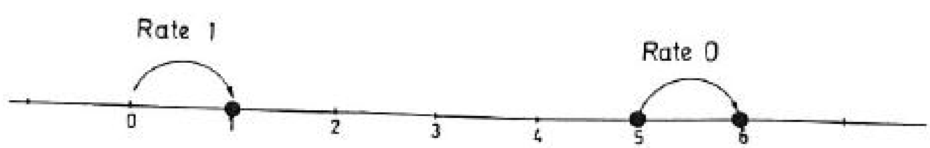


Figure 1. The simple exclusion process.

In Figure 1 we represent particles by black balls. The particle at the origin has rate θ for jumping because site 1 is empty. The particle at site 5 has rate zero as site 6 is occupied. As we will see, the essential properties of travelling waves of actual highways can be rigorously described using the simple exclusion process. The main feature of this process is the following: if one starts the system with a distribution that has densities ρ to the left, and λ to the right, $\rho < \lambda$, then there exists a random position that also obey local rules so that the system as seen from that position, uniformly in time, has densities λ to the right and ρ to the left, asymptotically.

On the way we review the proof of Liggett [l] that the set of all invariant measures is given by convex combinations of the translation invariant product measures and the blocking measures. The former are the measures ν_ρ, obtained by putting independently at each site a particle with probability ρ and no particle with probability $1-\rho$. The blocking measures are the measures that concentrate mass on configurations with particles to the right of a given site and no particles to its left. Clearly these configurations are invariant.

The simple exclusion process has been introduced by Spitzer in [spi]. The set of invariant measures was described by Liggett [l]. The hydrodynamical limit has been studied by Rost [r], Benassi and Fouque [bf1] and Andjel and Vares [av]. The existence of a microscopic shock was studied in the case of vanishing left density by Ferrari [f1], Wick [w], De Masi, Kipnis, Presutti and Saada [dkps] and Gärtner and Presutti [gp]. In the case of non vanishing left density, the existence of a microscopic shock was simulated by Boldrighini, Cosimi, Frigio and Nunes [bcfg] and proven by Ferrari, Kipnis and Saada [fks]. The present approach follows Ferrari [f2]. Bramson [b] and Lebowitz, Presutti and Spohn [lps] and Spohn [S] reviewed some of the results. Other related results are due to Kipnis [k] who proved a central limit theorem and law of large numbers for the position of a tagged particle and to De Masi and Ferrari [df] who computed the variance of the limiting Gaussian distribution.

2. The Burgers Equation

In this section we describe the (unviscous) Burgers equation and show how to find weak solutions when the initial condition is a shock. We briefly sketch the geometric approach of Lax [lax].

The inviscid Burgers equation is the hyperbolic equation (in what follows we take $\theta = 1$ in (1.1))

$$\frac{\partial u}{\partial t} = -\frac{\partial [u(1-u)]}{\partial r} \tag{2.1.a}$$

We consider the initial value problem $u(r,0) = u_0(r)$, where

$$u_0(r) = \rho 1\{r \le 0\} + \lambda 1\{r > 0\} \tag{2.1.b}$$

(shock initial conditions). The way to find the solutions is called the method of characteristics. If one calls $a(u) = \frac{\partial}{\partial u}(u(1-u)) = (1-2u)$, then the equation can be written

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial r} = 0$$

so that u is constant along trajectories $w(r,t)$ with $w(r,0) = r$, that propagate with speed $a(u)$. These trajectories are called characteristics. They are straight lines and allow to construct a solution of the equation for t small. If different characteristics meet, giving two different values to the same point, then the solution develops a discontinuity. Ours is the simplest case, when the discontinuity is present in the initial condition. Indeed, for $r > 0$, the characteristics starting

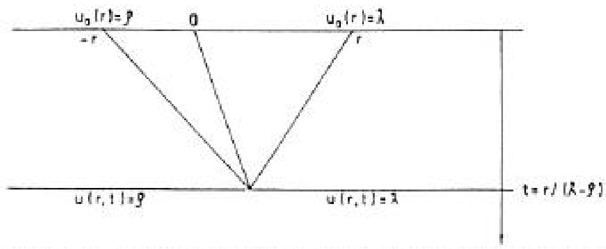


Figure 2. Shocks and characteristics in the Burgers equation.

at r and $-r$ have speed $(1-2\lambda)$ and $(1-2\rho)$ respectively and meet at time $t(r) = r/(\lambda-\rho)$. Calling $f(u) = u(1-u)$ and using the conservation law of the equation it is not difficult to show that the discontinuity propagates at velocity $v := (f(\lambda) - f(\rho))/(\lambda - \rho) = 1 - \lambda - \rho$.

In Figure 2 time is going down. We have drown the characteristics starting at r and $-r$ that go at velocity $1-2\lambda$ and $1-2\rho$ respectively. The center line is the shock that travels at velocity $1-\lambda-\rho$. The solution $u(r,t)$ is λ for $r > vt$ and ρ for $r < vt$ *i.e.* $u(r,t) = u_0(r - vt)$. This means that for all continuously differentiable test functions $\Phi(r,t)$,

$$\int\int\left(\frac{\partial\Phi}{\partial t}u + \frac{\partial\Phi}{\partial r}u(1-u)\right)dr\,dt = 0$$

Unfortunately the solution we gave above is not unique. But this is the one with physical interest because it comes as a limit when $b \to 0$ of the (unique) solution of the (viscous) Burgers equation

$$\frac{\partial u}{\partial t} = -\frac{\partial[u(1-u)]}{\partial r} + b\frac{\partial^2 u}{\partial r^2} \tag{2.2}$$

This solution is called entropic and we will see that this is the solution one gets when derives the equation as the hydrodynamical limit of the simple exclusion process.

3. The Simple Exclusion Process

The state space of the process is $\mathbf{X} := \{0,1\}^{\mathbb{Z}}$. Elements of $\mathbf{X}$ are functions that associate a number 0 or 1 to each integer. We call these elements configurations, denote them by greek letters η, ξ, σ, etc. and say, for a configuration η, that a site x is occupied by a particle if $\eta(x) = 1$ otherwise we say that x is empty. We identify a configuration η with the subset of $\mathbb{Z}$ $\{x : \eta(x) = 1\}$ of occupied sites.

Measures on $\mathbf{X}$ are characterized on cylindric functions, *i.e.* functions that depend on a finite number of sites.

The process was described in the introduction. We proceed now to construct it using the graphical representation. This construction has been widely used in other particle systems such as the contact process, the voter model, the symmetric simple exclusion process and the branching exclusion process. In all those processes the main tool was duality. Here it is coupling.

At each bond $(x, x+1)$ associate a Poisson point process (Ppp) with rate 1. Each of these processes is a sequence of times, that we call $\omega(x, n)$, $n \in \mathbb{N}$ with the property that $\{\omega(x,n) - \omega(x, n-1)\}_{n,x}$ is a family of mutually independent random variables with exponential distribution with parameter 1. That is, $P(\omega(x, n) - \omega(x, n-1) > t) = e^{-t}$. Neglect the set of probability zero where at least two of these times coincide, *i.e.* the set $\{\omega : \text{there exist } (x, n), (x', n') \text{ such that } \omega(x, n) = \omega(x', n')\}$. We say that an arrow going from x to $x+1$ is present at the times $\omega(x, n)$, $n \in \mathbb{N}$. Call ω a configuration of arrows and $(\Omega,\ \mathcal{F},\ P)$ the probability space induced by the Ppp described above.

Fix now a time $\bar{t}$. The set $\{\omega : \omega(x, 1) < \bar{t}, \text{ for all } x > 0\}$ has probability zero, as well as the event defined in the same way but with $x < 0$. This means that for almost all ω there is a pair of sites $x, x+1$ such that there are no arrows connecting them in the interval $(0, \bar{t})$. Repeating the same argument, we can say that with probability one there is a sequence of sites x_i, $i \in \mathbb{Z}$ such that there are no arrows connecting x_i and $x_i + 1$ in the time interval $(0, \bar{t})$. We consider only the ω belonging to this set of probability one. For each ω, we construct the process separately in the boxes $[x_i + 1, x_{i+1}] \cap \mathbb{Z}$. Of course the x_i are function of ω and $\bar{t}$.

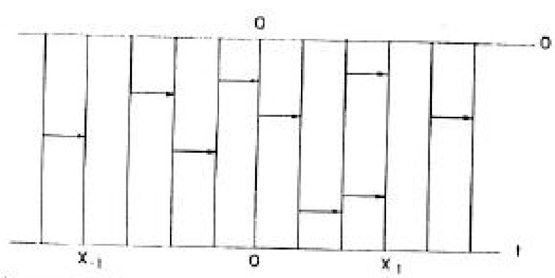


Figure 3. A typical realization of a ω with x_{-1} and x_1.

Now we put an initial configuration η at time zero, at the top of our graph –the time is flowing down– and construct η_t, $0 \leq t \leq \bar{t}$, as a function of η and ω. Since there are no arrows connecting different boxes, there is no interaction among them. Since the boxes have finite lenght, we are able to label the arrows inside each box by order of appearance. If the first arrow goes from, say, x to $x+1$, and before the arrow there is a particle at x and no particle at $x+1$, then the particle jumps from x to $x+1$ so that after the arrow there is a particle at $x+1$ and no particle at x. If before the arrow from x to $x+1$ there is a different event (two particles, two holes or a particle at $x+1$ and no particle at x), then nothing happens: the configuration after the arrow is exactly the same as before. Then we go to the second arrow and repeat the procedure up to the last arrow in the box and go to the next box constructing in this way $\eta^{\eta}_{\omega,t}$, $0 \le t \le \bar{t}$, with initial configuration η. For times greater than $\bar{t}$, use $\eta_{\bar{t}}$ as initial configuration and repeat the procedure to construct the process between $\bar{t}$ and $2\bar{t}$, and so on.

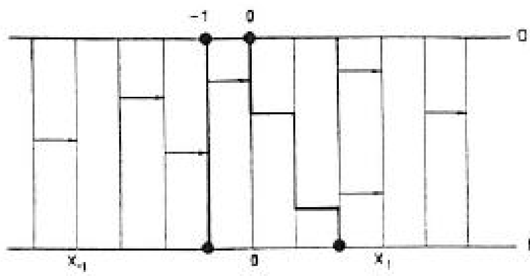


Figure 4. For the same ω, a realization of η_t.

In Figure 4 we have drawn the trajectories of two particles starting at sites 0 and 1. They interact when the particle at 0 can not use the arrow as 1 is occupied. We denote by $E_\eta f(\eta_t)$ or $Ef(\eta^\eta_t)$ the expected value of $f(\eta_t)$ with respect to P when the initial configuration is η. If ν is a measure on $\mathbf{X}$, we denote $E_\nu f(\eta_t) := \int d\nu(\eta) E_\eta f(\eta_t)$. If we define $S(t)$ as the operator $S(t)f(\eta) := E_\eta f(\eta_t)$, it is an exercise to prove that for f depending on a finite number of coordinates,

$$\frac{d}{dt} S(t)f(\eta)\big|_{t=0} = Lf(\eta); \quad \frac{d}{dt} S(t)f(\eta)\big|_{t=s} = S(s)Lf(\eta) = LS(s)f(\eta) \qquad (3.1)$$

where

$$Lf(\eta) = \sum_{x\in \mathbb{Z}} \eta(x)(1-\eta(x+1))[f(\eta^{x,x+1}) - f(\eta)] \qquad (3.2)$$

where $\eta^{x,y}(z)$ equals $\eta(x)$ for $z=y$, $\eta(y)$ for $z=x$ and $\eta(z)$ for $z \ne x,y$.

The operator L is called the generator of the process and plays the role that probability transition functions do in discrete time Markov processes. It describes the intuitive instantaneous behavior of the process. In the Liggett existence theorem for interacting particle systems the process is constructed starting from the generator and proving the existence of a semigroup $S(t)$ satisfying (3.1), from where $E_\eta f(\eta_t)$ is defined. The existence theorem of Hille Yoshida is used. The graphical approach goes back to Harris [h2] and Bertein and Galves [bg], see Liggett [L].

