THE EXPONENTIAL MATRIX: AN EXPLICIT FORMULA BY AN ELEMENTARY METHOD

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THE PARIS-ORSAY SYMPOSIUM

The 44rd Summer Symposium in Real Analysis and Applications

June 20 2022 - June 24 2022

Some comments.

(1) We show an explicit and very trivial formula for the exponential matrix (either real or complex) and for algebraic operators on infinite Banach spaces (either real or complex). The proof avoids Jordan canonical form, eigenvectors, resolution of any linear systems, matrix inversion, polynomial interpolation, complex integration, integration in Banach spaces and symbolic calculus.

(2) Besides the widely known ways to compute the exponential of a matrix (see wikipedia), two other ways are the so-called Putzer's method (shown in Apostol's Calculus) and an improvement of it given in an article by Kolodner. However, these two methods require solving a set of differential equations.

(3) The quite well-known proof for the exponential of an algebraic operator defined on an infinite complex Banach space B employs complex integration and symbolic calculus and thus does not apply if B is a real Banach space. This is commented in Rudin's Functional Analysis.

(4) The basic idea of the proof that follows relies on power series properties. This is "natural" since the exponential is defined by a power series.

PRELIMINARIES AND NOTATIONS

Let A be a $n \times n$ matrix (either real or complex) and $\det A$ be its determinant.

As is well-known, the computation of e^{tA} arises from the problem of finding a real curve $x : \mathbb{R} \to \mathbb{R}^n$ to the real constant coefficients linear system of ode's

$$\begin{cases} x'(t) = Ax(t) \\ x(0) = x_0, \end{cases}$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is a real and x_0 is fixed in $\mathbb{R}^n$. The unique solution is the curve $x(t) = e^{tA}x_0$. This real problem is best dealt in $\mathbb{C}$ and then, at last, we get at a real solution.

Let z be in $\mathbb{C}$. We assume the following.

Cayley-Hamilton Theorem. *Given a $n \times n$ real matrix A and $p(z) = \det(zI - A)$ its monic characteristic polynomial, we have*

$$\mathbf{p(A) = 0}.$$

Partial fraction decomposition. *Let f an q be everywhere convergent complex power series, and p and r be complex polynomials such that*

$$\mathbf{f}(\mathbf{z}) = \mathbf{q}(\mathbf{z})\mathbf{p}(\mathbf{z}) + \mathbf{r}(\mathbf{z}),$$

where p is monic and degree(r) <degree$(p) = n$. If $\lambda_1, \ldots, \lambda_m$ are the distinct zeros of $p(z)$, with respective multiplicities $m_1, \ldots, m_m$, we write

$$p(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_m)^{m_m}.$$

Then, there are n constants $C_{1,1}, \ldots, C_{1,m_1}, \ldots, C_{m,1}, \ldots, C_{m,m_m}$ such that

$$\frac{f(z)}{p(z)} = q(z) + \left[\frac{C_{1,1}}{z - \lambda_1} + \cdots + \frac{C_{1,m_1}}{(z - \lambda_1)^{m_1}}\right] + \cdots + \left[\frac{C_{m,1}}{z - \lambda_m} + \cdots + \frac{C_{m,m_m}}{(z - \lambda_m)^{m_m}}\right],$$

for all z outside $\{\lambda_1, \ldots, \lambda_m\}$. These constants are given by

$$C_{j,k} = \frac{g_j^{(m_j - k)}(\lambda_j)}{(m_j - k)!}, \text{ where } g_j(z) = \frac{f(z)(z - \lambda_j)^{m_j}}{p(z)}.$$

THE EXPONENTIAL OF A REAL MATRIX

Theorem. Let A be a $n \times n$ real matrix with characteristic polynomial $p(z) = (z-\lambda_1)^{m_1} \cdots (z-\lambda_m)^{m_m}$, with $\lambda_1, \ldots, \lambda_m$ the distinct zeros of p and $m_1, \ldots, m_m$ their respective algebraic multiplicities. For each $j = 1, \ldots, m$ and each $k = 1, \ldots, m_j$, let us consider the polynomial (a total of n polynomials)

$$p_{j,k}(z) = (z-\lambda_j)^{m_j-k} \prod_{l \neq j} (z-\lambda_l)^{m_l} \quad \left[= \frac{p(z)}{(z-\lambda_j)^k} \right].$$

Then, giving a real t, we have

$$\mathbf{e^{tA} = \sum C_{j,k} p_{j,k}(A)},$$

$$\text{where } C_{j,k} = \frac{1}{(m_j-k)!} \frac{d^{m_j-k}}{dz^{m_j-k}} \left\{ \frac{e^{tz}(z-\lambda_j)^{m_j}}{p(z)} \right\} \Big|_{z=\lambda_j}.$$

Proof.

