AN ALGORITHM FOR DIVIDING TWO POWER SERIES

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Professor Oswaldo Rio Branco de Oliveira

http://www.ime.usp.br/~oliveira oliveira@ime.usp.br

This text shows and proves the long division method*, a division table for two arbitrary power series, either real or complex. It also includes other ways of dividing, such as "the geometric series trick". Several (23) examples are given. The efficiency of such division table is highlighted, here, with the rather illustrating* Bernoulli Numbers.

1. Introduction

Let us denote by z the variable in the complex plane.

Let us consider two complex power series

$$\begin{cases} a(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \\ \text{and} \\ b(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots, \end{cases}$$

where $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$ are all complex coefficients, with the power series $b(z)$ satisfying the additional condition

$$b_0 \neq 0.$$

Let us assume that both series converge on a small neighborhood of the origin.

It's known that the *reciprocal function* (or *multiplicative inverse function*)

$$\frac{1}{b(z)} = \frac{1}{b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots}$$

may, on a possibly smaller neighborhood of the origin, be written as a convergent power series with complex coefficients.

Since the multiplication of convergent power series is also a convergent power series, the division of the power series $a(z)$ for the power series $b(z)$ is a power series that converges on a small neighborhood of the origin. Hence, we may write

$$\frac{a(z)}{b(z)} = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \text{for all } |z| \text{ small enough.}$$

Thus, for all z on a small enough neighborhood of the origin we have

$$\sum_{n=0}^{+\infty} a_n z^n = \left(\sum_{j=0}^{+\infty} b_j z^j \right) \left(\sum_{k=0}^{+\infty} c_k z^k \right).$$

We recall that the multiplication of two power series satisfy the following rule (basically, the "same" rule that applies to the multiplication of two polynomials)

$$\begin{aligned} &(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots)(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots) \\ &= b_0 c_0 + (b_0 c_1 + b_1 c_0) z + (b_0 c_2 + b_1 c_1 + b_2 c_0) z^2 + (b_0 c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0) z^3 + \cdots \end{aligned}$$

In short notation, we may write such infinite multiplication as

$$\left(\sum_{j=0}^{+\infty} b_j z^j\right)\left(\sum_{k=0}^{+\infty} c_k z^k\right) = \sum_{n=0}^{+\infty}\left(\sum_{j+k=n} b_j c_k\right) z^n.$$

These three last equations lead to the identity

$$\sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty}\left(\sum_{j+k=n} b_j c_k\right) z^n.$$

This identity, plus the Uniqueness Theorem for the Coefficients of a Power Series (see Oliveira [6, p. 7; 5, p. 15], Beardon [2, pp. 112–113]), presents the relations from where we can finally determine all the coefficients that we are looking for: $c_0, c_1, c_2, c_3, c_4, \ldots$. In fact, we can recursively determine the coefficients $c_0, c_1, c_2, c_3, c_4, c_5, \ldots$ (in this exact order) by the well known relations

$$\begin{cases} a_0 &= b_0c_0 \\ a_1 &= b_0c_1 + b_1c_0 \\ a_2 &= b_0c_2 + b_1c_1 + b_2c_0 \\ a_3 &= b_0c_3 + b_1c_2 + b_2c_2 + b_3c_0 \\ a_4 &= b_0c_4 + b_1c_3 + b_2c_3 + b_3c_1 + b_4c_0 \\ a_5 &= b_0c_5 + b_1c_4 + b_2c_3 + b_3c_2 + b_4c_1 + b_5c_0 \\ a_6 &= b_0c_6 + b_1c_5 + b_2c_4 + b_3c_3 + b_4c_2 + b_5c_1 + b_6c_0 \\ a_7 &= b_0c_7 + b_1c_6 + b_2c_5 + b_3c_4 + b_4c_3 + b_5c_2 + b_6c_1 + b_7c_0 \\ a_8 &= b_0c_8 + b_1c_7 + b_2c_6 + b_3c_5 + b_4c_4 + b_5c_3 + b_6c_2 + b_7c_1 + b_8c_0 \\ a_9 &= b_0c_9 + b_1c_8 + b_2c_7 + b_3c_6 + b_4c_5 + b_5c_4 + b_6c_3 + b_7c_2 + b_8c_1 + b_9c_0 \\ a_{10} &= b_0c_{10} + b_1c_9 + b_2c_8 + b_3c_7 + b_4c_6 + b_5c_5 + b_6c_4 + b_7c_3 + b_8c_2 + b_9c_1 + b_{10}c_0 \\ & \vdots \end{cases} .$$

Well, this really looks beautiful albeit quite cumbersome. Needless to say, the computations can be too tedious!

In view of this seemingly unpleasant strategy, we proceed by searching for a more suitable method for finding the much desired coefficients $c_{n's}$.

Remark. *The very important "Euclidean division" employed for dividing an infinite power series by a polynomial (that is the case if $b(z)$ is, in fact, a polynomial) can be seen at Oliveira [7].*

2. Notations

The notation that we are about to use for this so called long division process is the same one that we use for the Euclidean division for polynomials. In fact, this notation is the same one that we use since childhood for dividing natural numbers.

That is, by considering two natural numbers N and D, with $D \neq 0$, we write the Euclidean division as

$$\begin{array}{c|c} N & D \\ \hline R & Q \end{array} \quad \text{meaning} \quad \begin{cases} N = DQ + R \\ \text{or} \\ \frac{N}{D} = Q + \frac{R}{D}, \end{cases}$$

where N stands for the **numerator** or **dividend**, $D \neq 0$ stands for the **divisor**, the letter Q stands for the unique **quotient**, and the letter R stands for the unique **remainder**.

Analogously, the Euclidean division for polynomials allow us to divide an arbitrary polynomial $P(z)$ by a non null polynomial $Q(z)$, with the sole condition that the degree of $P(z)$ is greater or equal to the degree of $Q(z)$.

Such Euclidean polynomial division gives us an **unique** quotient polynomial $D(z)$, and an **unique** remainder polynomial $R(z)$ whose degree is strictly smaller than that of the divisor $Q(z)$, which is also called the denominator polynomial.

We then write

$$\begin{array}{c|c} P(z) & Q(z) \\ \hline R(z) & D(z) \end{array} \quad \text{meaning} \quad \begin{cases} P(z) = Q(z)D(z) + R(z) \\ \text{or} \\ \frac{P(z)}{Q(z)} = D(z) + \frac{R(z)}{Q(z)}, \\ \\ \text{with degree}(R) < \text{degree}(Q). \end{cases}$$

We recall that, as a convention, the zero polynomial has degree $-\infty$.