4. The Hydrodynamical Limit

We describe here the heuristic derivation of equations (2.1) from the process η_t and state the theorems of convergence. We use the following notations: for a measure on $\mathbf{X}$, $\nu f := \int d\nu(\eta) f(\eta)$ and $\nu S(t)$ is the measure defined by $\nu S(t) f = \int d\nu(\eta) S(t) f(\eta)$. The product measures ν_α are the measures defined by

$$\nu_\alpha f_A = \alpha^{|A|}$$

where $f_A(\eta) := \prod_{x\in A}\eta(x)$. This means that a configuration η picked from the distribution ν_α can be constructed in the following way: at each site of $\mathbb{Z}$ put a particle with probability α. Do this independently for each site. Define $\eta_t^\varepsilon(r) := \eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r)$, where $\varepsilon^{-1}r$ is an abuse of notation for integer part of $\varepsilon^{-1}r$. Applying (3.2) we get

$$\frac{d}{dt}E_\nu(\eta_t^\varepsilon(r)) = \varepsilon^{-1}E_\nu\left[-\eta_t^\varepsilon(r)(1-\eta_t^\varepsilon(r+\varepsilon)) + \eta_t^\varepsilon(r-\varepsilon)(1-\eta_t^\varepsilon(r))\right]$$

now, if there exist a limit $u(r,t) = \lim_{\varepsilon\to 0} E_\nu(\eta_t^\varepsilon(r))$ and the measure at time $\varepsilon^{-1}t$ is approximately a product measure, such that $E_\nu(\eta_t^\varepsilon(r)(1-\eta_t^\varepsilon(r+\varepsilon))$ converges, as $\varepsilon \to 0$, to $u(r,t)(1-u(r,t))$, then this limit must satisfy the Burgers equation (2.1).

In fact it is proven that if $u(r,t)$ is a solution of the Burgers equation (2.1) with $u(r,t) = u_0(r)$ and ν^ε is a family of product measures such that $\nu^\varepsilon(\eta(\varepsilon^{-1}r)) = u_0(r)$, then at the continuity points of $u(r,t)$,

$$\lim_{\varepsilon\to 0}\nu^\varepsilon S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}r} f = \nu_{u(r,t)} f \tag{4.1}$$

where $\tau_x\eta$ is the configuration defined by $\tau_x\eta(z) = \eta(z+x)$ and $\tau_x f(\eta) = f(\tau_x\eta)$. Eq. (4.1) has been proven first for initial profiles $u_0(r)$ that have only one discontinuity by Rost [r] in the non increasing case and extended by Benassi and Fouque [bf1] and Andjel and Vares [av] to the non decreasing case and for more general jump probabilities. Benassi, Fouque, Vares and Saada [bfvs] extended the result to any monotone profile and Landin [la2] to piecewise constant initial profiles presenting two discontinuities. We prove (4.1) when u_0 is a one step nondecreasing shock, like in (2.1) using the existence of a microscopic interface.

Related to that, the convergence of the density fields has been proven. Let Φ be a continuous function with compact support. Then,

$$\lim_{\varepsilon\to 0}\varepsilon\sum_{x\in\mathbb{Z}}\Phi(\varepsilon x)\eta_{\varepsilon^{-1}t}(x) = \int_{\mathbb{R}}\Phi(r)u(r,t)dr \tag{4.2}$$

P almost surely.

Notice that (4.1) gives the weak convergence of the process on the points of continuity of $u(r,t)$. Nothing is said about the points of discontinuity i.e. in $r = vt$. The expected result is the following

$$\nu_{\rho,\lambda} S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}v} = \tfrac{1}{2}\nu_\rho + \tfrac{1}{2}\nu_\lambda$$

This has been proven by Wick [w] in the case $\rho = 0$ for a related model that is isomorphic to this one and by De Masi, Kipnis, Presutti and Saada [dkps]. We give Wick's proof in Section 7. Then, using symmetry arguments, Andjel, Bramson and Liggett [abl] showed the result for the case $\lambda + \rho = 1$ *i.e.* $v = 0$. Presumably the study of the fluctuations of the microscopic shock allows one to treat this case also (see Section 13).

5. Invariant Measures

A measure ν is (time) invariant for the process if $\nu S(t) = \nu$ for all t. This means that if one starts the process with the initial measure ν and looks at the distribution at later times, one finds that the process is still distributed according to ν. Liggett [l] described the set of all invariant measures. This is a compact and convex set and is described by the extremal invariant measures. The product measures ν_α defined in the previous section are invariant for the process. To prove that, one has to show

$$\nu_\alpha S(t) f_A = \nu_\alpha f_A$$

or equivalently

$$\frac{d}{dt}\nu_\alpha S(t) f_A\Big|_{t=0} = 0$$

Suppose that A is a block of sites, $A = [a,b] \cap \mathbb{Z}$, $a, b \in \mathbb{Z}$. One has to prove that the rate of entering $\mathbf{A} := \{\eta : \eta(x) = 1, x \in A\}$ is the same as the rate of exiting. But this is true because the rate of entering is $\nu_\alpha\{\eta(a-1) = 1, \eta(a) = 0, \eta(a+1) = 1, \ldots, \eta(b) = 1\} = \alpha^{b-a}(1-\alpha)$ and the rate of exiting is $\nu_\alpha\{\eta(a) = 1, \ldots, \eta(b) = 1, \eta(b+1) = 0\} = \alpha^{b-a}(1-\alpha)$. The same argument can be applied when A consists in more than one block.

There are also other measures, called "blocking" measures. These are also product but not translation invariant. This means that the density depends on the site. We call them $\nu^{(n)}$. They give mass one to a single configuration:

$$\nu^{(n)}(\eta^{(n)}) = 1$$

where

$$\eta^{(n)}(x) := \begin{cases} 1, & \text{if } x \geq n \\ 0, & \text{if } x < n \end{cases}$$

The $\nu^{(n)}$ are translation of each other.

Theorem 5.1. (Liggett). Every invariant measure for the process is a convex combination of ν_α, $0 \leq \alpha \leq 1$ and $\nu^{(n)}$, $n \in \mathbb{Z}$.

This proof is based on coupling and we will sketch it after a discussion of coupling in the next section.

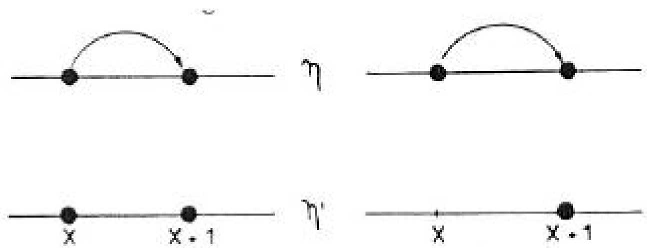
It is easy to verify that the set of invariant measures is convex and compact in the topology of weak convergence. Hence all invariant measure is a convex combination of the extremal points of the set, by the Krein Millman Theorem. A measure is said to be invariant extremal if it can not be written as a non trivial convex combination of other invariant measures. We will see later that the extremal invariant measures play a role in the proof of law of large numbers.

6. Coupling

Coupling is the main tool in this process. Almost all the results have been proved using coupling. There is a simple way of describing it using the graphical representation.

Let η and η' be two initial configurations. The coupled process is the process defined by $(\eta_{\omega,t}, \eta'_{\omega,t})$. In other words we are realizing the two processes with the same realization of the arrows. This implies that the coupled process is realized in the same probability space $(\Omega,\ \mathcal{F},\ P)$, and we can refer to probabilities or expectations with the same letters P and E. Since the particles of each copy follow the arrows without influence of the other copy, each marginal of the coupling has the distribution of the simple exclusion process. Intuitively the coupling works in the following way. Particles at site x of the configurations η_t and η'_t use the same arrows, so that they try to jump at the same time. If the destination site $x+1$ is empty in both configurations the jump is realized in both marginals, but if it is occupied for one of the marginals, then the jump is realized only for the other marginal.

In Figure 5 we see the configurations of η and η' before and after an arrow. One of the features of the coupling is that if $\eta \geq \eta'$ (coordinatewise) then $\eta_{\omega,t} \geq \eta'_{\omega,t}$ for almost all $\omega \in \Omega$, all $t \geq 0$. To prove this, it is sufficient to show that this is true for one arrow, as there is a finite number of arrows in each of the boxes where we construct the process. Suppose then that there is an arrow from x to $x+1$. There are the following possibilities, the top line represents the occupation numbers of the configuration η_t at sites x and $x+1$, before



before the arrow after the arrow

Figure 5. Coupling.

the arrow to the left and after the arrow to the right. The bottom line represents the same for the configuration η'. We ilustrate only the cases when $\eta \geq \eta'$:

11 11 11 11 11 11 11 11 10 01 10 01 01 01 01 01 00 00
10 01 01 01 11 11 00 00 10 01 00 00 01 01 00 00 00 00

We see that in any case, if before the jump the top configuration is greater than or equal to the bottom configuration, then the same is true after the jump. This property is called attractivity. This implies that

$$\nu \leq \mu \text{ implies } \nu S(t) \leq \mu S(t)$$

where $\nu \leq \mu$ is defined by the following equivalent statements

1. There exists $\tilde{\nu}$ on $\mathbf{X}^2$ with marginals ν and μ such that $\tilde{\nu}\{(\eta, \eta') : \eta \leq \eta'\} = 1$.

2. For all non decreasing f, $\nu f \leq \mu f$.

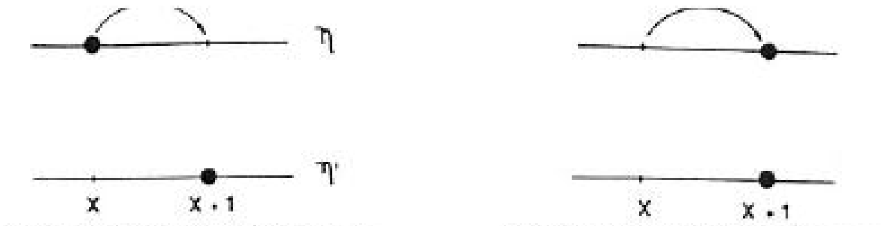
If $\rho \leq \lambda$, then $\nu_\rho \leq \nu_\lambda$. To see that, we construct a measure on $\mathbf{X}^2$ with marginals ν_ρ and ν_λ and concentrated on $\{(\eta, \eta') : \eta \leq \eta'\}$. We need a family $\{U_x\}_{x \in \mathbb{Z}}$ of random variables which are independent and uniformly distributed on $[0, 1]$. Define $\eta(x) = 1\{U_x \leq \rho\}$ and $\eta'(x) = 1\{U_x \leq \lambda\}$. The resulting distribution of (η, η') has the desired properties.

Now we sketch how Liggett uses the coupling to describe the set of invariant measures for the process in the translation invariant case. First prove that if ν and ν' are extremal invariant for $S(t)$ then there exists an extremal invariant measure $\tilde{\nu}$ for the coupled process with marginals ν and ν'. We say that there is a discrepancy at site x if $\eta_t(x) - \eta'_t(x) = \pm 1$. Let f_n be the number of changes of sign of $\eta_t(x) - \eta'_t(x)$, $-n \leq x \leq n$. Then one observes that under the coupling, if an arrow involving two discrepancies of different sign is present, then the discrepancies eliminate each other.

This is the key ingredient to prove that if $\tilde{\nu}$ is extremal invariant and translation invariant for the coupled process, then

$$\tilde{\nu}\{\eta \geq \eta'\} = 1 \quad \textit{or} \quad \tilde{\nu}\{\eta \leq \eta'\} = 1. \tag{6.1}$$

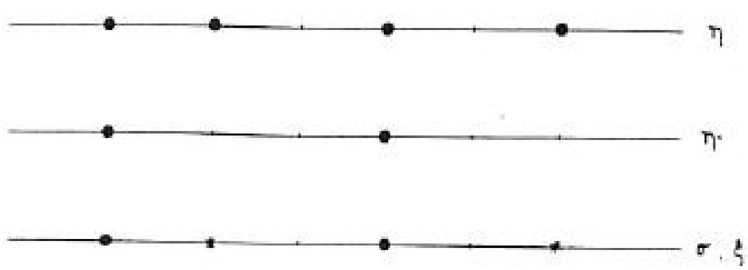
This implies that all extremal invariant and translation invariant measures are the ν_α's.



Discrepancies before the arrow No discrepancy after the arrow

Figure 6. Discrepancies eliminate each other.