Fixed $t \in \mathbb{R}$, the map $z \mapsto e^{tz}$ is given by a everywhere convergent power series. Dividing such power series by the polynomial $p(z)$ we find

$$\mathbf{e^{tz} = q(z)p(z) + r(z)},$$

$$\text{with } \begin{cases} q \text{ a everywhere convergent power series,} \\ r \text{ a polynomial with degree}(r) < \text{ degree}(p). \end{cases}$$

Hence,

$$\mathbf{e^{tA} = q(A)p(A) + r(A)}.$$

Cayley-Hamilton's gives $p(A) = 0$ and thus

$$\mathbf{e^{tA} = r(A)}.$$

The alluded partial fraction decomposition gives

$$\frac{r(z)}{p(z)} = \sum \frac{C_{j,k}}{(z-\lambda_j)^k} \text{ and } r(z) = \sum C_{j,k}\, p_{j,k}(z).$$

Hence,

$$\mathbf{e^{tA} = \sum C_{j,k} p_{j,k}(A)} \quad \square$$

TWO EXAMPLES

First Example. Let us compute e^{tA} for the real matrix

$$A = \begin{pmatrix} -1 & -3 & 3 \\ -6 & 2 & 6 \\ -3 & 3 & 5 \end{pmatrix}.$$

The characteristic polynomial is $p_A(z) = (z-2)(z+4)(z-8)$. Following the proven theorem and its notation we have $e^{tz} = q(z)p_A(z) + r(z)$ and

$$\frac{e^{tz}}{(z-2)(z+4)(z-8)} = q(z) + \frac{\alpha}{z-2} + \frac{\beta}{z+4} + \frac{\gamma}{z-8},$$

with q a convergent power series and $(\alpha, \beta, \gamma) = (-\frac{e^{2t}}{36}, \frac{e^{-4t}}{72}, \frac{e^{8t}}{72})$. Thus,

$$e^{tA} = -\frac{e^{2t}}{36}(A+4I)(A-8I) + \frac{e^{-4t}}{72}(A-2I)(A-8I) + \frac{e^{8t}}{72}(A-2I)(A+4I).$$

Second Example. Let us compute e^{tB} for the real matrix

$$B = \begin{pmatrix} 5 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{pmatrix}.$$

The characteristic polynomial is $p_B(z) = (z+1)(z-5)^2$ and, as it is not difficult to see, B is non-diagonalizable. Following the proven theorem and its notation we have $e^{tz} = q(z)p_B(z) + r(z)$ and

$$\frac{e^{tz}}{(z+1)(z-5)^2} = q(z) + \frac{\alpha}{z+1} + \frac{\beta}{(z-5)^2} + \frac{\gamma}{z-5},$$

with q a power series and $(\alpha, \beta, \gamma) = (\frac{e^{-t}}{36}, \frac{e^{5t}}{6}, \frac{(6t-1)e^{5t}}{36})$. Thus,

$$e^{tB} = \frac{e^{-t}}{36}(A-5I)^2 + \frac{e^{5t}}{6}(A+I) + \frac{(6t-1)e^{5t}}{36}(A+I)(A-5I).$$

EXPONENTIAL OF ALGEBRAIC OPERATORS

Here we extend the method above to more general situations.

- Complex matrices. Clearly, the method previously developed is extendable to a finite and complex square matrix.
- Complex Banach spaces. Given an infinite dimensional complex Banach space X and a continuous linear operator $T : X \to X$, we say that T is an algebraic operator if there exists a complex and monic polynomial $p_T(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \text{ where } n \geq 1$, such that

$$\mathbf{p_T(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I = 0}$$

where $I : X \to X$ is the identity operator.

Examples of algebraic operators are: *nilpotent* operators, *projections*, *idempotent* operators, *involution* operators and *finite rank* operators.

It is well defined the exponential operator

$$\mathbf{e^{tT}} = \sum_{\mathbf{n=0}}^{+\infty} \frac{\mathbf{(tT)^n}}{\mathbf{n!}} = \mathbf{I} + \mathbf{tT} + \frac{\mathbf{(tT)^2}}{\mathbf{2!}} + \frac{\mathbf{(tT)^3}}{\mathbf{3!}} + \cdots, \text{ for all } \mathbf{t} \in \mathbb{R}.$$

In this case, analogously to what we have commented for the exponential of a complex matrix, we obtain a formula for the operator e^{tT}.