3. The Algorithm (Long Division Algorithm)

In this section we show an algorithm that gives us the complex coefficients $c_0, c_1, c_2, c_3, c_4, \ldots$ related to the division

$$\frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots}{b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots} = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

The division algorithm for power series, also called Long Division Algorithm, that we will see has the following three characteristics

- First. It depends solely on the coefficients of the involved power series, and the order that these coefficients appear on the expressions of the related power series. Hence, in the algorithm, we don't need to explicit the monomials $z, z^2, z^3, z^4, \ldots$ (but we may write them down, if it is convenient).
- Second. In the last section we saw a way of finding the numerical coefficients $c_0, c_1, c_2, c_3, \ldots$, in this exact order. This exigence remains. That is, we will first find the independent term c_0, then the coefficient c_1 (of the term $c_1 z$), then the coefficient c_2 (of the term $c_2 z^2$), and so on.
- Third. The algorithm is merely **formal**. This has two consequences.
    - The fact that the algorithm gives to us the coefficients $c_0, c_1, c_2, \ldots$, does not guarantee at all the convergence of the power series $c(z) = c_0 + c_1 z + c_2 z^2 + \cdots$. Furthermore, the algorithm does not guarantee, by itself, the numerical identity $a(z) = b(z)c(z)$ at a point z. What guarantee this identity are the results mentioned in the introduction.
    - The algorithm can be applied to two arbitrary power series, either real or complex ones, either convergent or not. The computations are merely formal. Thus, given two completely arbitrary power series $a(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $b(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, with the sole condition $b_0 \neq 0$, the algorithm gives the **unique** coefficients $c_0, c_1, c_2, \ldots$ of a power series $c(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ that **formally satisfy**

    $$a_0 + a_1 z + a_2 z^2 + \cdots = (b_0 + b_1 z + b_2 z^2 + \cdots)(c_0 + c_1 z + c_2 z^2 + \cdots).$$

    In other words, we are merely saying that the coefficients $c_0, c_1, c_2, \ldots$ satisfy the relations $a_0 = b_0 c_0$, $a_1 = b_0 c_1 + b_1 c_0$, $a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0, \ldots$.

ALGORITHM DIVISION FOR $\dfrac{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots}{b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots} = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$

The algorithm is given by

$$\left|\begin{array}{cccccc|cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & \ldots & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ & & & & & & -- & -- & -- & -- & -- & -- \\ 0 & a_1 - c_0 b_1 & a_2 - c_0 b_2 & a_3 - c_0 b_3 & a_4 - c_0 b_4 & \ldots & c_0 & c_1 & c_2 & c_3 & c_4 & \cdots \\ 0 & 0 & a_2 - c_0 b_2 - c_1 b_1 & a_3 - c_0 b_3 - c_1 b_2 & a_4 - c_0 b_4 - c_1 b_3 & \ldots & & & & & & \\ 0 & 0 & 0 & a_3 - c_0 b_3 - c_1 b_2 - c_2 b_1 & a_4 - c_0 b_4 - c_1 b_3 - c_2 b_2 & \ldots & & & & & & \\ 0 & 0 & 0 & 0 & a_4 - c_0 b_4 - c_1 b_3 - c_2 b_2 - c_3 b_1 & \ldots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & & & & & & \end{array}\right|,$$

where the coefficients $c_0, c_1, c_2, c_3, c_4, \ldots$ are given recursively by

$$\left\{\begin{array}{l} c_0 = \frac{a_0}{b_0} \\ c_1 = \frac{a_1 - c_0 b_1}{b_0} \\ c_2 = \frac{a_2 - c_0 b_2 - c_1 b_1}{b_0} \\ c_3 = \frac{a_3 - c_0 b_3 - c_1 b_2 - c_2 b_1}{b_0} \\ c_4 = \frac{a_4 - c_0 b_4 - c_1 b_3 - c_2 b_2 - c_3 b_1}{b_0} \\ \quad \vdots \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} 0 = a_0 - c_0 b_0 \\ 0 = a_1 - c_0 b_1 - c_1 b_0 \\ 0 = a_2 - c_0 b_2 - c_1 b_1 - c_2 b_0 \\ 0 = a_3 - c_0 b_3 - c_1 b_2 - c_2 b_1 - c_3 b_0 \\ 0 = a_4 - c_0 b_4 - c_1 b_3 - c_2 b_2 - c_3 b_1 - c_4 b_0 \\ \quad \vdots \end{array}\right. .$$

4. Kid Examples

Example 1. Write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1}{1+z}.$$

Solution. By employing the division algorithm for power series (long division algorithm), we find

$$\begin{array}{c|l}
1 & 1+z \\ \hline
-z & 1 \quad -z \quad +z^2 \quad -z^3 \quad +z^4 \quad -z^5 \quad \cdots \\
z^2 & \\
-z^3 & \\
z^4 & \\
-z^5 & \\
\vdots &
\end{array}.$$

[As a mere remark, we notice that the left column below the horizontal line, the column of the remainders, originates from the very short computations

$$\begin{cases}
-z &= 1 - (1)(1+z) \\
z^2 &= -z - (-z)(1+z) \\
-z^3 &= z^2 - (z^2)(1+z) \\
z^4 &= -z^3 - (-z^3)(1+z) \\
-z^5 &= z^4 - (z^4)(1+z) \\
&\vdots
\end{cases},$$

where $(1), (-z), (z^2), (-z^3)$, and (z^4) indicate monomials appearing in the line of the quotient, the second line on the right of the vertical line. The polynomial 1 is in the position of the dividend while the polynomial $1+z$ is the divisor.]

Hence, we have the development

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + \cdots.$$

As is well known, this development is valid for all z such that $|z| < 1$ ♣

Example 2. Write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1+z^2}{1+z}.$$

Solution. By employing the division algorithm for power series (long division algorithm), we find

$$\begin{array}{c|cccccccc} 1+z^2 & 1+z & & & & & & \\ \hline -z+z^2 & 1 & -z & +2z^2 & -2z^3 & +2z^4 & -2z^5 & \cdots \\ 2z^2 & & & & & & & \\ -2z^3 & & & & & & & \\ 2z^4 & & & & & & & \\ -2z^5 & & & & & & & \\ \vdots & & & & & & & \end{array}.$$

[As a mere remark, we notice that the left column below the horizontal line, the column of the remainders, originates from the short computations

$$\begin{cases} -z+z^2 & = & 1+z^2 & - & (1)(1+z) \\ 2z^2 & = & -z+z^2 & - & (-z)(1+z) \\ -2z^3 & = & 2z^2 & - & (2z^2)(1+z) \\ 2z^4 & = & -2z^3 & - & (-2z^3)(1+z) \\ -z^5 & = & z^4 & - & (2z^4)(1+z) \\ & \vdots & & & \end{cases},$$

where $(1), (-z), (2z^2), (-2z^3)$, and $(2z^4)$ indicate monomials appearing in the line of the quotient, the second line on the right of the vertical line. The polynomial $1+z^2$ is the dividend while the polynomail $1+z$ is the divisor.]