Later it will be useful to read the coupling in the following different way. Assume that (η, η') is the initial pair of configurations for the coupled process and that $\eta' \leq \eta$. Call $\xi(\eta, \eta')$ the set of sites $\{x : \eta(x) > \eta'(x)\}$ and $\sigma(\eta, \eta')$ the set $\{x : \eta(x) = \eta'(x)\}$.



•: are σ-particles (first class)
$*$: are ξ-particles (second class)

Figure 7. Another way of looking at the coupling.

Also call $\xi_t := \xi(\eta_t, \eta'_t)$ and $\sigma_t := \sigma(\eta_t, \eta'_t)$. Then we can recover (η_t, η'_t) from (σ_t, ξ_t). On the other hand this last coupled process has an interesting property. First, at each site there is at most one particle, either ξ or σ. Second, when an arrow appears involving two sites with the same kind of particles (ξ-ξ or σ-σ), the same rules as for the (non coupled) process η_t hold. But when the arrow involves σ-ξ particles the rules are different. If the arrow goes from the σ particle to the ξ particle, then the particles interchange positions. If the arrow goes from the ξ to the σ particle, then nothing happens. In some way we can say that the σ particles "do not see" the ξ particles that must live in the sites not occupied by the σ particles. For this reason it is said that the σ particles are "first class particles" and the ξ particles are "second class", or that the σ particles have "priority" over the ξ particles. This priority is denoted $\sigma_t \vdash \xi_t$. This process can be constructed directly, just fixing the initial configurations σ and ξ and the priorities $\sigma_t \vdash \xi_t$.

In the same way, if one considers a three way coupling with initial configurations (η, η', η''), $\eta \geq \eta' \geq \eta''$, and call $\sigma(\eta, \eta', \eta'')$ the set $\{x : \eta(x) + \eta'(x) + \eta''(x) =$

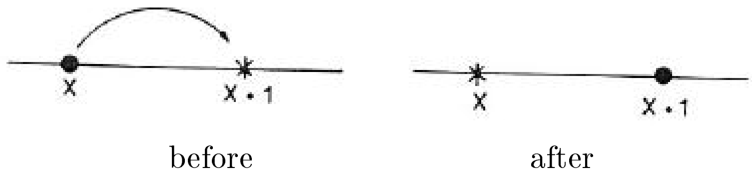


Figure 8. The σ - ξ version of Figure 5.

$3\}$, $\xi(\eta, \eta', \eta'')$ the set $\{x : \eta(x) + \eta'(x) + \eta''(x) = 2\}$ and $\gamma(\eta, \eta', \eta'')$ the set $\{x : \eta(x) + \eta'(x) + \eta''(x) = 1\}$, we have a process $(\sigma_t, \xi_t, \gamma_t)$ with priorities $\sigma_t \vdash \xi_t \vdash \gamma_t$ from which one can recover the original three way process.

7. The Semi-infinite Case

In this section we consider the measure $\nu_{0,\lambda}$, that is, the product measure with densities 0 and λ to the left and right of the origin, respectively. For convenience we assume also that there is a particle at the origin, and define $\nu' := \nu(.|\eta(0) = 1)$. Hence our initial measure is $\nu'_{0,\lambda}$. Call $X(t)$ the position at time t of the particle that is initially at the origin. We keep track of the position of that particle considering the process $(\eta_t, X(t))$ in $\{(\eta, z) : \eta \in \mathbf{X},\ z \in \mathbb{Z},\ \eta(z) = 1\}$. The generator of this process is the following

$$\bar{L}f(\eta, z) = \sum_{x, x+1 \neq z} \eta(x)(1 - \eta(x+1))[f(\eta^{x,x+1}, z) - f(\eta, z)] \\ + (1 - \eta(z+1))[f(\eta^{z,z+1}, z+1) - f(\eta, z)]$$

Now we are interested in the process as seen from the tagged particle. Hence we consider the process $\eta'_t := \tau_{X(t)}\eta_t$. This process has the following generator

$$L'f(\eta) = \sum_{x, x+1 \neq 0} \eta(x)(1 - \eta(x+1))[f(\eta^{x,x+1}) - f(\eta)] \\ + (1 - \eta(1))[f(\tau_1\eta^{0,1}) - f(\eta)]$$

For this process there is a translation each time that the tagged particle moves, in such a way that the tagged particle is always at the origin. The position of the tagged particle can be recovered from this process by defining $X(t) :=$ number of translations of the system in the time interval $[0, t]$. The following remarkable result is the key tool in this section. It comes from queuing theory. The connection between the simple exclusion process and a system of queues has been established by Kesten (see [spi], [L] and also [k], [f1], [df] and [dkps]). In words one can think that each particle is a server in a series of infinitely many queuing systems, where each system consits of a server and a queue. The holes are thought of as customers. Each time that a particle jumps over a hole, it is served and enters in the next system. The Burke's theorem says that a single queue with Poisson arriving times at rate a and exponential service times at any rate $b > a$ has Poisson exiting times at the same rate a. In this way the arriving time of the next system is the same as the one for the previous one and the exiting time of any system is Poisson. In our context, we have:

Theorem 7.1. Burke's Theorem. If the initial measure is $\nu'_{0,\lambda}$ then $\tau_{X(t)}\eta_t$ and $X(t)$ are independent. Indeed

$$E(f(\tau_{X(t)}\eta_t)|X(t)) = \nu'_{0,\lambda}f \text{ for all } t \geq 0$$

and $X(t)$ is a nearest neighbor totally asymmetric random walk with parameter $1-\lambda$.

Proof. The measure $\nu'_{0,\lambda}$ is invariant for the process $\eta'_t := \tau_{X(t)}\eta_t$. Define the reversed process η^*_s, $0 \leq s \leq t$, with respect to $\nu'_{0,\lambda}$, by

$$E_{\nu'_{0,\lambda}} f(\eta^*_s) := \int d\nu'_{0,\lambda}(\eta) E(f(\eta'_{t-s})|\eta'_t = \eta)$$

Since the measure $\nu'_{0,\lambda}$ is stationary, η^*_s is a stationary Markov process with that measure as invariant. The generator L^* is the adjoint of the operator L' on $\mathbf{L}_2(\nu'_{0,\lambda})$ and is given by

$$\begin{aligned} L^* f(\eta) = &\sum_{x,x-1\neq 0} \eta(x)(1-\eta(x-1))[f(\eta^{x,x-1}) - f(\eta)] \\ &+ (1-\lambda)[f(\tau_{-1}\eta^{0,-1}) - f(\eta)] \end{aligned}$$

To prove that L^* is the adjoint of L' in $\mathbf{L}_2(\nu'_{0,\lambda})$ one has to prove that $\nu'_{0,\lambda}(fLg) = \nu'_{0,\lambda}(gL^*f)$. This follows from the following identities:

$$\begin{aligned} &\int \nu'_{0,\lambda}(d\eta)(\eta(x)(1-\eta(x+1))f(\eta)g(\eta^{x,x+1})) \\ &\quad = \int \nu'_{0,\lambda}(d\eta)(\eta(x+1)(1-\eta(x))f(\eta^{x,x+1})g(\eta)) \end{aligned}$$

for $x, x+1 \neq 0$, and

$$\int \nu'_{0,\lambda}(d\eta)((1-\eta(1))f(\eta)g(\tau_1\eta^{0,1})) = \int \nu'_{0,\lambda}(d\eta)(1-\lambda)f(\tau_{-1}\eta^{0,-1})g(\eta))$$

Now observe that out of the origin η^*_t is a simple exclusion process with jumps only from right to left. Furthermore the leftmost particle jumps to the left not with rate one but with rate $1-\lambda$, producing a translation of the system so that we keep the configuration in the set with one particle at the origin and no particles to its left. Call $X^*(t)$ the number of translations of η^*_t in the interval $[0,t]$. Then $(\eta'_t, X(t)-X(0)) = (\eta^*_0, X^*(0)-X^*(t))$ in distribution. Now, since the law of $-X^*(t)$ is that of a Ppp of parameter $1-\lambda$ and is independent of η^*_0, so is the law of $X(t)$. Since $\eta^*_0 = \eta_t$, this finishes the proof. ♣

The next result is just a consequence of the fact that $X(t)$ is a Poisson point process and can be found in introductory books on stochastic processes.

Corollary 7.2. Law of large numbers and Central limit theorem. The following hold

$$\lim_{t\to\infty} \frac{X(t)}{t} = 1-\lambda,\ P_{\nu_{0,\lambda}}\ a.s.$$

$$\lim_{\varepsilon\to 0} \frac{X(\varepsilon^{-1}t)-(1-\lambda)\varepsilon^{-1}t}{\sqrt{(1-\lambda)\varepsilon^{-1}t}} = \mathcal{N}(0,1)$$

Corollary 7.3. Hydrodynamics. The following holds

$$\lim_{\varepsilon\to 0} \nu_{0,\lambda}S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}r} = \begin{cases} \nu_0 & \text{if } r < 1-\lambda \\ \nu_\lambda & \text{if } r > 1-\lambda \end{cases}$$

Proof. Write

$$\tau_{\varepsilon^{-1}r}\nu'_{0,\lambda}S(\varepsilon^{-1}t)f = E_{\nu'_{0,\lambda}}\tau_{\varepsilon^{-1}r-X(\varepsilon^{-1}t)}f(\tau_{X(\varepsilon^{-1}t)}\eta_{\varepsilon^{-1}t})$$

Now, by Burke's theorem, $\tau_{X(t)}\eta_t$ has distribution $\nu'_{0,\lambda}$ for all t independently of $X(t)$. On the other hand, by the law of large numbers for $X(t)$, $\varepsilon^{-1}r - X(\varepsilon^{-1}t)$ converges almost surely, as $\varepsilon \to 0$, to $-\infty$ if $r > vt$ and to ∞ if $r < vt$. This proves the Corollary. ♣

Corollary 7.4. Dynamical phase transition. The following holds

$$\lim_{\varepsilon\to 0} \nu_{0,\lambda}S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}(1-\lambda)t+\varepsilon^{-1/2}r} = (1-g(r))\nu_0 + g(r)\nu_\lambda$$

where $g(r) := (2\pi)^{-1/2}\int_{-\infty}^{r}\exp(-x^2/2)dx$ is the standard Gaussian distribution function.

Proof. The idea is the same as in the proof of the previous corollary. The difference is that in order to show that $|X(\varepsilon^{-1}t) - \varepsilon^{-1}vt - \varepsilon^{-1/2}r|$ diverges one needs to use the fact that $X(t)$ is roughly a Gaussian random variable with variance (of the order of) t. After this it suffices to notice that $g(r)$ is just the probability that $X(\varepsilon^{-1}t)$ is to the left of $\varepsilon^{-1}vt + \varepsilon^{-1/2}r$ and apply Burke's theorem again. ♣

8. The Tagged Particle Process

This is the process $(\eta_t, X(t))$ on the state space $\mathbf{X} \times \mathbb{Z}$. If $X(0) = x$ then $\eta_0(x) = 1$, so that $X(t)$ describes the position of the particle that at time 0 was at x. The generator for this process is

$$\begin{aligned}\bar{L}f(\eta, z) = \sum_{x,x+1\neq z} &\eta(x)(1-\eta(x+1))[f(\eta^{x,x+1}, z) - f(\eta, z)] \\ &+ (1-\eta(z+1))[f(\eta^{z,z+1}, z+1) - f(\eta, z)]\end{aligned}$$

In this process $\eta_t(X(t)) = 1$ for all t. Notice that by the translation invariance of P (i.e. $\tau_x\omega$ has the same probability as ω), we get

$$E_{(\eta,x)}\tau_u f(\eta_t, X(t)) = E_{\tau_u(\eta,x)} f(\eta_t, X(t)) \tag{8.1}$$

where $\tau_u(\eta, x) := (\tau_u\eta, x-u)$. Now define the process as seen from the tagged particle

$$\eta'_t = \tau_{X(t)}\eta_t$$

For this process $\eta'_t(0) = 1$ for all t. The generator is

$$\begin{aligned}L'f(\eta) = \sum_{x,x+1\neq 0} &\eta(x)(1-\eta(x+1))[f(\eta^{x,x+1}) - f(\eta)] \\ &+ (1-\eta(1))[f(\tau_1\eta^{0,1}) - f(\eta)]\end{aligned}$$

Let $S(t)$ be the semigroup for η_t and $S'(t)$ the semigroup for η'_t. Let $\mathcal{S}$ be the set of translation invariant measures, *i.e.* $\mathcal{S} := \{\mu \text{ on } \mathbf{X} : \mu = \tau_x\mu \text{ for all } x\}$. Let $\mu \in \mathcal{S}$ with $\mu(\eta(0)) > 0$. Define $\mu' := \mu(.|\eta(0) = 1)$. That is, for any cylindric f,

$$\mu' f = \frac{1}{\mu(\eta(0))}\int d\mu(\eta)\eta(0)f(\eta).$$

The next result is due to Harris [h1] (see also Port Stone [ps] and Ferrari [f1]). It says that the distribution of the process as seen from the tagged particle is the same as the distribution of the process as seen from the origin conditioned to have a particle at the origin.