- Real Banach spaces. The definition of an algebraic operator $T : X \to X$, with X a real Banach space, is analogous to the previous one. The monic polynomial p_T, such that $p_T(T) = 0$, has real coefficients and fixed a real t we may write

$$\mathbf{e^{tz} = q_T(z)p_T(z) + r_T(z)}, \text{ for all complex } \mathbf{z},$$

with q_T a everywhere convergent power series with real coefficients and r_T a polynomial, with real coefficients and whose degree is smaller than that of p_T.
Thus, we may write

$$e^{tT} = r_T(T).$$

By employing the partial fraction decomposition we may write

$$\frac{\mathbf{r_T(z)}}{\mathbf{p_T(z)}} = \sum_{\substack{1\le j\le\mu \\ 1\le k\le\mu_j}} \frac{\alpha_{\mathbf{j,k}}}{(\mathbf{z}-\mathbf{z_j})^{\mathbf{k}}} + \sum_{\substack{1\le j\le\mu \\ 1\le k\le\mu_j}} \frac{\beta_{\mathbf{j,k}}}{(\mathbf{z}-\overline{\mathbf{z_j}})^{\mathbf{k}}} + \sum_{\substack{1\le l\le\nu \\ 1\le k\le\nu_l}} \frac{\gamma_{\mathbf{l,k}}}{(\mathbf{z}-\mathbf{x_l})^{\mathbf{k}}},$$

where the polynomial p_T has complex roots $z_1, \overline{z_1}, \ldots, z_\mu, \overline{z_\mu}$ and real roots $x_1, \ldots, x_\nu$ (all the roots are distinct and the algebraic multiplicities of these are, respectively, $\mu_1, \mu_1, \ldots, \mu_\mu, \mu_\mu, \nu_1, \nu_2, \ldots \nu_\nu$), with

$$\mathrm{degree}(p_T) = 2(\mu_1 + \cdots + \mu_\mu) + \nu_1 + \cdots + \nu_\nu = n,$$

and all the coefficients $\alpha_{j,k}, \beta_{j,k}$, and $\gamma_{l,k}$ are unique complex constants.

In what follows, we omit the sets where the indices j, k, and l take values.

Each $\gamma_{\mathbf{l},\mathbf{k}}$ is real. In fact, since the map $z \mapsto e^{tz}(z - x_l)^{\nu_l}$ may be developed as a power series with real coefficients and the polynomial p_T has real coefficients, it follows that

$$\gamma_{l,k} = \frac{1}{(\nu_l - k)!} \frac{d^{\nu_l - k}}{dz^{\nu_l - k}} \left\{ \frac{e^{tz}(z - x_l)^{\nu_l}}{p_T(z)} \right\} \Big|_{z = x_l} \in \mathbb{R}.$$

We have $\beta_{\mathbf{j,k}} = \overline{\alpha_{\mathbf{j,k}}}$ for each j and k. This follows from the derivatives of the functions (and the derivatives of their conjugates)

$$\varphi(z) = \frac{e^{tz}(z - z_j)^{\mu_j}}{p_T(z)} \text{ and } \psi(z) = \frac{e^{tz}(z - \overline{z_j})^{\mu_j}}{p_T(z)}.$$

Thus far, we have

$$\frac{r_T(z)}{p_T(z)} = \sum \frac{\alpha_{j,k}(z - \overline{z_j})^k + \overline{\alpha_{j,k}}(z - z_j)^k}{(z - z_j)^k(z - \overline{z_j})^k} + \sum \frac{\gamma_{l,k}}{(z - x_l)^k}.$$

Conclusion. The expansion of the map $u_{j,k}(z) = \alpha_{j,k}(z - \overline{z_j})^k + \overline{\alpha_{j,k}}(z - z_j)^k$ is a polynomial with real coefficients. We write

$$r_T(z) = \sum u_{j,k}(z) \frac{p_T(z)}{(z - z_j)^k (z - \overline{z_j})^k} + \sum \gamma_{l,k} \frac{p_T(z)}{(z - x_l)^k}.$$

Eliminating singularities, with clear identifications we may write

$$\mathbf{r_T} = \sum \mathbf{u_{jk}} \mathbf{v_{jk}} + \sum \gamma_{\mathbf{lk}} \mathbf{w_{lk}},$$

with u_{jk}, v_{jk} and w_{lk} polynomials with real coefficients and each γ_{lk} real.

Summing up, and since $e^{tT} = r_T(T)$, we arrive at

$$\mathbf{e^{tT}} = \sum \mathbf{u_{j,k}(T) v_{j,k}(T)} + \sum \gamma_{\mathbf{l,k}} \mathbf{w_{l,k}(T)} \quad \square$$

REFERENCES

(1) Oliveira, O., *The Exponential Matrix: An Explicit Formula By An Elementary Method.*, Real Analysis Exchange **46** (1), 2021, pp. 99–106.

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