Hence, we have the development

$$\frac{1+z^2}{1+z} = 1 - z + 2z^2 - 2z^3 + 2z^4 - 2z^5 + \cdots.$$

This development is valid for all z such that $|z| < 1$. Please, check ♣

Example 3. Write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1+z}{1+z^2}.$$

Solution. By employing the long division algorithm, we find

$$\begin{array}{c|cccccccccccc} 1+z & 1+z^2 & & & & & & & & & & & \\ \hline z-z^2 & 1 & +z & -z^2 & -z^3 & +z^4 & +z^5 & -z^6 & -z^7 & +z^8 & +z^9 & -z^{10} & \cdots \\ -z^2-z^3 & & & & & & & & & & & & \\ -z^3+z^4 & & & & & & & & & & & & \\ z^4+z^5 & & & & & & & & & & & & \\ z^5-z^6 & & & & & & & & & & & & \\ -z^6-z^7 & & & & & & & & & & & & \\ -z^7+z^8 & & & & & & & & & & & & \\ z^8+z^9 & & & & & & & & & & & & \\ z^9-z^{10} & & & & & & & & & & & & \\ -z^{10}-z^{11} & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & \end{array}.$$

[The column of the remainders originates from

$$\begin{cases} z-z^2 & = & 1+z & - & (1)(1+z^2) \\ -z^2-z^3 & = & z-z^2 & - & (z)(1+z^2) \\ -z^3+z^4 & = & -z^2-z^3 & - & (-z^2)(1+z^2) \\ z^4+z^5 & = & -z^3+z^4 & - & (-z^3)(1+z^2) \\ z^5-z^6 & = & z^4+z^5 & - & (z^4)(1+z^2) \\ -z^6-z^7 & = & z^5-z^6 & - & (z^5)(1+z^2) \\ -z^7+z^8 & = & -z^6-z^7 & - & (-z^6)(1+z^2) \\ z^8+z^9 & = & -z^7+z^8 & - & (-z^7)(1+z^2) \\ z^9-z^{10} & = & z^8+z^9 & - & (z^8)(1+z^2) \\ -z^{10}-z^{11} & = & z^9-z^{10} & - & (z^9)(1+z^2) \\ & & \vdots & & \end{cases},$$

where $(1), (z), (-z^2), (-z^3), (z^4), (z^5), (-z^6), (-z^7), (z^8), (z^9)$ and $(-z^{10})$ indicate monomials in the quotient. Yet, $1+z^2$ is the dividend and $1+z$ is the divisor.]
Hence, we have the development (a convergent one, for all $|z|<1$).

$$\frac{1+z}{1+z^2} = 1 + z - z^2 - z^3 + z^4 + z^5 - z^6 - z^7 + z^8 + z^9 - z^{10} + \cdots \clubsuit$$

5. Examples

Example 4. Write down the first five non null terms of the power series, centered at the origin, of the tangent function

$$\tan(z) = \frac{\sin z}{\cos z}.$$

Solution. By employing the division algorithm for power series (the long division algorithm), we find

$$
\begin{array}{cccccccc|l}
 & z & -\frac{z^3}{3!} & +\frac{z^5}{5!} & -\frac{z^7}{7!} & +\frac{z^9}{9!} & \cdots & & 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\frac{z^8}{8!}+\cdots \\ \cline{9-9}
- & z & -\frac{z^3}{2!} & +\frac{z^5}{4!} & -\frac{z^7}{6!} & +\frac{z^9}{8!} & \cdots & & z+\frac{2}{3!}z^3+\frac{16}{5!}z^5+\frac{272}{7!}z^7+\frac{7936}{9!}z^9+\cdots \\ \cline{1-7}
 & 0 & +\frac{2z^3}{3!} & -\frac{4z^5}{5!} & +\frac{6z^7}{7!} & -\frac{8z^9}{9!} & \cdots & & \\
- & & +\frac{2z^3}{3!} & -\frac{20z^5}{5!} & +\frac{70z^7}{7!} & -\frac{168z^9}{9!} & \cdots & & \\ \cline{1-7}
 & & 0 & +\frac{16z^5}{5!} & -\frac{64z^7}{7!} & +\frac{160z^9}{9!} & \cdots & & \\
- & & & +\frac{16z^5}{5!} & -\frac{336z^7}{7!} & +\frac{2016z^9}{9!} & \cdots & & \\ \cline{1-7}
 & & & 0 & +\frac{272z^7}{7!} & -\frac{1856z^9}{9!} & \cdots & & \\
- & & & & +\frac{272z^7}{7!} & -\frac{9792z^9}{9!} & \cdots & & \\ \cline{1-7}
 & & & & 0 & +\frac{7936z^9}{9!} & \cdots & &
\end{array}.
$$

Or the following short presentation, simply built by omitting all the subtractions indicated in the remainders column that is right above,

$$
\begin{array}{cccccc|l}
z & -\frac{z^3}{3!} & +\frac{z^5}{5!} & -\frac{z^7}{7!} & +\frac{z^9}{9!} & \cdots & 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\frac{z^8}{8!}+\cdots \\ \hline
0 & +\frac{2z^3}{3!} & -\frac{4z^5}{5!} & +\frac{6z^7}{7!} & -\frac{8z^9}{9!} & \cdots & z+\frac{2}{3!}z^3+\frac{16}{5!}z^5+\frac{272}{7!}z^7+\frac{7936}{9!}z^9+\cdots \\
 & 0 & +\frac{16z^5}{5!} & -\frac{64z^7}{7!} & +\frac{160z^9}{9!} & \cdots & \\
 & & 0 & +\frac{272z^7}{7!} & -\frac{1856z^9}{9!} & \cdots & \\
 & & & 0 & +\frac{7936z^9}{9!} & \cdots &
\end{array}.
$$

We may also write

$$\tan(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \frac{62}{2835}z^9 + \cdots \clubsuit$$

Example 5. Write down the sixth non null term of the power series, centered at the origin, of the tangent function

$$\tan(z) = \frac{\sin z}{\cos z}.$$

Solution. Following Example 4 and taking advantage that $\sin z$ is an odd function we may conveniently highlight the monomials $z, z^3, z^5, z^7, \ldots$ in its expansion.

Hence, one may also write the division table for $\tan(z)$ in the following way.

z	z^3	z^5	z^7	z^9	z^{11}	$\cdots$
1	$-\frac{1}{3!}$	$+\frac{1}{5!}$	$-\frac{1}{7!}$	$+\frac{1}{9!}$	$-\frac{1}{11!}$	$\cdots$
-1	$-\frac{1}{2!}$	$+\frac{1}{4!}$	$-\frac{1}{6!}$	$+\frac{1}{8!}$	$-\frac{1}{10!}$	$\cdots$
0	$+\frac{2}{3!}$	$-\frac{4}{5!}$	$+\frac{6}{7!}$	$-\frac{8}{9!}$	$+\frac{10}{11!}$	$\cdots$
$-$	$+\frac{2}{3!}$	$-\frac{20}{5!}$	$+\frac{70}{7!}$	$-\frac{168}{9!}$	$+\frac{330}{11!}$	$\cdots$
	0	$+\frac{16}{5!}$	$-\frac{64}{7!}$	$+\frac{160}{9!}$	$-\frac{320}{11!}$	$\cdots$
$-$		$+\frac{16}{5!}$	$-\frac{336}{7!}$	$+\frac{2016}{9!}$	$-\frac{7392}{11!}$	$\cdots$
		0	$+\frac{272}{7!}$	$-\frac{1856}{9!}$	$+\frac{7072}{11!}$	$\cdots$
$-$			$+\frac{272}{7!}$	$-\frac{9792}{9!}$	$+\frac{89760}{11!}$	$\cdots$
			0	$+\frac{7936}{9!}$	$-\frac{82688}{11!}$	$\cdots$
$-$				$+\frac{7936}{9!}$	$-\frac{436480}{11!}$	$\cdots$
				0	$+\frac{353792}{11!}$	$\cdots$

$$\frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots}{z + \frac{2z^3}{3!} + \frac{16z^5}{5!} + \frac{272z^7}{7!} + \frac{7936z^9}{9!} + \frac{353792z^{11}}{11!} + \cdots}$$

We may also write

$$\tan(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \frac{62}{2835}z^9 + \frac{1382}{155925}z^{11} + \cdots.$$

Thus, the sixth non null term of the power series expansion of $\tan(z)$ is

$$\frac{353792}{11!}z^{11} = \frac{1382}{155925}z^{11} \clubsuit$$

Example 6. Write down the first four non null terms of the power series, centered at the origin, of the function

$$f(z) = \frac{z}{\sin z} = \frac{z}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}.$$

[The very important synchronic function is defined by $(\sin z)/z$.]

