$1^{\underline{a}}$ Prova de MAT133 - Cálculo II - IQUSP
13/09/2013

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N$^{\underline{o}}$USP : ____
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É necessário justificar todas as passagens.
Boa Sorte!

1. Calcule a área de $A = \left\{ (x,y) \in \mathbb{R}^2 : \ x > 0 \ \text{ e } \ \frac{1}{x^2} \leq y \leq 5 - 4x^2 \right\}$.

Solução.

Impondo

$$\frac{1}{x^2} = 5 - 4x^2$$

encontramos

$$4x^4 - 5x^2 + 1 = 0 = 4(x^2-1)\left(x^2 - \frac{1}{4}\right).$$

Donde obtemos, já que temos $x > 0$ (por hipótese),

$$x = 1 \ \text{ e } \ x = \frac{1}{2}.$$

Concluímos então que

$$\text{área}(A) = \int_{\frac{1}{2}}^{1} \left(5 - 4x^2 - \frac{1}{x^2}\right) dx = \frac{5}{2} - 4\left(\frac{1}{3} - \frac{1}{24}\right) - 1 = \frac{1}{3} \ \blacksquare$$

2. Calcule

(a) $\displaystyle\int \sqrt{-x^2+2x+2}\,dx.$

(b) $\displaystyle\int \frac{x^2}{x^3-6x^2+11x-6}\,dx.$

Solução.

(a) Temos,

$$\int \sqrt{-x^2+2x+2}\,dx = \int \sqrt{3-(x-1)^2}\,dx.$$

Substituindo $y = \frac{x-1}{\sqrt{3}}$ temos $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$. Logo,

$$\int \sqrt{3-(x-1)^2}\,dx = \int \sqrt{3-3y^2}\,\sqrt{3}\,dy = 3\int \sqrt{1-y^2}\,dy.$$

Agora, substituindo $y = \sin\theta$, com $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, obtemos

$$3\int \sqrt{1-y^2}\,dy = 3\int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = 3\int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta =$$

$$= \int \cos^2\theta\,d\theta = 3\int \frac{\cos 2\theta + 1}{2}\,d\theta = \frac{3\sin 2\theta}{4} + \frac{\theta}{2} + c =$$

$$= \frac{3\sin\theta\cos\theta}{2} + \frac{3\theta}{2} + c = \frac{3y\sqrt{1-y^2}}{2} + \frac{3\arcsin y}{2} + c =$$

$$= \frac{3}{2}\left[\left(\frac{x-1}{\sqrt{3}}\right)\sqrt{1-\frac{(x-1)^2}{3}}\right] + \frac{3}{2}\arcsin\left(\frac{x-1}{\sqrt{3}}\right) + c\,.$$

(b) Temos,

$$\int \frac{x^2}{x^3-6x^2+11x-6}\,dx = \int \frac{x^2}{(x-1)(x-2)(x-3)}dx.$$

Ainda, pelo método das frações parciais,

$$\frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Pelo método de **Heaviside** deduzimos que

$$A = \frac{1}{2},\; B = -4 \;\text{ e }\; C = \frac{9}{2}.$$

Consequentemente,

$$\int \frac{x^2}{x^3-6x^2+11x-6}\,dx = \frac{\ln|x-1|}{2} - 4\ln|x-2| + \frac{9\ln|x-3|}{2} \blacksquare$$

3. Calcule as integrais definidas abaixo:

(a) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx.$

(b) $\displaystyle\int_{\pi/3}^{\pi/2} \operatorname{sen}^3 x\,dx.$

Solução.

(a) Temos,

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_0^3 \frac{x+1-1}{\sqrt{x+1}}\,dx = \int_0^3 \left(\sqrt{x+1} - \frac{1}{\sqrt{x+1}}\right)dx =$$

$$= \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}}\right]\Big|_0^3 = \frac{2}{3}(8-1) - 2(2-1) = \frac{8}{3}.$$

(b) Temos $\operatorname{sen}^3 x = \operatorname{sen} x(1-\cos^2 x) = \operatorname{sen} x - \cos^2 x \operatorname{sen} x$. Logo,

$$\int_{\pi/3}^{\pi/2} \operatorname{sen}^3 x\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{sen} x - \cos^2 x \operatorname{sen} x)\,dx = \left(-\cos x + \frac{\cos^3 x}{3}\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{2} - \frac{1}{3.2^3} = \frac{11}{24} \blacksquare$$

4. Calcule:

   (a) $\displaystyle\int \frac{x^4}{x^3-8}\,dx.$

   (b) $\displaystyle\int \frac{x^2+1}{\sqrt{2x-2}}\,dx.$

Solução.

(a) Temos,

$$\int \frac{x^4}{x^3-8}\,dx = \int \frac{x(x^3-8)+8x}{x^3-8}\,dx = \int x\,dx + \int \frac{8x}{x^3-8}\,dx.$$

Pelo método das frações parciais temos

$$\frac{8x}{(x-2)(x^2+2x+4)} = \frac{8x}{(x-2)[(x+1)^2+3]} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

Donde segue $A = \frac{16}{12} = \frac{4}{3}$, $C = \frac{8}{3}$ e $B = -\frac{4}{3}$. Logo,

$$\int \frac{x^4}{x^3-8}\,dx = \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{4}{3}\int \frac{x-2}{(x+1)^2+3}\,dx =$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{2}{3}\int \frac{2x-4}{(x+1)^2+3}\,dx =$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{2}{3}\int \frac{2(x+1)-6}{(x+1)^2+3}\,dx =$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{2}{3}\int \frac{2(x+1)}{(x+1)^2+3}\,dx + 4\int \frac{dx}{(x+1)^2+3} =$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{2}{3}\ln|(x+1)^2+3| + \frac{4}{3}\int \frac{dx}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1} =$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{2}{3}\ln|(x+1)^2+3| + \frac{4}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + c.$$

(b) Temos,

$$\int \frac{x^2+1}{\sqrt{2x-2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{[(x-1)+1]^2+1}{\sqrt{x-1}}\,dx =$$

$$= \frac{1}{\sqrt{2}}\int \left[(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + 2(x-1)^{-\frac{1}{2}}\right]dx$$

$$= \frac{1}{\sqrt{2}}\left[\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 4(x-1)^{\frac{1}{2}}\right] + c \blacksquare$$

5. Calcule

   (a) $\int_0^1 x^2 e^x \, dx$.