Theorem 8.2. Let μ be translation invariant and $\mu(\eta(0)) > 0$, then $\mu' S'(t) = (\mu S(t))'$.

Proof. First show, as an exercice, that $\mu S(t)(\eta(0)) = \mu(\eta(0))$. For this it suffices to show that $\mu L(\eta(0)) = 0$. By the definition of $(.)'$,

$$\begin{aligned}
(\mu S(t))' f &= \frac{1}{\mu S(t)(\eta(0))} \int d(\mu S(t))(\eta) f(\eta)\eta(0) \\
&= \frac{1}{\mu(\eta(0))} \int d\mu(\eta) \sum_{x\in\mathbb{Z}} \eta(x) E_{(\eta,x)}[f(\eta_t) 1\{X(t) = 0\}] \\
&= \frac{1}{\mu(\eta(0))} \int d\mu(\eta) \sum_{x\in\mathbb{Z}} \eta(x) E_{(\tau_x\eta,0)}[f(\tau_{-x}\eta_t) 1\{X(t) = -x\}] \\
&= \frac{1}{\mu(\eta(0))} \sum_{x\in\mathbb{Z}} \int d\mu(\eta)\eta(x) E_{(\tau_x\eta,0)}[f(\tau_{-x}\eta_t) 1\{X(t) = -x\}]
\end{aligned} \tag{8.3}$$

where the third identity follows from the translation invariance of the process (8.1). Now writing $\eta(x) = (\tau_x\eta)(0)$ and using the translation invariance of μ, the last member of (8.3) equals

$$= \frac{1}{\mu(\eta(0))} \int d\mu(\eta)\eta(0) E_{(\eta,0)} \sum_{x\in\mathbb{Z}} f(\tau_{-x}\eta_t) 1\{X(t) = -x\} = \mu' S'(t) f$$

Note: all interchanges of sums with integrals are justified as f is cylindric (hence bounded). ♣

Corollary 8.4. If μ is invariant for $S(t)$ and translation invariant, then μ' is invariant for $S'(t)$. In particular we have that ν'_ρ is invariant for the process as seen from the tagged particle.

Proof. $\mu' S'(t) = (\mu S(t))' = \mu'$. ♣

Remark. Identify a configuration η with the ordered set $\{x_i\}_{i\in\mathbb{Z}}$ of occupied sites such that x_0 is the first occupied site to the right of the origin of η. It has been proven [f1] that $\{\nu'_\rho : 0 \le \rho \le 1\}$ is the set of extremal invariant and "translation invariant" measures –the measures for which η and $\tau_{x_i}\eta$ are identically distributed.

There are also other invariant measures that concentrate on the set $\{\eta : \sum_{x<0}\eta(x) < \infty\}$. These can be constructed in the following way: let ξ be a configuration distributed according to ν'_λ. Now call x_i the position of the i-th ξ particle ($x_0 = 0$). Let $\eta = \eta(\xi)$ be defined by $\eta(x) = 1\{x \ge 0\}\xi(x)$. Then the distribution of η is $\nu'_{0,\lambda}$. Call $\mu^{[i]}$ the distribution of $\tau_{x_i}\eta$. It is proven in [f1] that all invariant measures for the process as seen from the tagged particle are convex combinations of ν'_α and $\mu^{[i]}$, but we don't use it in this paper.

Now, calling $G(t)$ the position of the leftmost particle of η_t so constructed and $X(t)$ the position of the tagged particle of ξ we get that $G(t) \equiv X(t)$. Hence, if the law of large numbers or the central limit theorem holds for one of these positions then it also holds for the other.

9. Laws of Large Numbers

To prove laws of large numbers we use the ergodic theorem. We follow Kipnis [k] and Saada [s1] [s2]. The Birkhoff ergodic theorem guarantees that if μ is invariant, then for all cylindric f, $(1/t)\int_0^t f(\eta_s^\eta)ds$ converges P_μ a.s. to a limit, where we used the notation η_s^η to indicate the random configuration obtained at time s when the initial configuration is η.

We say that P_μ is ergodic if for all cylindric f,

$$\hat{f}(\eta) := \lim_{t\to\infty} t^{-1}\int_0^t f(\eta_s^\eta)ds = \mu f \quad P_\mu \ a.s.$$

Lemma 9.1. If μ is extremal invariant for η_t then P_μ is ergodic.

Proof. We follow Rosenblatt [rb]. Assume that P_μ is not ergodic. Then there exists $c > 0$ and a cylindric function f such that $\mu(\mathbf{A}) = \beta$, $0 < \beta < 1$, where $\mathbf{A} := \{\eta : \hat{f}(\eta) > c\}$. But $\mathbf{A}$ is a.s. closed for the motion, *i.e.* for all $\eta \in \mathbf{A}$, $P_\eta(\eta_t \in \mathbf{A}) = 1$. The same is true for $\mathbf{A^c}$. Hence $\mu(.|\mathbf{A})$ and $\mu(.|\mathbf{A^c})$ are invariant and we can write $\mu = \beta\mu(.|\mathbf{A}) + (1-\beta)\mu(.|\mathbf{A^c})$. This implies that μ is not extremal.
♣

Theorem 9.2. Let $F(t)$ be an additive functional of η_t. That is, $\text{law}(F(t+s) - F(t)|\eta_u, u \le t) = \text{law}(F(t+s) - F(t)|\eta_t)$. Assume that μ is extremal invariant for the process and that $\lim_{h\to 0} h^{-1}E_\mu(F(t+h) - F(t))^2 < \infty$. Then

$$\lim_{t\to\infty} \frac{F(t)}{t} = \mu\psi \quad P_\mu \ \ a.s.$$

where ψ is the compensator of $F(t)$ defined by

$$\psi(\eta) := \lim_{h\to 0} \frac{E(F(t+h) - F(t)|\eta_t = \eta)}{h}$$

We say that $\psi(\eta)$ is the instantaneous increment of $F(t)$ when the configuration at time t is η.

Proof. Define

$$M(t) := F(t) - \int_0^t \psi(\eta_s)ds \tag{9.3}$$

It is easy to see that $M(t)$ is a martingale because the instantaneous increment of $F(t)$ is $\psi(\eta_t)$ which is exactly the same as the instantaneous increment of the integral in (9.3). Now, since $M(t)$ is a martingale, $t^{-1}E_\mu M(t)^2$ is independent of t and we can write

$$\frac{E_\mu M(t)^2}{t} = \lim_{h\to 0} \frac{E_\mu(M(t+h) - M(t))^2}{h} = \lim_{h\to 0} \frac{E_\mu(F(t+h) - F(t))^2}{h}$$

If this last limit is finite, we call it $\mu\Phi$ and we can apply the convergence theorem for martingales (Breiman [B]) to conclude that

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad P_\mu \quad a.s.$$

So that

$$\lim_{t\to\infty} \frac{F(t)}{t} = \lim_{t\to\infty} t^{-1} \int_0^t \psi(\eta_s)ds = \mu\psi \quad P_\mu \quad a.s.$$

where the second identity follows from Lemma 9.1. ♣

Let's see some applications of this theorem.

Corollary 9.4. Law of large numbers for the flux. Let $F(t) := \#\{i : x_i(0) \le 0$ and $x_i(t) > 0\} - \#\{i : x_i(0) > 0$ and $x_i(t) \le 0\}$, where $x_i(t)$ is the position of the i-th particle of η_t. Then

$$\lim_{t\to\infty} \frac{F(t)}{t} = \lambda(1-\lambda) \quad P_{\nu_\lambda} \ a.s.$$

Proof. We apply the theorem to the process η_t. We saw before that ν_λ is extremal invariant. On the other hand $\psi(\eta) = \eta(0)(1-\eta(1))$, $\nu_\lambda\psi = \lambda(1-\lambda)$ and $\nu_\lambda\Phi = \lambda(1-\lambda) < \infty$. ♣

The next corollary has been proven in Section 7, using Burke's theorem. We give a proof that does not use that result.

Corollary 9.5. Law of large numbers for the tagged particle.

$$\lim_{t\to\infty} \frac{X(t)}{t} = (1-\lambda) \quad P_{\nu'_\lambda} \ a.s.$$

Proof. We apply the theorem to the process $\tau_{X(t)}\eta_t$. It was proved by [f1] that ν'_λ is extremal for $\tau_{X(t)}\eta_t$. We have that $\psi(\eta) = (1-\eta(1))$, $\nu_\lambda\psi = (1-\lambda)$, $\lim_{h\to 0} E_{\nu_\lambda}(X(t+h)-X(t))^2/h = (1-\lambda)$. Hence $EM(t)^2 = t(1-\lambda) < \infty$ and the Theorem applies. ♣

Corollary 9.6. Law of large numbers for the flux throught a random position. Let $U(t)$ be a birth process with rate w independent of η_t , i.e. $P(U(t)=k) = e^{-wt}(wt)^k/k!$. Let $F(t)$ be the net flux throught $U(t)$: $F(t) := \#\{i : x_i(0) \le 0$ and $x_i(t) > U(t)\} - \#\{i : x_i(0) > 0$ and $x_i(t) \le U(t)\}$. Then

$$\lim_{t\to\infty} \frac{F(t)}{t} = \lambda(1-\lambda) - w \quad P_{\nu_\lambda} \ a.s.$$

Proof. We apply the theorem to the process $\tau_{U(t)}\eta_t$. First we claim that if μ is invariant for this process, then μ is translation invariant. It is easy to show that if μ is invariant then $\tau_x\mu$ is also invariant. To prove the claim it suffices to show that the processes with initial measure μ and $\tau_x\mu$ converge to the same measure. To do that we couple two versions of the process in the following way: we choose the same initial configuration for the two processes according to μ and consider two versions of $U(t)$: $U_0(t)$ with initial point 0 and $U_x(t)$ with initial point x. These two versions are independent up to $T :=$ first time that $U_0(t) = U_x(t)$. After T, they continue together. In that way $\tau_{U_0(t)}\eta_t$ and $\tau_{U_x(t)}\eta_t$ are the processes with initial measure μ and $\tau_x\mu$, respectively. Since $U_0(t)$ and $U_x(t)$ are independent random walks, they will meet with probability one and the two processes will coincide after T. Since the initial measures are invariant they must be the same. This proves the claim.