Solution. We remark that in order to employ the long division algorithm we may divide, or we may not divide, the numerator and the denominator by z^1. Let us choose not to divide.

Computing factorials, we write

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \cdots.$$

By employing the division algorithm for power series we find

$$
\begin{array}{cccccc|l}
 & z & & & & & z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \cdots \\ \cline{7-7}
- & z & -\frac{z^3}{6} & +\frac{z^5}{120} & -\frac{z^7}{5040} & \cdots & 1 + \frac{z^2}{6} + \frac{7z^4}{360} + \frac{31z^6}{15120} + \cdots \\ \cline{1-6}
 & 0 & +\frac{z^3}{6} & -\frac{z^5}{120} & +\frac{z^7}{5040} & \cdots & \\
- & & +\frac{z^3}{6} & -\frac{z^5}{36} & +\frac{z^7}{720} & \cdots & \\ \cline{1-6}
 & & 0 & +\frac{7z^5}{360} & -\frac{6z^7}{5040} & \cdots & \\
- & & & +\frac{7z^5}{360} & -\frac{7z^7}{2160} & \cdots & \\ \cline{1-6}
 & & & 0 & +\frac{31z^7}{15120} & \cdots &
\end{array}.
$$

Thus, we arrive at the expression

$$\frac{z}{\sin z} = 1 + \frac{z^2}{6} + \frac{7z^4}{360} + \frac{31z^6}{15120} + \cdots \clubsuit$$

Example 7. Write down the first four non null terms of the series expansion, in powers of z and centered at the origin, of the hyperbolic cosecant function

$$f(z) = \operatorname{cosech}(z) = \frac{1}{\sinh(z)}.$$

Solution. First, we notice that

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \frac{z^9}{9!} + \cdots.$$

Thus,

$$\frac{1}{\sinh(z)} = \frac{1}{z}\left(\frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \frac{z^6}{7!} + \frac{z^8}{9!} + \cdots}\right).$$

Moreover, we have $\sinh(0) = 0$, which implies that we won't find a power series expansion. Instead, we will need the power z^{-1}, a negative power of z.

Putting z into evidence in the denominator and by employing the division algorithm for power series (the long division algorithm), we find

$$\begin{array}{cccccc|l}
 & 1 & & & & & 1 + \frac{z^2}{6} + \frac{z^4}{120} + \frac{z^6}{5040} + \cdots \\ \cline{7-7}
- & 1 & +\frac{z^2}{6} & +\frac{z^4}{120} & +\frac{z^6}{5040} & \cdots & 1 - \frac{z^2}{6} + \frac{7z^4}{360} - \frac{31z^6}{15120} + \cdots \\ \cline{1-6}
 & 0 & -\frac{z^2}{6} & -\frac{z^4}{120} & -\frac{z^6}{5040} & \cdots & \\
- & & -\frac{z^2}{6} & -\frac{z^4}{36} & -\frac{z^6}{720} & \cdots & \\ \cline{1-6}
 & & 0 & +\frac{7z^4}{360} & +\frac{z^6}{840} & \cdots & \\
- & & & +\frac{7z^4}{360} & +\frac{7z^6}{2160} & \cdots & \\ \cline{1-6}
 & & & 0 & -\frac{31z^6}{15120} & \cdots & \\
- & & & & -\frac{31z^6}{15120} & \cdots & \\ \cline{1-6}
 & & & & 0 & \cdots &
\end{array}.$$

Finally, we obtain the expansion (also called a Laurent series, since it contains negative powers of z)

$$\operatorname{cosech}(z) = \frac{1}{z} - \frac{z}{6} + \frac{7z^3}{360} - \frac{31z^5}{15120} + \cdots.$$

The domain of convergence. We notice that we have $\sinh(z) = 0$ if and only if $e^z - e^{-z} = e^{-z}(e^{2z} - 1) = 0$. Writing $z = x + iy$, with x and y real numbers, we have

$$\sinh(z) = 0 \Leftrightarrow e^{2x+2yi} = 1 = e^{0+2\pi i} \Leftrightarrow e^{2x}e^{2yi} = 1e^{2\pi i}.$$

Thus, $z = x + iy$ satisfies $\sinh(z) = 0$ if and only if $x = 0$ and $y = n\pi$, for some $n \in \mathbb{Z}$. Hence, the expansion that we found converges on the punctured disk

$$\{z \in \mathbb{C} : 0 < |z| < \pi\}.$$

6. Polynomial Division: Euclidean X Long

Let us compare, giving two polynomials, their two possible divisions. We compare the one provided by the division algorithm for power series vis à vis the very usual polynomial division (the Euclidean division algorithm for polynomials).

Considering two polynomials $P(z)$ and $Q(z)$, we comment and give examples for the following three characteristics regarding these two divisions of P by Q.

- The case degree(P) <degree(Q). Then, the Euclidean division of $P(z)$ by the polynomial $Q(z)$ is impossible. However, the long division is applicable. See Example 1 and Example 3.

- The case degree(P) ≥degree(Q). Then, the Euclidean division of $P(z)$ by $Q(z)$ is possible and we may write

$$\begin{cases} P(z) = Q(z)D(z) + R(z) \\ \text{or} \\ \frac{P(z)}{Q(z)} = D(z) + \frac{R(z)}{Q(z)}, \\ \\ \text{with degree}(R) < \text{degree}(Q). \end{cases}$$

  We then have two possibilities.

  (1) If $R \neq 0$, then the long division algorithm of P by Q gives an infinite power series that is not a polynomial. That is, it gives an infinite power series with an infinite number of non zero coefficients.

  (2) If $R = 0$ (that is, R is the zero polynomial), then the long division of P by Q gives an infinite power series that is in fact a polynomial. Furthermore, the Euclidean division and the long division agree.

We already saw some examples (Examples 1, 2, and 3) where we applied the algorithm for dividing two infinite power series to the task of dividing two polynomials. Now, let us go over some other three examples, also related to the polynomial division.