   (b) $\int_0^{\frac{1}{2}} \arcsin x \, dx$.

Solução.

(a) Integrando por partes obtemos,

$$\int_0^1 x^2 e^x \, dx = x^2 e^x \Big|_0^1 - \int_0^1 2x e^x \, dx = e - 2\Big[x e^x \Big|_0^1 - \int_0^1 e^x \, dx\Big] =$$

$$= e - 2[e - (e-1)] = e - 2.$$

(b) Integrando por partes obtemos,

$$\int_0^{\frac{1}{2}} 1. \arcsin x \, dx = x \arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x \frac{1}{\sqrt{1-x^2}} \, dx =$$

$$= \frac{\pi}{12} + \frac{1}{2} \int_0^{\frac{1}{2}} \frac{(-2x)}{\sqrt{1-x^2}} \, dx = \frac{\pi}{12} + (1-x^2)^{\frac{1}{2}} \Big|_0^{\frac{1}{2}} =$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \blacksquare$$

6. Calcule $\displaystyle\int \frac{x^5+x+1}{x^4-4}\,dx$.

Solução.

Temos,

$$\int \frac{x^5+x+1}{x^4-4}\,dx = \int \left(x+\frac{5x+1}{x^4-4}\right)\,dx =$$

$$= \frac{x^2}{2}+\int \frac{5x+1}{(x^2-2)(x^2+2)}\,dx = \frac{x^2}{2}+\int \frac{5x+1}{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)}\,dx.$$

Pelo Método das Frações Parciais escrevemos

$$\frac{5x+1}{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)} = \frac{A}{x-\sqrt{2}}+\frac{B}{x+\sqrt{2}}+\frac{Cx+D}{x^2+2}.$$

Pelo Método de Heaviside encontramos

$$A=\frac{5\sqrt{2}+1}{8\sqrt{2}}=\frac{10+\sqrt{2}}{16},\quad B=\frac{-5\sqrt{2}+1}{-8\sqrt{2}}=\frac{10-\sqrt{2}}{16},$$

$$-\frac{1}{4}=-\frac{10+\sqrt{2}}{16\sqrt{2}}+\frac{10-\sqrt{2}}{16\sqrt{2}}+\frac{D}{2}=-\frac{1}{8}+\frac{D}{2}\Longrightarrow D=-\frac{1}{4}.$$

Substituindo $x=1$ encontramos

$$-2=\frac{A}{1-\sqrt{2}}+\frac{B}{1+\sqrt{2}}+\frac{C+D}{3}\quad \text{e então}$$

$$-6(1-\sqrt{2})(1+\sqrt{2}) = 3A(1+\sqrt{2})+3B(1-\sqrt{2})+(C+D)(1-\sqrt{2})(1+\sqrt{2}).$$

Logo,

$$6=3(A+B)+3\sqrt{2}(A-B)-(C+D)\;\Longrightarrow C=-\frac{5}{4}.$$

Assim sendo,

$$\int \frac{x^5+x+1}{x^4-4}\,dx =$$

$$=\frac{x^2}{2}+\frac{10+\sqrt{2}}{16}\ln|x-\sqrt{2}|+\frac{10-\sqrt{2}}{16}\ln|x^2+\sqrt{2}|-\frac{1}{4}\int\frac{5x+1}{x^2+2}\,dx.$$

Porém,

$$\int\frac{5x+1}{x^2+2}\,dx=\frac{5}{2}\int\frac{2x\,dx}{x^2+2}+\int\frac{dx}{x^2+2}=\frac{5}{2}\ln(x^2+2)+\frac{1}{2}\int\frac{dx}{(\frac{x}{\sqrt{2}})^2+1}=$$

$$=\frac{5}{2}\ln(x^2+2)+\frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+c.$$

Resposta final.

$$\int \frac{x^5+x+1}{x^4-4}\,dx=\frac{x^2}{2}+\frac{10+\sqrt{2}}{16}\ln|x-\sqrt{2}|+\frac{10-\sqrt{2}}{16}\ln|x^2+\sqrt{2}|+$$

$$-\frac{5}{8}\ln(x^2+2)-\frac{1}{4\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+c\,\blacksquare$$

7. Calcule $\displaystyle\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx$.

Solução.

Seja $y = x^2 + 1$, com $0 \leq x \leq \sqrt{3}$. Então $1 \leq y \leq 4$ e $x = \sqrt{y-1}$. Ainda,

$$\frac{dy}{dx} = 2x.$$

Assim,

$$\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx = \int_0^{\sqrt{3}} x^2\sqrt{x^2+1}\,xdx = \frac{1}{2}\int_1^4 (y-1)\sqrt{y}\,dy =$$

$$= \frac{1}{2}\int_1^4 (y^{\frac{3}{2}} - y^{\frac{1}{2}})\,dy = \frac{1}{2}\left(\frac{2}{5}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{3}{2}}\right)\Big|_1^4 =$$

$$= \left(\frac{32}{5} - \frac{8}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right) = \frac{31}{5} - \frac{7}{3} = \frac{58}{15} \blacksquare$$