Now, using the method of Liggett described in Section 6, one proves that all extremal invariant measures for this process are the product measures with density ρ, $0 \le \rho \le 1$. Once proved that ν_λ is extremal invariant for this process, the rest follows as in the previous corollaries. ♣

10. Microscopic Shock in the General Case

We saw that in the case of vanishing left density the position of the leftmost particle defines a microscopic shock. Indeed the system as seen from the shock has a measure $\mu' S'(t) = \nu'_{0,\lambda}$ for all times. Such strong statment is not true in the case of non vanishing left density. We introduce a weaker definition of a microscopic shock:

Definition 10.1. We say that a random position $X(t)$ is a microscopic shock for η_t if calling μ_t the distribution of $\tau_{X(t)}\eta_t$, the following weak limits hold uniformly in t:

$$\lim_{x \to +\infty} \tau_x \mu_t = \nu_\lambda, \quad \lim_{x \to -\infty} \tau_x \mu_t = \nu_\rho$$

When the left density does not vanish it is not obvious how to define the position of a microscopic shock. One can try to tag a particle and follow it, but one immediately realizes that the tagged particle has the wrong velocity: in regions of ν_ρ has velocity $(1-\rho)$ and in regions fo ν_λ has velocity $(1-\lambda)$. Nevertheless, in some sense, the idea of considering a last particle used for vanishing left density also works in the general case. We couple two processes. The σ process with initial measure ν_ρ and the $\eta = \sigma + \gamma$ process with initial measure $\nu_{\rho,\lambda}$. At time 0 we couple the initial configurations as in Section 5: Let $\{U_x\}_{x\in\mathbb{Z}}$ be a sequence of independent identically distributed random variables with distribution uniform in $[0,1]$. Given a realization of those variables we define $\sigma(x) = 1\{U(x) \le \rho\}$ and $\gamma(x) = 1\{\rho < U_x \le \lambda\}.1\{x \ge 0\}$. The two processes use the same arrows, hence the process σ_t coincides with the set of sites occupied by the two marginals and γ_t the set of sites occupied only by the marginal $\sigma + \gamma$. The reader can check that when an arrow appear from site x to site y, the following can happen:

1. The site x is occupied by a σ or γ particle and y is empty. Then the particle jumps.
2. The site x is occupied by a σ or γ particle and y is occupied by a σ particle. Then nothing happens.
3. The site x is occupied by a γ particle and y is occupied by a γ particle. Then nothing happens.
4. The site x is occupied by a σ particle and y is occupied by a γ particle. Then the particles interchange positions: after the arrow $\sigma(y) = 1$ and $\gamma(x) = 1$.

We say that the σ particles are first class and the γ are second class particles. We denote this priority by $\sigma_t \vdash \gamma_t$.

Our program is to prove that the law of the system as seen from the leftmost γ particle satisfies (10.1). When $\rho = 0$ the η process coincides with the γ process and we saw in Section 7 that this is the case. For $\rho > 0$ we call $X(t)$ the position of the leftmost γ particle. We have that $\sigma_t \vdash (\gamma_t \setminus X(t)) \vdash X(t)$ and since $\eta_t = \sigma_t + \gamma_t$, we have $(\eta_t \setminus X(t)) \vdash X(t)$, where we identify the position $X(t)$ with the configuration having a particle at $X(t)$ and no particles at the other sites. So the process $(\eta_t, X(t))$ is well defined without using the σ and γ processes.

Theorem 10.2. Assume that η_0 has distribution $\nu_{\rho,\lambda}$ and $X(0) = 0$. Then $X(t)$ is a shock for η_t.

The proof of this theorem uses properties of translation invariant systems. To get a related translation invariant system we couple the processes (σ_t, γ_t) with a third process ζ_t in such a way that $\sigma_t + \gamma_t + \zeta_t$ at time zero has distribution ν_λ that is translation invariant. The way of doing that is to use the uniform variables U_x: define $\zeta(x) = 1\{\rho < U_x \le \lambda\}.1\{x < 0\}$. Call $\xi_t = \gamma_t + \zeta_t$. Then the process (σ_t, ξ_t) has initial product distribution with marginals ν_ρ and $\nu_{\lambda-\rho}$ and a translation invariant distribution for all t. We say that a measure π_2 on $\mathbf{X}^2$ has the good marginals if $\int d\pi_2(\sigma,\xi) f(\sigma) = \nu_\rho f$ and $\int d\pi_2(\sigma,\xi) f(\sigma+\xi) = \nu_\lambda f$. If the process (σ_t, ξ_t) has initial distribution with the good marginals, then at any time t the distribution of the process has the good marginals.

Assume now that $\xi(0) = 1$, so that $X(t)$ is the position of the ξ particle that at time zero is at the origin. Now we have the following result the proof of which is the same as the one of Theorem 8.2:

Theorem 10.3. Let $\lambda > \rho$, π_2 be a translation invariant measure, $\pi_2' = \pi_2(.|\xi(0) = 1)$ and $S_2'(t)$ be the semigroup of the process $\tau_{X(t)}(\sigma_t, \xi_t)$, then

$$\pi_2' S_2'(t) = (\pi_2 S_2(t))' \tag{10.4}$$

Equation (10.4) is crucial. It expresses the distribution of the process as seen from $X(t)$ in terms of the distribution of the process without shifting.

Lemma 10.5. If π_2 is a measure with the good marginals then for any cylindric function f on $\mathbf{X}$,

$$\begin{aligned} \lim_{x\to\pm\infty} \int d\pi_2'(\sigma,\xi)\tau_x f(\sigma) &= \nu_\rho f \\ \lim_{x\to\pm\infty} \int d\pi_2'(\sigma,\xi)\tau_x f(\sigma+\xi) &= \nu_\lambda f \end{aligned} \tag{10.6}$$

Proof. We use our knowledge that the σ and $\sigma + \xi$ marginals of π_2 are ν_ρ and ν_λ respectively. Call $g(x) = \tau_x f$. Since f is cylindric function, $g(x)$ is uniformly bounded and for any sequence x_n, there exists a subsequence $x_{n(k)}$ such that

$\pi_2' g(x_{n(k)})$ is convergent. We want to show that the limit is always $\nu_\rho f$. Since $\pi_2' g(x_{n(k)})$ is convergent, so is the Cesaro limit

$$\lim_{K\to\infty} (1/K) \sum_{k=1}^{K} \pi_2' g(x_{n(k)}) \tag{10.7}$$

To compute this limit we take yet another subsequence with the property that for all k, $n(k+1) - n(k) > b - a$, where we let $[a, b]$ be a (finite) interval containing the set of sites determining f. For this subsequence we have that, under π_2, $g(x_{n(k)})$ and $g(x_{n(k+1)})$ are independent. Then, since the σ marginal of π_2 is ν_ρ, the law of large numbers imply that

$$\lim_{K\to\infty} (1/K) \sum_{k=1}^{K} g(x_{n(k)}) = \nu_\rho f \quad \pi_2 \ a.s.$$

Since π_2' is absolutely continuous with respect to π_2, the same is true π_2' a.s. By dominated convergence we have that the limit in (10.7) must be $\nu_\rho f$ and we have proved the first line of (10.6). Analogously one proves the second line. ♣

Proof of Theorem 10.2. Let π_2 be the product measure with the good marginals. Let (σ, ξ) be distributed according to π_2'. Then, defining η by $\eta(x) = \sigma(x) + \xi(x)1\{x \ge 0\} = \sigma(x) + \gamma(x)$, we have that η has distribution $\nu'_{\rho,\lambda}$, $X(0) = 0$ and $\eta_t(x) = \sigma_t(x) + \xi_t(x)1\{x \ge X(t)\}$. Now, $\tau_{X(t)}\eta_t(x) = \tau_{X(t)}(\sigma_t(x) + \xi_t(x))1\{x \ge 0\}$. On the other hand by (10.4), $\tau_{X(t)}(\sigma_t, \xi_t)$ has distribution $(\pi_2 S_2(t))'$ which is absolutely continuous with respect to $\pi_2 S_2(t)$, a measure with the good marginals. Hence Lemma 10.5 implies that, uniformly in t,

$$\lim_{x\to+\infty} \int d(\nu'_{\rho,\lambda} S'(t))(\eta) \tau_x f(\eta) = \lim_{x\to+\infty} \int d(\pi_2' S_2'(t))(\sigma, \xi) \tau_x f(\sigma + \xi) = \nu_\lambda f$$

by Lemma 10.5. Analogously,

$$\lim_{x\to-\infty} \int d(\nu'_{\rho,\lambda} S'(t))(\eta) \tau_x f(\eta) = \lim_{x\to-\infty} \int d(\pi_2' S_2'(t))(\sigma, \xi) \tau_x f(\sigma) = \nu_\rho f \quad ♣$$

11. Law of Large Numbers for the Shock

In this section we prove that $X(t)/t$ converges almost surely to $v = 1 - \rho - \lambda$. Given $U(t) \in I\!R$ we define the σ flux throught $U(t)$ by $F_\sigma(t) := \#\{i : z_i(0) <$

$0, z_i(t) \geq U(t)\} - \#\{i : z_i(0) \geq 0, z_i(t) < U(t)\}$, where $z_i(t)$ is the position of the i-th particle of σ_t. For the uniqueness of the representation we fix $z_0(0)$ as the position of the first particle to the right of the origin. Analogously we define $F_{\sigma+\xi}(t)$ and $F_\xi(t)$. If π_2 is a measure with the good marginals, then σ_t and $\sigma_t + \xi_t$ are simple exclusion processes with (extremal invariant) measure ν_ρ and ν_λ respectively. Hence, by Corollary 9.6, if $U(t)$ is a Poisson point process of parameter w, then

$$\lim_{t\to\infty} \frac{F_{\sigma+\xi}(t)}{t} = \lambda(1-\lambda) - w\lambda$$

$$\lim_{t\to\infty} \frac{F_\sigma(t)}{t} = \rho(1-\rho) - w\rho$$

But $F_{\sigma+\xi}(t) = F_\sigma(t) + F_\xi(t)$, hence

$$\begin{aligned}\lim_{t\to\infty} \frac{F_\xi(t)}{t} &= [\lambda(1-\lambda) - \rho(1-\rho] - w(\lambda-\rho) \\ &= (\lambda-\rho)(v-w)\end{aligned}$$

where $v = 1-\lambda-\rho$. The limit is negative for $w > v$ and positive for $w < v$. On the other hand, $F_{(.)}(t)$ is non increasing in $U(t)$ and for $U(t) = X(t)$, due to the exclusion interaction, $F_\xi(t) \equiv 0$. Hence

$$\lim_{t\to\infty} \frac{X(t)}{t} = v, \quad P_{\nu_{\rho,\lambda}}\ a.s.\clubsuit \tag{11.1}$$

As a consequence of the law of large numbers we can prove the hydrodynamics announced in Section 4:

Theorem 11.2. Let $u(r,t)$ be the following solution of the Burgers equation (2.1):

$$u(r,t) = \begin{cases} \lambda & \text{for } r > vt \\ \rho & \text{for } r < vt. \end{cases}$$

Then in the continuity points of $u(r,t)$,

$$\lim_{\varepsilon\to 0} \nu_{\rho,\lambda} S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}r} f = \nu_{u(r,t)} f$$

Proof. Let $A \subset \mathbb{Z}$ be a finite set and let $f_A(\eta) := \prod_{x\in A} \eta(x)$.

$$\begin{aligned}&\int d\nu_{\rho,\lambda}(\eta) E_\eta \tau_{\varepsilon^{-1}r} f_A(\eta_{\varepsilon^{-1}t}) \\ &= \int d\nu_2'(\sigma,\xi) E_{(\sigma,\xi)} \tau_{\varepsilon^{-1}r} f_A(\sigma_{\varepsilon^{-1}t} + \gamma_{\varepsilon^{-1}t}) \\ &= \int d\nu_2'(\sigma,\xi) E_{(\sigma,\xi)} \tau_{\varepsilon^{-1}r - X(\varepsilon^{-1}t)} f_A(\sigma'_{\varepsilon^{-1}t} + \gamma'_{\varepsilon^{-1}t}) \\ &= \int d\nu_2'(\sigma,\xi) E_{(\sigma,\xi)} \sum_x \tau_{\varepsilon^{-1}r - x} f_A(\sigma'_{\varepsilon^{-1}t} + \gamma'_{\varepsilon^{-1}t}) 1\{X(\varepsilon^{-1}t) = x\}\end{aligned} \tag{11.3}$$

where E_η, is the expected value of the process with initial configuration η, etc. and $\gamma_t(z) = \xi_t(z)1\{z \geq X(t)\}$. Now, consider a positive number a such that $|r - vt| > a$, and decompose the sum in the last line of (11.3) in three parts: $\{x : |\varepsilon^{-1}r - x| \leq a\}$, $\{x : \varepsilon^{-1}r - x > a\}$ and $\{x : \varepsilon^{-1}r - x < -a\}$. The integral of the first of those three sums goes to zero by (11.1) and dominated convergence. By Lemma 10.5, the second part converges to $\nu_\lambda f_A$, if $r > vt$ and to zero otherwise and the third part converges to $\nu_\rho f_A$ if $r < vt$ and to zero otherwise. ♣

The following result can be proven in a similar way.

Theorem 11.4. Convergence of the density fields. For any smooth function Φ with compact support, the following holds

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} \Phi(\varepsilon x) \eta_{\varepsilon^{-1}t}(x) = \int_{\mathbb{R}} \Phi(r) u(r,t) dr$$

$P_{\nu_{\rho,\lambda}}$ almost surely.