Example 8. (Compare with the long division method applied in Example 2.) Find the Euclidean division for

$$f(z) = \frac{1+z^2}{1+z}.$$

Solution. We have

$$\begin{array}{rr|l}
 & z^2+1 & z+1 \\ \cline{3-3}
- & z^2+z & z-1 \\ \cline{1-2}
 & -z+1 & \\
- & -z-1 & \\ \cline{1-2}
 & 2 &
\end{array}.$$

That is, we have

$$z^2+1 = (z-1)(z+1)+2,$$

or

$$\frac{z^2+1}{z+1} = z-1+\frac{2}{z+1}.$$

The quotient is $Q(z) = z-1$ and the remainder is $R(z) = 2$♣

Example 9. Find the long division for

$$f(z) = \frac{z^6+7z^5+7z^4-35z^3-55z^2+35z+60}{z^2+7z+12}.$$

Solution. We have

$$\begin{array}{rrrrrrrr|lll}
 & 60 & +35z & -55z^2 & -35z^3 & +7z^4 & +7z^5 & +z^6 & 12 & +7z & +z^2 \\ \cline{9-11}
- & 60 & +35z & +5z^2 & & & & & 5 & -5z^2 & +z^4 \\ \cline{1-8}
 & 0 & 0 & -60z^2 & -35z^3 & +7z^4 & +7z^5 & +z^6 & & & \\
- & & & -60z^2 & -35z^3 & -5z^4 & & & & & \\ \cline{1-8}
 & & & 0 & 0 & +12z^4 & +7z^5 & +z^6 & & & \\
- & & & & & +12z^4 & +7z^5 & +z^6 & & & \\ \cline{1-8}
 & & & & & 0 & 0 & 0 & & &
\end{array}.$$

That is, we have

$$z^6+7z^5+7z^4-35z^3-55z^2+35z+60 = (z^4-5z^2+5)(z^2+7z+12).$$

I invite the reader to do this division by the Euclidean algorithm♣

The following example highlights the difference between the long division and the Euclidean division.

Example 10. (Comparing divisions.) Let us divide in two different ways the polynomial $P(z)$ by the polynomial $Q(z)$, where $P(z) = z^4 + z^3 + z^2 + z + 1$, which we also write $P(z) = 1 + z + z^2 + z^3 + z^4$, and $Q(z) = z + 1$, which we also write $Q(z) = 1 + z$. By applying the Euclidean division algorithm and the long division algorithm, in this order, show that we have the following formulas

$$\begin{cases} \frac{z^4+z^3+z^2+z+1}{z+1} = (z^3 + z) + \frac{1}{z+1} \qquad \text{or} \quad P(z) = (z^3 + z)Q(z) + 1, \\ \\ \text{and} \\ \\ \frac{1+z+z^2+z^3+z^4}{1+z} = (1 + z^2 + z^4) - \frac{z^5}{1+z} \quad \text{or} \quad P(z) = (1 + z^2 + z^4)Q(z) - z^5. \end{cases}$$

Solution. It is obvious that

$$P(z) = z^4 + z^3 + z^2 + z + 1 = (z + 1)(z^3 + z) + 1 = (z^3 + z)Q(z) + 1,$$

and thus we very easily have the Euclidean division searched for.

Now, let us apply the long division algorithm. We have

$$\begin{array}{rrrrrrr|lll} & 1 & +z & +z^2 & +z^3 & +z^4 & & 1 & +z & \\ \cline{8-10} - & 1 & +z & & & & & 1 & +z^2 & +z^4 \\ \hline & 0 & 0 & z^2 & +z^3 & +z^4 & & & & \\ - & & & z^2 & +z^3 & & & & & \\ \hline & & & 0 & 0 & +z^4 & & & & \\ - & & & & & +z^4 & +z^5 & & & \\ \hline & & & & & 0 & -z^5 & & & \end{array}.$$

This truncated long division algorithm shows that

$$\frac{1 + z + z^2 + z^3 + z^4}{1 + z} = 1 + z^2 + z^4 - \frac{z^5}{1 + z}$$

or, writing in another way,

$$P(z) = 1 + z + z^2 + z^3 + z^4 = (1 + z^2 + z^4)Q(z) - z^5 \clubsuit$$

7. Avoiding The Long Division (The Geometric Series Trick)

A very practical and general way of avoiding the long division algorithm is to employ the geometric series and some very nice properties of the absolutely convergent series (in particular, those of the convergent power series).

To start with, let us consider a convergent power series

$$f(z) = 1 - a_1 z - a_2 z^2 - a_3 z^3 - \cdots,$$

in some open ball centered at the origin. We are allowed to write

$$\frac{1}{f(z)} = \frac{1}{1-(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 - \cdots)} = 1 + \sum_{N=1}^{+\infty} (a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^N.$$

By the multiplication rule for convergent power series it immediately follows that $(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^N$, for each $N \geq 1$, is a convergent power series in which the smallest power of z showing off is z^N. Thus, we may write

$$(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^N = \sum_{k=N}^{+\infty} b_{(k,N)} z^k,$$

where each $b_{(k,N)}$ is a complex coefficient. It is not hard to see that the coefficient $b_{(k,N)}$ is given by a homogeneous polynomial in several variables, with natural numbers as coefficients (hence, positive coefficients), of order N, and evaluated at the point $(a_1, a_2, \ldots, a_{(k-N)+1})$ [thus, such polynomial is in $(k-N)+1$ variables].

Hence, we have

$$\frac{1}{f(z)} = 1 + \sum_{N=1}^{+\infty} \sum_{k=N}^{+\infty} b_{(k,N)} z^k.$$

Next, we argue that it is allowed to change this order of summation.

We first remark that the power series $F(z) = (1 - |a_1|z - |a_2|z^2 - |a_3|z^3 - \cdots)$ converges absolutely and in the same region as $f(z) = (1 - a_1 z - a_2 z^2 - a_3 z^3 - \cdots)$. Therefore, analogously to what we have done right above, we may write $1/F(z) = 1 + \sum_{N=1}^{+\infty} \sum_{k=N}^{+\infty} B_{(k,N)} z^k$, converging absolutely in some small neighborhood of the origin and each $B_{(k,N)} \geq 0$. It is obvious the inequality $|b_{(k,N)}| \leq B_{(k,N)}$. Such inequality implies the absolutely convergence of the double series $\sum_{N=1}^{+\infty} \sum_{k=N}^{+\infty} b_{k,N} z^k$. From this it follows that we may change the order of summation in this double summation (see Oliveira [5], Apostol [1], Gamelin [3], Lang [4]).

Thus, we have

$$\sum_{N\geq 1}\sum_{k\geq N} b_{(k,N)} z^k = \sum_{k\geq 1}\left[\sum_{1\leq N\leq k} b_{(k,N)}\right] z^k.$$

Hence, by defining the coefficients

$$c_k = \sum_{1\leq N\leq k} b_{(k,N)},$$

we have proven that

$$\begin{aligned}
\frac{1}{f(z)} &= \frac{1}{1 - a_1 z - a_2 z^2 - a_3 z^3 - \cdots} \\
&= 1 + \sum_{N=1}^{+\infty} (a_1 z + a_2 z^2 + a_3 z^3 + \cdots)^N \\
&= 1 + \sum_{N=1}^{+\infty}\sum_{k=N}^{+\infty} b_{(k,N)} z^k \\
&= 1 + \sum_{N\geq 1}\sum_{k\geq N} b_{(k,N)} z^k \\
&= 1 + \sum_{k\geq 1}\left[\sum_{1\leq N\leq k} b_{(k,N)}\right] z^k \\
&= 1 + \sum_{k=1}^{+\infty}\left[\sum_{N=1}^{k} b_{(k,N)}\right] z^k \\
&= 1 + \sum_{k=1}^{+\infty} c_k z^k \clubsuit
\end{aligned}$$

Remark. It is good to point out that this geometric series method is a "hands on" job. In other words, while such method is good to provide the first non null terms of such division, it has the unpleasant disadvantage of not providing a too nice table for the coefficients we are searching for. As a matter of fact, we have

$$\begin{aligned}
\sum_{N=1}^{+\infty}(a_1 z+a_2 z^2+a_3 z^3+a_4 z^4+\cdots)^N &= (a_1 z+a_2 z^2+a_3 z^3+a_4 z^4+\cdots)+(a_1 z+a_2 z^2+a_3 z^3+a_4 z^4+\cdots)^2 \\
&\quad +(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^3 + (a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^4 + \cdots \\
&= a_1 z + (a_2 + a_1^2) z^2 + (a_3 + 2a_1 a_2 + a_1^3) z^3 + [a_4 + (2a_1 a_3 + a_2^2) + 3a_1^2 a_2 + a_1^4] z^4 + \cdots.
\end{aligned}$$

8. Examples (Using The Geometric Series Trick)

Example 11. (Compare with Example 2.) By using a geometric series, write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1+z^2}{1+z}.$$

Solution. By the well known geometric formula for $(1+z)^{-1}$ it follows that

$$\begin{aligned}
\frac{1+z^2}{1+z} &= (1+z^2)(1-z+z^2-z^3+z^4-z^5+z^6-z^7+\cdots)\\
&= (1-z+z^2-z^3+z^4-z^5+\cdots)+z^2(1-z+z^2-z^3+z^4-z^5+\cdots)\\
&= (1-z+z^2-z^3+z^4-z^5+\cdots)+(z^2-z^3+z^4-z^5+z^6-z^7+\cdots)\\
&= (1-z+z^2-z^3+z^4-z^5+\cdots)+(0+0+z^2-z^3+z^4-z^5+z^6-z^7+\cdots)\\
&= 1-z+2z^2-2z^3+2z^4-2z^5+2z^6-2z^7+2z^8-2z^9+\cdots.
\end{aligned}$$

This development is valid for all $|z|<1$ ♣

Example 12. (Compare with Example 3.) By using a geometric series, write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1+z}{1+z^2}.$$

Solution. By the geometric formula for $(1+z^2)^{-1}$ it follows that

$$\begin{aligned}
\frac{1+z}{1+z^2} &= (1+z)(1-z^2+z^4-z^6+z^8-z^{10}\ +\cdots)\\
&= (1-z^2+z^4-z^6+z^8-z^{10}\ +\cdots)+(z-z^3+z^5-z^7+z^9-z^{11}+\cdots)\\
&= \quad (1+0-z^2+0+z^4+0-z^6+0+z^8+0\ -z^{10}\ +\cdots)\\
&\quad +(0+z+0-z^3+0+z^5+0-z^7+0\ +z^9\ +0\ \ -z^{11}+0+\cdots)\\
&= 1+z-z^2-z^3+z^4+z^5-z^6-z^7+z^8+z^9-z^{10}-z^{11}+\cdots.
\end{aligned}$$

This development is valid for all $|z|<1$ ♣

9. Examples, using the Big-oh Notation $\mathcal{O}(\cdot)$

Let us consider two complex functions, $f(z)$ and $g(z)$, defined on a neighborhood of a point z_0, but not necessarily at $z = z_0$. In other words, this means that the maps f and g are defined at every point of an open ball centered at z_0, with the possible exception of its center.

Definition and Notation. We say that $f(z)$ is big-oh of $g(z)$ as $z \longrightarrow z_0$, if there exists a constant $C > 0$ such that we have

$$|f(z)| \leq C|g(z)| \text{ as } z \longrightarrow z_0$$

or, equivalently,

$$|f(z)| \leq C|g(z)| \text{ for all } z \text{ near } z_0, \text{ but } z \neq z_0.$$

We then write

$$\begin{cases} f(z) = \mathcal{O}(g(z)) \text{ as } z \longrightarrow z_0, \\ \text{or} \\ f(z) = \mathcal{O}(g(z)). \end{cases}$$

Example 13 (A Guide Example). Let us consider

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

an arbitrary complex power series convergent on a neighborhood of the origin, and an arbitrary integer $N \geq 1$. It is not hard to see that the so called "tail"

$$T(z) = a_N z^N + a_{N+1} z^{N+1} + \cdots$$

is big-oh of z^N as $z \longrightarrow 0$. In fact, by writing

$$|T(z)| = |z|^N |a_N + a_{N+1} z^1 + a_{N+2} z^2 + a_{N+3} z^3 + \cdots|,$$

we see that there is a constant $C > 0$ (e.g., $C = |a_N| + 1$) such that we have

$$|T(z)| \leq C|z|^N, \text{ for all } z \text{ near } 0.$$

This means that

$$T(z) = \mathcal{O}(z^N) \text{ as } z \longrightarrow 0.$$

With some abuse of notation, we may write

$$\boxed{A(z) = a_0 + a_1 z + \cdots + a_{N-1} z^{N-1} + \mathcal{O}(z^N)} \clubsuit$$

Example 14. Calculate the terms through order seven of the power series expansion about $z = 0$ of the secant function

$$\sec(z) = \frac{1}{\cos z}.$$

Solution. It is well known that near the origin we have (see Example 13)

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \mathcal{O}(z^8).$$

Hence,

$$\begin{aligned}
\frac{1}{\cos z} &= \frac{1}{1 - \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \mathcal{O}(z^8)\right)} \\
&= 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \mathcal{O}(z^8)\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \mathcal{O}(z^8)\right)^2 \\
&\quad + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \mathcal{O}(z^8)\right)^3 + \mathcal{O}(z^8) \\
&= 1 + \frac{z^2}{2!} + \left(-\frac{z^4}{4!} + \frac{z^4}{2!2!}\right) + \left(\frac{z^6}{6!} - \frac{2z^6}{2!4!} + \frac{z^6}{2!2!2!}\right) + \mathcal{O}(z^8) \\
&= 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \left(\frac{z^6}{720} - \frac{z^6}{24} + \frac{z^6}{8}\right) + \mathcal{O}(z^8) \\
&= 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \frac{(1 - 30 + 90)z^6}{720} + \mathcal{O}(z^8).
\end{aligned}$$

Thus, we find

$$\frac{1}{\cos z} = 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \frac{61}{720}z^6 + \mathcal{O}(z^8) \clubsuit$$

Remark. As can be easily verified, the function $\cos z$ vanishes at $z = \pi/2$ but does not vanish for $|z| < \pi/2$. Hence, it follows that the radius of convergence of the power series for $\sec z = 1/\cos z$ is $r = \pi/2$ (see Oliveira [6, p.18]).

Example 15. (Compare with Example 4 and Example 5.) Find the coefficients of z^n, for $n \le 7$, in the power series expansion of

$$\tan z = \frac{\sin z}{\cos z}$$

near the origin.