12. Second Class Particles and Characteristics

We saw in Section 2 that in regions where the solution u of (2.1) is constant, the characteristics have speed $1 - 2u$, and that the shock forms at the meeting point of the characteristics with different speeds. In this section we study the microscopic counterpart of this phenomenon. We first show that if a second class particle is added at time zero at a given site, and the first class particles are distributed according to ν_α, then, calling $W(t)$ the position at time t of the second class particle, we have that

$$\lim_{t \to \infty} \frac{W(t)}{t} = 1 - 2\alpha, \quad P_{\hat{\nu}_\alpha} \ a.s. \tag{12.1}$$

Denote by $W(\varepsilon^{-1}r, \varepsilon^{-1}t)$ the position at time $\varepsilon^{-1}t$ of a second class particle that at time zero is at site (integer part of) $\varepsilon^{-1}r$. Then we prove that

$$\lim_{\varepsilon \to 0} \varepsilon W(\varepsilon^{-1}r, \varepsilon^{-1}t) = \begin{cases} w(r,t), & \text{for } t < t(r) \\ vt, & \text{for } t > t(r) \end{cases} \quad P_{\hat{\nu}_{\rho,\lambda}} \ \ a.s. \tag{12.2}$$

where $w(r,t)$ is the characteristic starting at r and $t(r)$ is the time when $w(r,t)$ and $w(-r,t)$ meet, as defined in Section 2:

$$w(r,t) = \begin{cases} r + (1 - 2\lambda)t & \text{if } r > 0 \\ r + (1 - 2\rho)t & \text{if } r < 0 \end{cases}$$

and $t(r) = |r|/(\lambda - \rho)$.

Proof of (12.1). We couple the process $(\eta_t, W(t))$ (the process with only one second class particle located a $W(t)$) and initial measure ν_α with the process $(\sigma_t, \xi_t, X(t))$ with initial measure $\nu_{\rho,\lambda}$, $\rho < \lambda$ and a ξ particle at the origin. Take first $\rho = \alpha$ (hence $\lambda > \alpha$). Initially the two processes are coupled in such a way that $\eta_0 = \sigma_0$ and $W(0) = X(0) = 0$. Since the processes are using the same arrows and $\sigma_t \vdash \xi_t$, we have that $\eta_t \equiv \sigma_t$ for all t. On the other hand,

$$W(t) \geq X(t) \text{ for all } t \text{ almost surely.} \tag{12.3}$$

Equation (12.3) holds at time 0. To prove it for all times it then suffices to show that if $X(t-) = W(t-) = x$, no arrow involving x at time t has the effect that $W(t) < X(t)$. There are the following possibilities:

– the arrow goes from x to $x+1$ and $\eta_{t-}(x+1) = \sigma_{t-}(x+1) = 1$, then $X(t) = W(t) = x$.

– the arrow goes from x to $x+1$ and $\eta_{t-}(x+1) = \sigma_{t-}(x+1) = 0$, then $W(t) = x+1$ and $X(t) = x + (1 - \xi_{t-}(x+1))$.

– the arrow goes from $x-1$ to x and $\eta_{t-}(x-1) = \sigma_{t-}(x-1) = 1$, then $W(t) = x-1$ and $X(t) = x-1$.

– the arrow goes from $x-1$ to x and $\eta_{t-}(x-1) = \sigma_{t-}(x-1) = 0$, then $W(t) = x$ and $X(t) = x$.

In all cases, after the arrow $W(t) \geq X(t)$. This implies that

$$\liminf_{t\to\infty} \frac{W(t)}{t} \geq \lim_{t\to\infty} \frac{X(t)}{t} = 1 - \rho - \lambda \quad a.s. \tag{12.4}$$

for all $\lambda > \alpha$, where the identity is the law of large numbers for the shock (11.1).

Now, taking $\lambda = \alpha$ $(\rho < \alpha)$, $\eta_t = \sigma_t + \xi_t$, a similar argument shows that $W(t) \leq X(t)$ almost surely and that

$$\limsup_{t\to\infty} \frac{W(t)}{t} \leq \lim_{t\to\infty} \frac{X(t)}{t} = 1 - \rho - \lambda \quad a.s. \tag{12.5}$$

Since (12.4) holds for $\rho = \alpha$ and all $\lambda > \alpha$ and (12.5) holds for $\lambda = \alpha$ and all $\rho < \alpha$ this completes the proof of (12.1). ♣

Proof of (12.2). For each r and ε couple $(\eta_t, W(\varepsilon^{-1}r, t))$ with initial measure $\nu_{\rho,\lambda}$ with $(\eta_t, X(t))$, with initial measure $\hat\nu_\alpha$. Observe that after $T := \inf\{t : X(t) = W(\varepsilon^{-1}r, t)\}$, we have that $X(t) \equiv W(t)$. Hence one only needs to prove that $\lim_{\varepsilon\to 0}\varepsilon T = t(r)$. This follows from the law of large numbers for $X(t)$ (11.1) and the following law of large numbers for $W(t)$:

$$\lim_{\varepsilon\to 0}\varepsilon W(\varepsilon^{-1}r,\varepsilon^{-1}t) = r + (1-2\alpha)t \quad P_{\nu_\alpha}\ a.s. \tag{12.6}$$

that can be shown as (12.1). ♣

13. Shock Fluctuations

We first show that a perturbation on the initial condition translates as time goes to infinity into a shift of the shock position. For any given configuration η and site $y \in \mathbb{Z}$, let $\eta^{y|i}$ be defined by ($i \in \{0,1\}$):

$$\eta^{y|i}(x) = \begin{cases} \eta(x) & \text{for } x \neq y \\ i & \text{for } x = y. \end{cases}$$

Let $X(\eta,t)$ be the random position of the shock when the initial configuration is η, *i.e.* the position of a second class particle initially at 0. Define $r^+ := (\lambda-\rho)$, $r^- := -(\lambda-\rho)$.

Theorem 13.1. For all $\varepsilon > 0$ it holds

$$\lim_{t\to\infty}\sup_{(r^-+\varepsilon)t<y<(r^+-\varepsilon)t}\left|E_{\nu_{\rho,\lambda}}(X(\eta^{y|0},t)-X(\eta^{y|1},t))-(\lambda-\rho)^{-1}\right| = 0 \tag{13.2}$$

$$\lim_{t\to\infty}\frac{1}{t}\sum_{y=0}^{r^+t}E_{\nu_{\rho,\lambda}}(X(\eta^{y|0},t)-X(\eta^{y|1},t)) = 1 \tag{13.3}$$

$$\lim_{t\to\infty}\frac{1}{t}\sum_{y=r^-t}^{0}E_{\nu_{\rho,\lambda}}(X(\eta^{y|0},t)-X(\eta^{y|1},t)) = 1 \tag{13.4}$$

Proof. To catch the idea we first prove it for $\rho = 0$. It is convenient to consider an initial configuration ξ distributed according to ν_λ and η_t defined by $\eta_t(x) = \xi_t(x)1\{x \geq X(t)\}$, where $X(t) = X(\eta,t)$. Call $X^i(t) := X(\eta^{y|i},t)$. We have already seen that the two processes will differ at only one site. Let $W(t)$ be the site where the processes with initial configuration $\eta^{y|1}$ and $\eta^{y|0}$ differ by time t. Call $x_n(t)$ the positions at time t of the particles of the process starting at $\xi^{y|0}$, with $x_0(0) = 0$. The shock in both configurations is $X^1(t) = X^0(t) = x_0(t)$ for all times $t \leq T_1 := \inf\{t : W(t) < x_0(t)\}$. After T_1, due to the exclusion interaction $X^1(t) = W(t) < x_0(t)$. Call $T_2 := \inf\{t : W(t) = x_{-1}(t)\}$. Now T_2 is finite with probability one. After T_2, $W(t) \equiv x_{-1}(t)$. Since $\nu'_\lambda(x_0 - x_{-1}) = 1/\lambda$ we get the theorem in this case.

When $\rho > 0$ the shock is given also by $x_0(t)$, the 0-th ξ-particle. With the same argument we show that, after T_2, $W(t) \equiv x_{-1}(t)$. Equations (13.3) and (13.4) follow because the expectation of each term is bounded by $\lambda - \rho$. ♣

Define (if the limit exists)

$$D = \lim_{t\to\infty} t^{-1} E_{\nu'_{\rho,\lambda}} (X(t) - vt)^2$$

and

$$\bar{D} := \frac{\rho(1-\rho) + \lambda(1-\lambda)}{\lambda - \rho}.$$

The next result gives $\bar{D}$ as a lowerbound for D.

Theorem 13.5. $D \geq \bar{D}$

The proof of this theorem is a corollary of the next result. In order to state it define

$$D(t) := E_{\nu'_{\rho,\lambda}} (X(t) - vt)^2,$$

$$I(t) := \int d\nu'_{\rho,\lambda}(\eta) E \left(X(\eta, t) - \frac{n_0(\eta, r^+t) - n_1(\eta, r^-t)}{\lambda - \rho} \right)^2,$$

where $X(\eta, t)$ and $r^{\pm}$ are defined at the begining of Section 13 and $n_0(\eta, x) := \sum_{y=0}^{x} (1 - \eta(y))$ is the number of empty sites of η between 0 and x and $n_1(\eta, x) := \sum_{y=x}^{0} \eta(y)$ is the number of η particles between the origin and $x < 0$.

Theorem 13.6. The following holds

$$\lim_{t\to\infty} \frac{D(t)}{t} = \bar{D} + \lim_{t\to\infty} \frac{I(t)}{t} \tag{13.7}$$

if the limits exist. If not (13.7) holds with lim substituted by either lim sup or lim inf.

Proof. Summing and substracting vt, $I(t)$ equals

$$\begin{aligned} &\int d\nu'_{\rho,\lambda}(\eta) E \left(X(\eta, t) - vt \right)^2 + \int d\nu'_{\rho,\lambda}(\eta) \left(\frac{n_0(\eta, r^+t)}{\lambda - \rho} - (1-\lambda)t \right)^2 \\ &\quad + \int d\nu'_{\rho,\lambda}(\eta) \left(\frac{n_1(\eta, r^-t)}{\lambda - \rho} - \rho t \right)^2 \\ &\quad - 2 \int d\nu'_{\rho,\lambda}(\eta) E \left(X(\eta, t) \left(\frac{n_0(\eta, r^+t)}{\lambda - \rho} - (1-\lambda)t - \frac{n_1(\eta, r^-t)}{\lambda - \rho} + \rho t \right) \right) \end{aligned} \tag{13.8}$$

where we have used that $n_0(\eta, r^+t)$ and $n_1(\eta, r^-t)$ are independent under $\nu'_{\rho,\lambda}$. Dividing by t and taking $t \to \infty$, the first term gives $\lim(D(t)/t)$, and the second and third terms give $\bar{D}$. Then it suffices to show that dividing by t and taking $t \to \infty$ the last term equals $-2\bar{D}$. Using the definition of $n_i(.,.)$, the expectation in the last term in (13.8) equals

$$\frac{1}{\lambda-\rho} E\left(\sum_{x=0}^{r^+t} X(\eta,t)(1-\eta(x)-(1-\lambda)) - \sum_{x=r^-t}^{0} X(\eta,t)(\eta(x)-\rho)\right) \tag{13.9}$$

Integrating the first term of (13.9),

$$\begin{aligned}
&-\frac{1}{\lambda-\rho}\int d\nu'_{\rho,\lambda}(\eta)\sum_{x=0}^{r^+t} E\left(X(\eta,t)(\eta(x)-\lambda)\right)\\
&= -\frac{1}{\lambda-\rho}\int d\nu'_{\rho,\lambda}(\eta)\sum_{x=0}^{r^+t}\big[E\left(X(\eta,t)|\eta(x)=1\right)\lambda\\
&\qquad -\lambda\big(E\left(X(\eta,t)|\eta(x)=1\right)\lambda + E\left(X(\eta,t)|\eta(x)=0\right)(1-\lambda)\big)\big]
\end{aligned}$$

this equals

$$\begin{aligned}
&= -\frac{1}{\lambda-\rho}\lambda(1-\lambda)\int d\nu'_{\rho,\lambda}(\eta)\\
&\quad\times\sum_{x=0}^{r^+t}\big[E\left(X(\eta,t)|\eta(x)=1\right) - E\left(X(\eta,t)|\eta(x)=0\right)\big] \qquad (13.10)\\
&= -\frac{1}{\lambda-\rho}\lambda(1-\lambda)\int d\nu'_{\rho,\lambda}(\eta)\sum_{x=0}^{r^+t} E\left(X(\eta^{x|1},t) - X(\eta^{x|0},t)\right)
\end{aligned}$$