Solution. By Example 14, and the notation in Example 13, we have

$$\begin{aligned}
\frac{\sin z}{\cos z} &= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \mathcal{O}(z^9)\right)\left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \frac{61}{720}z^6 + \mathcal{O}(z^8)\right) \\
&= z + \left(\frac{1}{2} - \frac{1}{6}\right)z^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)z^5 \\
&\quad + \left(\frac{61}{720} - \frac{5}{144} + \frac{1}{240} - \frac{1}{5040}\right)z^7 + \mathcal{O}(z^9) \\
&= z + \left(\frac{3-1}{6}\right)z^3 + \left(\frac{25-10+1}{120}\right)z^5 \\
&\quad + \left(\frac{427-175+21-1}{5040}\right)z^7 + \mathcal{O}(z^9) \\
&= z + \frac{1}{3}z^3 + \frac{16}{120}z^5 + \frac{272}{5040}z^7 + \mathcal{O}(z^9) \\
&= z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \mathcal{O}(z^9).
\end{aligned}$$

Thus, we have

$$\frac{\sin z}{\cos z} = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \mathcal{O}(z^9) \clubsuit$$

Remark. Regarding the radius of convergence for the power series for the function $\tan z$, see Remark to Example 14.

Example 16. (Compare with Example 6.) Calculate the terms through order seven of the power series expansion about $z = 0$ of the function

$$f(z) = \frac{z}{\sin z}.$$

Solution. We have, on a neighborhood of the origin, excepting the origin itself,

$$\begin{aligned}
\frac{z}{\sin z} &= \frac{z}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \mathcal{O}(z^9)} \\
&= \frac{1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \mathcal{O}(z^8)\right)} \\
&= 1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \mathcal{O}(z^8)\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \mathcal{O}(z^8)\right)^2 \\
&\quad + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \mathcal{O}(z^8)\right)^3 + \mathcal{O}(z^8).
\end{aligned}$$

Collecting even powers of z we arrive at

$$\begin{aligned}
\frac{z}{\sin z} &= 1 + \frac{z^2}{3!} + \left(-\frac{1}{5!} + \frac{1}{3!3!}\right) z^4 + \left(\frac{1}{7!} - \frac{2}{3!5!} + \frac{1}{3!3!3!}\right) z^6 + \mathcal{O}(z^8) \\
&= 1 + \frac{z^2}{6} + \left(-\frac{1}{120} + \frac{1}{36}\right) z^4 + \left(\frac{1}{7!} - \frac{84}{3!7!} + \frac{840}{3!3!7!}\right) z^6 + \mathcal{O}(z^8) \\
&= 1 + \frac{z^2}{6} + \frac{-3 + 10}{360} z^4 + \left(\frac{1}{7!} - \frac{14}{7!} + \frac{140}{3!7!}\right) z^6 + \mathcal{O}(z^8) \\
&= 1 + \frac{z^2}{6} + \frac{7}{360} z^4 + \frac{1}{7!}\left(1 - 14 + \frac{70}{3}\right) z^6 + \mathcal{O}(z^8) \\
&= 1 + \frac{z^2}{6} + \frac{7}{360} z^4 + \frac{1}{7!}\left(\frac{31}{3}\right) z^6 + \mathcal{O}(z^8).
\end{aligned}$$

Thus, we have

$$\frac{z}{\sin z} = 1 + \frac{z^2}{6} + \frac{7}{360} z^4 + \frac{31}{15120} z^6 + \mathcal{O}(z^8) \clubsuit$$

Example 17. Show that

$$\frac{e^z}{1+z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{3}{8}z^4 - \frac{11}{30}z^5 + \cdots + a_n z^n + \cdots,$$

where

$$a_n = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right], \text{ for all } n \geq 2.$$

Solution. We have, for all $|z| < 1$,

$$\frac{e^z}{1+z} = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \frac{z^k}{k!} + \cdots\right)\left(1 - z + z^2 + \cdots + (-1)^j z^j + \cdots\right).$$

Hence, by the multiplication rule for power series, and noticing that the resulting coefficient of z^1 is zero, we arrive at

$$\frac{e^z}{1+z} = 1 + \sum_{N=2}^{+\infty}\left(\sum_{j+k=N}\frac{(-1)^j}{k!}\right)z^N.$$

Next, we notice that for each $N \geq 2$ we may write

$$\begin{aligned}
\sum_{j+k=N}\frac{(-1)^j}{k!} &= \sum_{0\leq k\leq N}\frac{(-1)^{N-k}}{k!} \\
&= \frac{(-1)^N}{0!} + \frac{(-1)^{N-1}}{1!} + \frac{(-1)^{N-2}}{2!} + \cdots + \frac{(-1)^{N-N}}{N!} \\
&= (-1)^N\left[\frac{(-1)^{-2}}{2!} + \frac{(-1)^{-3}}{3!} + \cdots + \frac{(-1)^{-N}}{N!}\right] \\
&= (-1)^N\left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^N}{N!}\right] \clubsuit
\end{aligned}$$

10. Examples (Miscellaneous Methods)

Example 19. (Compare with Example 1.) Use the formula for the sum of a finite geometric sequence and write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1}{1+z}.$$

Solution. Let us apply the well known formula

$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}, \text{if } r \neq 1.$$

Thus, for each $|z| < 1$ we have

$$\begin{aligned} 1 + z + z^2 + z^3 + \cdots &= \lim_{n\to+\infty} (1 + z + \cdots + z^n) \\ &= \lim_{n\to+\infty} \frac{1-z^{n+1}}{1-z} \\ &= \frac{1}{1-z}. \end{aligned}$$

Hence, we have the following power series expansion

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + \cdots, \quad \text{for all } |z| < 1♣$$

Example 20. (Compare with Example 2 and Example 16.) Use a polynomial identity to write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1}{1+z}.$$

Solution. Let us apply the following polynomial identity

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1).$$

Thus, for each $|z| < 1$ we have

$$\begin{aligned} 1 + z + z^2 + z^3 + \cdots &= \lim_{n\to+\infty} (1 + z + \cdots + z^n) \\ &= \lim_{n\to+\infty} \frac{z^{n+1}-1}{z-1} \\ &= \frac{1}{1-z}. \end{aligned}$$

Hence, we have the following power series expansion

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + \cdots, \quad \text{for all } |z| < 1♣$$

Example 21. (Compare with Example 2 and Example 11.) By way of rewriting a polynomial, write $f(z)$ as a power series centered at the origin, where

$$f(z) = \frac{1+z^2}{1+z}.$$

Solution. Let us write

$$1+z^2 = (1+z)^2 - 2z.$$

Hence,

$$\begin{aligned}
\frac{1+z^2}{1+z} &= \frac{(1+z)^2 - 2z}{1+z} \\
&= 1+z-\frac{2z}{1+z} \\
&= 1+z-\frac{2(1+z)-2}{1+z} \\
&= 1+z-2+\frac{2}{1+z} \\
&= -1+z+2\left(\frac{1}{1+z}\right).
\end{aligned}$$

By a well known geometric formula it follows that

$$\begin{aligned}
\frac{1+z^2}{1+z} &= -1+z+2(1-z+z^2-z^3+z^4+\cdots) \\
&= 1-z+2(z^2-z^3+z^4-z^5+\cdots).
\end{aligned}$$

This development is valid for all z in the unit ball centered at the origin♣

11. Bernoulli Numbers

In this section we show two examples highlighting how advantageous can be the algorithm (the division table) presented in this text.