Dividing by t and taking the limit as $t \to \infty$ of the first term of (13.9), we get using (13.3) on (13.10) that

$$-\frac{1}{\lambda-\rho}\lim_{t\to\infty}\frac{1}{t}\int d\nu'_{\rho,\lambda}(\eta)\left(\sum_{x=0}^{r^+t} X(\eta,t)(\eta(x)-\lambda)\right) = \frac{\lambda(1-\lambda)}{\lambda-\rho}$$

and analogously using (5.4),

$$-\frac{1}{\lambda-\rho}\lim_{t\to\infty}\frac{1}{t}\int d\nu'_{\rho,\lambda}(\eta)\sum_{x=r^-t}^{0} X(\eta,t)(\eta(x)-\rho) = \frac{\rho(1-\rho)}{\lambda-\rho}$$

This implies the Theorem. ♣

Remarks. From Theorem 13.6 we conclude that the diffusion coefficient of the shock is the same as the conjectured diffusion coefficient if and only if the position of the shock at time t is given –in the scale $\sqrt{t}$– by $(\lambda-\rho)^{-1}$ times the number of holes between 0 and $r^{+}t$ minus the number of particles between 0 and $r^{-}t$. In any case, $I(t)$ is non negative, then $\bar{D}$ is always a lower bound and this proves Theorem 13.5. When $\rho = 0$, $X(t)$ has the distribution of a plain tagged particle in the simple exclusion process with density λ In this case it is known that $D := \lim_{t\to\infty} t^{-1}E(X(t) - EX(t))^2 = \bar{D} = (1-\lambda)$ (Corollary 7.2). This implies that $\lim_{t\to\infty} I(t)/t = 0$, hence in the scale $\sqrt{t}$ the position of $R(t)$ is determined by the initial configuration in the sense discussed above. This was proved by Gärtner and Presutti [gp] using a different method.

We finish this section by mentioning a couple of open problems.

Prove that there exists a microscopic shock in more dimensions or for a jump function that allows to go further than to the nearest neighbors. The argument of this approach does not work even for the case of two parallel lines with a symmetric dynamics for jumps between the two lines and asymmetric jumps inside each line. Landim [la1] proved the existence of the hydrodynamical limit in high dimensions for some initial conditions.

General initial conditions. Benassi, Fouque, Saada and Vares [bfsv] have proved that the hydrodynamical limit can be taken when the initial profile is monotone and the jump probabilities are more general. Prove the hydrodynamical limit for any initial profile.

Fluctuations. It is expected that the fluctuations arround the deterministic hydrodynamic limit depend on the initial configuration as the fluctuations of the shock do. In the case $\rho = 0$ this has been studied by [bf2].

14. The Nearest Neighbor Asymmetric Simple Exclusion Process. General Case.

Almost all the results described above have been proven for the process where the particles can also jump backwards. In this case one assumes that the particles jump at rate p to the right nearest neighbor and with rate q the the left one. We assume $p+q=1$ and $p>q$.

The graphical representation works in the same way. The only difference is that now we have different Ppp per bond $(x, x+1)$. One associated to arrows going from x to $x+1$ with rate p and the other associated to arrows going from $x+1$ to x at rate q. As before we can couple two or more versions of the process making the different versions to follow the same arrows.

The set of extremal invariant measures is $\{\nu_\alpha : 0 \le \alpha \le 1\} \cup \{\nu^{(n)} : n \in \mathbb{Z}\}$. The measure $\nu^{(n)}$ concentrates on the set $\mathbf{A}_n := \{\eta : \sum_{x\ge 0}(1-\eta(x)) - \sum_{x<0}\eta(x) = n\}$. They are defined by $\nu^{(n)} := \nu^{[k]}(.|\mathbf{A}_n)$ for all k, where $\nu^{[k]}$ is also product, have marginals

$$\nu^{[k]}(\eta(x)) = \frac{(p/q)^{x-k}}{1-(p/q)^{x-k}} \tag{14.1}$$

and are even reversible for the process. They approach exponentially fast the densities 0 and 1 to the left and right of the origin respectively, so that, under $\nu^{(n)}$, the origin is a shock for $\rho = 0$ and $\lambda = 1$. Now we construct a reversible measure for the process $(\eta_t, R(t))$, where $R(t)$ stands for the position of the second class particle at time t. Put a second class particle at i with probability

$$m(i) := M(p/q)\left((1+(p/q)^{i-\frac{1}{2}})(1+(q/p)^{i+\frac{1}{2}})\right)^{-1} \tag{14.2}$$

where $M(p/q)$ is a normalizing constant making $\sum m(i) = 1$. For the other sites decide that a η particle is present at site j with probability

$$m(j|i) = \begin{cases} (p/q)^{j-\frac{1}{2}}/(1+(p/q)^{j-\frac{1}{2}}) & \text{if } j < i \\ (p/q)^{j+\frac{1}{2}}/(1+(p/q)^{j+\frac{1}{2}}) & \text{if } j > i \end{cases}$$

independently of everything. Since the measure concentrates on a denumerable state space, that statement follows from a routine computation. It is clear from (14.2) that the second class particle will remain tight, hence it also is a shock in this case.

In the case $0 \le \rho < \lambda \le 1$ one follows the same steps as in the case $p = 1$. So we construct a process $(\sigma_t, \xi_t, X(t))$, with $\sigma_t \vdash \xi_t$ and $X(t)$ is a tagged ξ particle. Then one calls $x_i(t)$ the positions of the ξ particles, assuming that $x_0(t) \equiv X(t)$. Finally one chooses the i-th ξ_t particle to be $R(t)$ with probability $m(i)$ and label the others ξ particles independently in the following way: the j-th particle is labeled γ with probability $m(j|i)$ otherwise it is labeled ζ. The remarkable property of this distribution is that the labeling remains invariant for later times. Hence the density of γ particles vanishes to the left of $X(t)$ and the density of ζ particles vanishes to the right of $X(t)$ exponentially fast. Furthermore $X(t) - R(t)$ remains tight. Since our original process can be recovered by writing $\eta_t = \sigma_t + \gamma_t$ as before one can prove that either $X(t)$ or $R(t)$ are microscopic shocks. The advantage of $R(t)$ is that it can be defined directly as a second class particle with respect to η_t, while for defining $X(t)$ one needs to use the process (σ_t, ξ_t).

With this observation and a little care one can prove all the other results. We refer to [fks] and [f2] for details.

15. The Weakly Asymmetric Simple Exclusion Process

We have studied the hydrodynamical limit of a process by rescaling time and the space in a convenient way. The rescaling is done with a parameter ε. Another possibility is to consider not a single process, but a family of processes depending on the parameter ε. Indeed this is what is done to derive the (viscous) Burgers equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}(u(1-u)) = 0 \tag{15.1}$$

This kind of limit is called kinetic. In this case the family consists of asymmetric simple exclusion processes as defined in the previous section with $p = \frac{1+\varepsilon}{2}$ and $q = \frac{1-\varepsilon}{2}$. Since the asymmetry $p - q$ goes to 0 with the scaling parameter ε, the resulting (family of) process(es) is called weakly asymmetric. It was introduced by De Masi, Presutti and Scacciatelli [dps] and studied by Gartner [g], Dittrich [d] Gartner and Dittrich [gd] and Ferrari, Kipnis and Saada [fks]. The results are quite complete and reinforce the conjectures for the simple exclusion process we did in the previous sections.

Let L^ε be defined by $L^\varepsilon = (1 - \varepsilon/2)L_0 + \varepsilon L$, where L is the generator of the simple exclusion process defined in (3.2) and

$$L_0 f = \sum_x \tfrac{1}{2}[f(\eta^{x,x+1}) - f(\eta)]$$

is the generator of a symmetric exclusion process. Denoting by $S^\varepsilon(t)$ the semigroup corresponding to the generator L^ε, and by ν^ε a family of product mesures with marginals given by $\nu^\varepsilon(\eta(\varepsilon^{-1}r)) = u_0(r)$ the kinetic limit is given by

$$\lim_{\varepsilon\to 0} \nu^\varepsilon S^\varepsilon(\varepsilon^{-1}t)\tau_{[\varepsilon^{-1/2}r]} f = \nu_{u(r,t)} f$$

(local equilibrium), where $u(r,t)$ is the solution of (15.1) with inital condition $u(r,0) = u_0(r)$. This result was proven by De Masi, Presutti and Scacciatelli [dps] and by Gartner [g]. Notice that the scaling is different from the one we used to derive the unviscous Burgers equation. To obtain the Laplacian $\partial^2 u/\partial^2 r$ one needs to scale space as square root of time. Since we are looking at time ε^{-1} and L is multiplied by ε, the asymmetry is not rescaled but it appears anyway in the macroscopic limit as a transport term. The same authors proved the law of large numbers for the density fields: Let Φ be a smooth function with compact support. Then, calling η_t^ε the process with generator L^ε,

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} \Phi(\varepsilon x)\eta^\varepsilon_{\varepsilon^{-1}t}(x) = \int_{\mathbb{R}} \Phi(r)u(r,t)dr$$

$P^\varepsilon_{\nu^\varepsilon}$ almost surely. The stationary case was study by Ferrari, Kipnis and Saada [fks]: For each ε there exists a position $X(t)$ such that the process as seen from $X(t)$ has a law μ^ε with the property that for all ε, $\mu^\varepsilon \sim \nu_{\rho,\lambda}$. Furthermore

$$\lim_{\varepsilon\to 0} \mu^\varepsilon(\eta(\varepsilon^{-1/2}r)) = u(r)$$

where

$$u(r) := \rho + \frac{\lambda-\rho}{1+e^{-2r(\lambda-\rho)}}$$

is the stationary travelling wave solution of the Burgers equation (15.1) with asymptotic densities ρ and λ. Also the density fields converge and the hydrodynamical limit is performed for this family of initial measures. The equilibrium case in a finite macroscopic box was studied by De Masi, Ferrari and Vares [dfv].

A stronger result has been proved by Dittrich [d], who exibits a function ξ_t of the initial configuration η_0, such that for any test function ψ,

$$\lim_{\varepsilon\to 0} \varepsilon \sum_x (\eta^\varepsilon_{\varepsilon^{-1}t}(x) - \xi_{\varepsilon^{-1}t}(x))\psi(\varepsilon x) = 0$$

In other words, in that scale the motion is determined by the initial configuration. The approach also allows to study the shock wave case. In this case he proves that the shock fluctuates as a Brownian motion with diffusion coefficient given by $\bar{D}$, defined in Section 13 and that these fluctuations depend only on the initial configuration.

An interesting problem is to decide what happens with the second class particle in this limit. Calling $R(t)$ the position of the second class particle at time t, with $R(0) = 0$, the rescaling of (14.2) gives

$$\lim_{\varepsilon\to 0} P_{\mu^\varepsilon}(R^\varepsilon(t) - vt \le \varepsilon^{-(1/2)}r) = M\int_{-\infty}^{r} \frac{\lambda-\rho}{(1+\exp(-2s(\lambda-\rho)))(1+\exp(2s(\lambda-\rho)))} ds$$

where $M = \lim_{\varepsilon\to 0} M((\frac{1}{2}+\varepsilon)/(\frac{1}{2}-\varepsilon))$ is the limit of the normalizing constants of (14.2). We conjecture that the limiting motion of the second class particle is a Ornstein Uhlenbeck process with an appropriate drift with stationary measure given above.

De Masi, Presutti and Scacciatelli [dps] have studied the fluctuation fields and proved that they converge to a generalized Ornstein Uhlenbeck process. Ravishankar [ra] studied the d-dimensional case.

16. The Boghosian Levermore Cellular Automaton

In this section we review recent results of Ferrari and Ravishankar [fr] on a deterministic version of a probabilistic cellular automaton introduced by Boghosian and Levermore [bl]. This is a dynamical system with random initial condition. The simplicity of the model allows to prove all the known results and conjectures for the asymmetric simple exclusion process.

A configuration of the one dimensional Boghosian Levermore Cellular automaton (BCLA) is an arrangement of particles with velocities $+1$ and -1 on $\mathbb{Z}$, satisfying the exclusion condition that there is at most one particle with a given velocity ($+1$ or -1) at each site. We denote a configuration by $\eta \in \{0,1\}^{\mathbb{Z}\times\{-1,+1\}} := \mathbf{X}$, the state space. If $\eta(x,s) = 1$ we say that there is a particle with velocity s at site x, where $x \in \mathbb{Z}$ and $s \in \{-1,+1\}$.

Dynamics. The time is discrete and the dynamics is given in two steps:

1. Collision: for a given η let $C\eta$ be the configuration

$$C\eta(x,+1) = 1\{\eta(x,1) + \eta(x,-1) \geq 1\} = \max\{\eta(x,1), \eta(x,-1)\}$$
$$C\eta(x,-1) = 1\{\eta(x,1) + \eta(x,-1) = 2\} = \min\{\eta(x,1), \eta(x,-1)\}$$

In other words, if there is no particle or two particles at x, then nothing happens. If there is only one particle at x, then this particle adopts velocity 1.

2. Advection. This part of the dynamics moves each particle along its velocity to a neighboring site in unit time. The operator A is defined by

$$A\eta(x,s) = \eta(x-s,s).$$

Defining $T := AC$, the dynamics is given by

$$\eta_{t+1} = T\eta_t$$

We say that a measure μ on $\mathbf{X}$ is stationary for the process if $\mu T = \mu$. Cheng, Lebowitz and Speer [cls] have noticed that this dynamics acts independently in the space-time sublattices $\{(x,t) : x+t \text{ is even}\}$ and $\{(x,t) : x+t \text{ is odd}\}$. Any translation invariant measure concentrating in one of the sets $\{\eta : \eta(x,1) = 1\}$, $\{\eta : \eta(x,-1) = 0\}$ is stationary for the process. Also there are non translation invariant measures, the measures ν^n giving mass 1/2 to η^n and $T\eta^n$ where $\eta^n(x,\pm 1) = 1\{x \geq n\}$. Observe that the configuration η^n is two steps invariant (i.e. $\eta^n = T^2\eta^n$). The problem of determining if the measures introduced above are sufficient to describe the set of all invariant measures for the process is open. We remark that in contrast with simple exclusion, there are stationary translation invariant ergodic measures that are not product measures.

The hydrodynamical limit has been performed for initial product measures with densities 0 and ρ for particles with velocity -1 and $+1$ respectively at negative sites, and with densities λ and 1 for particles with velocities -1 and $+1$, respectively to the right of the origin. Hence the particle density per site to the left of the origin is ρ and to its right is $1+\lambda$. The limiting density per site satisfies the equation

$$\frac{\partial u}{\partial t} + F(u) = 0$$

where

$$F(u) = u1\{0 \le u \le 1\} + (2-u)1\{1 \le u \le 2\}$$

This equation has only two characteristics, 1 and -1 according to the density being bigger or smaller than 1. The definition of the microscopic shock is the same as for the simple exclusion process. Just take two configurations that differ at only one velocity at only one site. At latter times they will also differ at only one site that we call second class particle, as it gives priority to other particles that attempt to occupy its place.

When the initial configuration is taken from the product measure described above, it turns out that the position of the second class particle can be expressed as a sum of a random number of independent and identically distributed random variables, each of which is a difference of two geometric random variables. Furthermore the number of summands is independent of the summands. Hence, as a corollary of this result we can prove laws of large numbers and central limit theorems for the position of the shock. Another result is that the position of the shock at any given time is independent of the configuration at that time as seen from the shock. This gives a way to prove the hydrodynamical limit described above.

The results can be extended to initial measures with more than one step and to decreasing profiles and also to the probabilistic cellular automata when the C rule is applied with probability p and is not applied with probability $1-p$ [frv]. The weakly asymmetric case was studied by [lop].

17. Other Cellular Automata

In this section we show that the BLCA is isomorphic to two simple exclusion type of automata in $\{0,1\}^{\mathbb{Z}}$ and a type of sand-pile automaton.
The asymmetric simple exclusion cellular automaton. For a given configuration $\xi \in \{0,1\}^{\mathbb{Z}}$, define $B_1\xi$ as the configuration

$$\begin{aligned} B_1\xi(2z+1) &= 1\{\xi(2z)+\xi(2z+1) \ge 1\} = \max\{\xi(2z), \xi(2z+1)\} \\ B_1\xi(2z) &= \xi(2z)+\xi(2z+1) - 1\{\xi(2z)+\xi(2z+1) \ge 1\} \\ &= \min\{\xi(2z), \xi(2z+1)\} \end{aligned}$$

and $B_2\xi$ as the configuration

$$B_2\xi(2z) = 1\{\xi(2z-1)+\xi(2z)\geq 1\}$$
$$B_2\xi(2z-1) = \xi(2z-1)+\xi(2z) - 1\{\xi(2z-1)+\xi(2z)\geq 1\}$$

Now define the asymmetric simple exclusion cellular automaton (ASECA):

$$\xi_{2t} = B_1\xi_{2t-1} \text{ and } \xi_{2t+1} = B_2\xi_{2t} \tag{17.1}$$

In words, at even times, all particles occupying even sites that can jump to the right (i.e. that the succesive odd site is empty), jump to the right. At odd times, the particles occupying the odd sites do the same.

We prove that this is isomorphic to a subsystem of the BLCA. As observed before, the BLCA consists in two independent subsystems: $\{\eta(x,s,t) : x+t \text{ odd }\}$ and $\{\eta(x,s,t) : x+t \text{ even }\}$. Consider the subsystem $\{\eta(x,s,t) : x+t \text{ odd }\}$ and define the configuration ξ_t by

$$\xi_t(2x+1) = \begin{cases} \eta(2x+1,-1,t) & \text{for } t \text{ even} \\ \eta(2x,1,t) & \text{for } t \text{ odd} \end{cases} \tag{17.2.a}$$

and

$$\xi_t(2x) = \begin{cases} \eta(2x-1,1,t) & \text{for } t \text{ even} \\ \eta(2x,-1,t) & \text{for } t \text{ odd} \end{cases} \tag{17.2.b}$$

We have the following result which proof is straightforward (see Figure 9)

Lemma 17.3. The transformation (17.2) defines an isomorfism between the subsystem $\{\eta(x,s,t) : x+t \text{ odd }\}$ and $\xi_t, t\in \mathbb{Z}$, such that ξ_t is the asymmetric simple exclusion cellular automaton, with distribution described by (17.1).

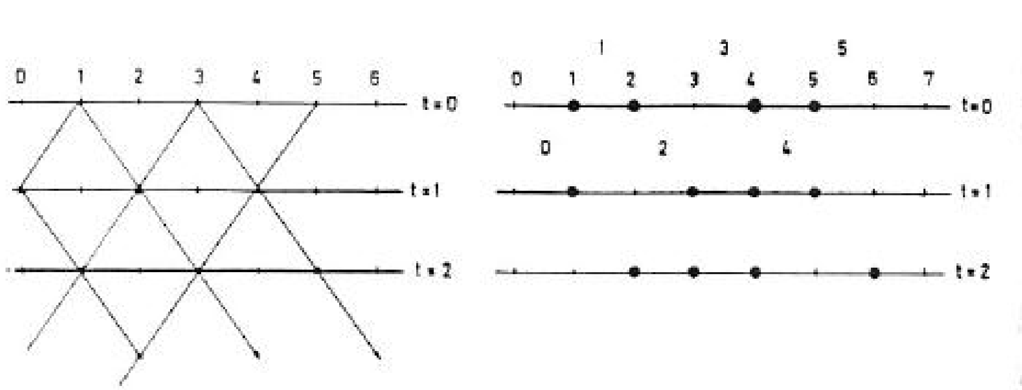


Figure 9. Isomorphism between the BLCA and the ASECA.

The automaton 184. This automaton was classified by Wolfram [wo], and studied by Krug and Spohn [ks]. For a configuration $\gamma \in \{0,1\}^{\mathbb{Z}}$, let $B\gamma$ be the configuration defined by

$$B\gamma(z) = \begin{cases} 1 & \text{if } \gamma(z-1) = 1 \text{ and } \gamma(z) = 0 \\ 0 & \text{if } \gamma(z) = 1 \text{ and } \gamma(z+1) = 0 \\ \gamma(z) & \text{otherwise} \end{cases}$$

In words, $B\gamma$ is the configuration obtained when all particles of γ allowed to jump one unit to the right do it. Define the automaton by $\gamma_t = B\gamma_{t-1}$. Assume now that at time 0, all even sites are empty. In this case this is isomorphic to ξ_t with the same initial configuration. On the other hand, if all even sites are occupied, it is also isomorphic to ξ_t. Nevertheless, for other configurations this system is not isomorphic, presenting a richer structure. But for those tipe of initial conditions the results proved for the BLCA hold.

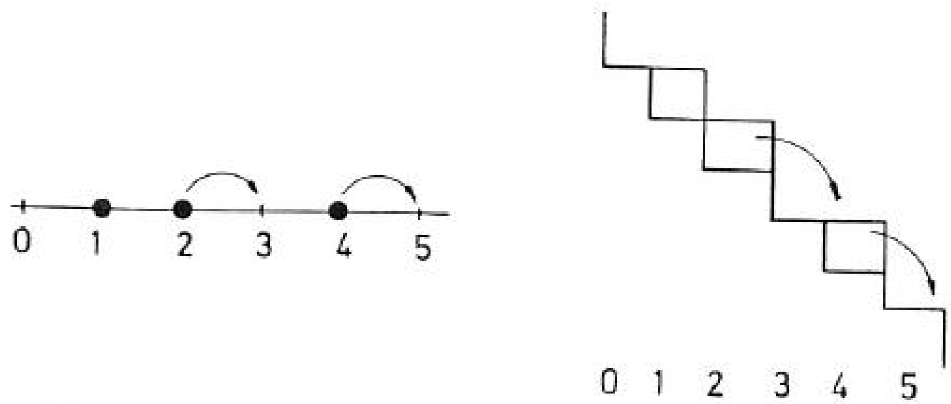


Figure 10. Isomorphism between the automaton 184 and a sand-pile.

Sand piles. We consider now an infinite version of an automaton introduced by Bak [b] and studied by Goles [g]. Consider a process ζ_t on the state space $\{\zeta \in \mathbb{Z}^{\mathbb{Z}} : \zeta(x) \geq \zeta(x+1)\}$. Define $C : \zeta \mapsto C\zeta$ as follows

$$C\zeta(x) = \zeta(x) + 1\{\zeta(x-1) - \zeta(x) \geq 2\} - 1\{\zeta(x) - \zeta(x+1) \geq 2\}$$

In words, we can think that at each integer there is a pile of grains of sand. At each time each pile is ready to give one of its grains to its right neighboring pile. But this only happens if after that the receiving pile is not higher than the the one that is giving the grain. This operations are all done in parallel. The automaton is defined by

$$\zeta_t = C\zeta_{t-1}$$

It turns out that for some initial configurations, this automaton is isomorphic to the Automaton 184 described above. This has been established in [fgv]. Let γ be a configuration of $\{0,1\}^{\mathbb{Z}}$ and define $\zeta = \zeta(\gamma)$ as the configuration

$$\zeta(x) = -x + \gamma(x) \tag{18.10}$$

Then it is easy to see that

$$C\zeta(x) = -x + B\gamma(x)$$

so that we get that $\zeta_t(x) = -x + \gamma_t(x)$ for all t, all x. This implies that all the results for the hydrodynamics and shocks hold for this model for this kind of initial conditions. Other initial conditions are under investigation by [fgv].

It is not hard to see that a particle system can be constructed using the same law. Assume that at each site we have a Poisson point process of rate one, independent of everything. When the clock rings at site x, it attempts to give a grain to site $x+1$, but it does so only if $\zeta(x) - \zeta(x+1) \geq 2$. This graphical construction is useful to show the existence of such a particle system because as in the simple exclusion case, at each given time we can make a (random) partition of the space in finite boxes that do not interact among them. One can show that the transformation (18.10) makes this system isomorphic to the simple exclusion process.

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