Example 22 (Bernoulli Numbers). Define the Bernoulli numbers by

$$\frac{z}{2}\cot\left(\frac{z}{2}\right) = 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - B_3\frac{z^6}{6!} - B_4\frac{z^8}{8!} - \cdots.$$

Answer the following questions.

(a) Why there are no odd terms in this series?

(b) What is the radius of convergence of the series?

(c) Find the first five Bernoulli numbers, by using the "geometric series trick".

Solution.

(a) The function z is odd and so is the function $\cot z$. Thus, $z\cot(z)$ is even.

(b) It is the same as the radius of convergence of the function $g(z) = z/\sin z$, with $g(0) = 1$. We have $g(z) \neq 0$, if $|z| < \pi$, and $\sin\pi = 0$. Hence, the radius of convergence is $r = \pi$. (See Oliveira [6, p. 18].)

(c) We have

$$\begin{aligned} z\cot z &= z\frac{\cos z}{\sin z} \\ &= z\left[\frac{\cos z}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{(11)!} + \mathcal{O}(z^{13})}\right] \\ &= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{(10)!} + \mathcal{O}(z^{12})}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{(11)!} + \mathcal{O}(z^{12})} \end{aligned}$$

Hence, we compute

$$\begin{aligned} z\cot z = &\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{(10)!} + \mathcal{O}(z^{12})\right) \\ &\cdot\left[1 + \sum_{N=1}^{5}\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \frac{z^8}{9!} + \frac{z^{10}}{(11)!} - \mathcal{O}(z^{12})\right)^N + \mathcal{O}(z^{12})\right] \end{aligned}$$

Simplifying a little, we may write

$$\begin{aligned} z\cot z &= \left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\frac{z^8}{8!}-\frac{z^{10}}{(10)!}\right) \\ &\quad\times\left[1+\sum_{N=1}^{5}\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}-\frac{z^8}{9!}+\frac{z^{10}}{(11)!}\right)^N\right]+\mathcal{O}(z^{12}). \end{aligned}$$

At this point, we relate the coefficients of the power series for $(z/2)\cot(z/2)$ with the ones of the power series for $z\cot(z)$. In doing so, it follows that the coefficients of z^2, z^4, z^6, z^8, and z^{10} of these power series satisfy the identities

$$\left\{\begin{aligned}
-2^2\tfrac{B_1}{2!} &= \tfrac{1}{3!}-\tfrac{1}{2!} = -\tfrac{2}{6} = -\tfrac{1}{3} \\
&= -\tfrac{2^2}{2!}\left(\tfrac{1}{6}\right) \\
-2^4\tfrac{B_2}{4!} &= \left(-\tfrac{1}{5!}+\tfrac{1}{3!3!}\right)-\tfrac{1}{2!3!}+\tfrac{1}{4!} = \tfrac{-3+10-30+15}{360} = \tfrac{-8}{360} = -\tfrac{8}{(15)4!} \\
&= -\tfrac{2^4}{4!}\left(\tfrac{1}{30}\right) \\
-2^6\tfrac{B_3}{6!} &= \left(\tfrac{1}{7!}-\tfrac{2}{3!5!}+\tfrac{1}{3!3!3!}\right)+\left(\tfrac{1}{2!5!}-\tfrac{1}{2!3!3!}\right)+\tfrac{1}{4!3!}-\tfrac{1}{6!} \\
&= \tfrac{1}{6!}\left(\tfrac{1}{7}-2+\tfrac{10}{3}\right)+\tfrac{1}{6!}(3-10)+\tfrac{5}{6!}-\tfrac{1}{6!} \\
&= \tfrac{1}{6!}\left(\tfrac{1}{7}-2+\tfrac{10}{3}+3-10+5-1\right) \\
&= \tfrac{1}{6!}\left(\tfrac{3+70}{21}-5\right) = \tfrac{1}{6!}\left(\tfrac{73-105}{21}\right) = -\tfrac{1}{6!}\tfrac{32}{21} \\
&= -\tfrac{2^6}{6!}\left(\tfrac{1}{42}\right) \\
-2^8\tfrac{B_4}{8!} &= \left(-\tfrac{1}{9!}+\tfrac{2}{3!7!}+\tfrac{1}{5!5!}-\tfrac{3}{3!3!5!}+\tfrac{1}{3!3!3!3!}\right)+\left(-\tfrac{1}{2!7!}+\tfrac{2}{2!3!5!}-\tfrac{1}{2!3!3!3!}\right) \\
&\quad+\left(-\tfrac{1}{4!5!}+\tfrac{1}{4!3!3!}\right)-\tfrac{1}{6!3!}+\tfrac{1}{8!} \\
&= \tfrac{1}{8!}\left(-\tfrac{1}{9}+\tfrac{16}{3!}+\tfrac{8\times7\times6}{5!}-\tfrac{3\times8\times7\times6}{3!3!}+\tfrac{8\times7\times5}{9}\right) \\
&\quad+\tfrac{1}{8!}\left(-\tfrac{8}{2!}+\tfrac{2\times8\times7}{2!}-\tfrac{8\times7\times5}{3}\right)+\tfrac{1}{8!}\left(-\tfrac{8\times7\times6}{4!}+\tfrac{8\times7\times5}{3!}\right)-\tfrac{56}{8!3!}+\tfrac{1}{8!} \\
&= \tfrac{1}{8!}\left(-\tfrac{1}{9}+\tfrac{8}{3}+\tfrac{14}{5}-28+\tfrac{280}{9}\right)+\tfrac{1}{8!}\left(-4+56-\tfrac{280}{3}\right) \\
&\quad+\tfrac{1}{8!}\left(-14+\tfrac{140}{3}\right)-\tfrac{28}{8!3}+\tfrac{1}{8!} \\
&= \tfrac{1}{8!}\left(-\tfrac{1}{9}+\tfrac{8}{3}+\tfrac{14}{5}-28+\tfrac{280}{9}-4+56-\tfrac{280}{3}-14+\tfrac{140}{3}-\tfrac{28}{3}+1\right) \\
&= \tfrac{1}{8!}\left(\tfrac{-1+24+280-840+420-84}{9}+\tfrac{14}{5}+11\right) \\
&= \tfrac{1}{8!}\left(-\tfrac{201}{9}+\tfrac{14}{5}+11\right) = \tfrac{1}{8!}\left(-\tfrac{293}{15}+11\right) \\
&= \tfrac{1}{8!}\left(-\tfrac{128}{15}\right) \\
&= -\tfrac{2^8}{8!}\left(\tfrac{1}{30}\right) \\
-2^{10}\tfrac{B_5}{(10)!} &= \ldots.
\end{aligned}\right.$$

Thus, the first four Bernoulli numbers are

$$B_1 = \frac{1}{6},\quad B_2 = \frac{1}{30},\quad B_3 = \frac{1}{42},\quad \text{and}\quad B_4 = \frac{1}{30}.$$

I leave to the courageous reader to show that $B_5 = 5/66$♣

Example 23 (Bernoulli Numbers, compare with Example 22). Define the Bernoulli numbers by

$$\frac{z}{2}\cot\left(\frac{z}{2}\right) = 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - B_3\frac{z^6}{6!} - B_4\frac{z^8}{8!} - \cdots.$$

Find the first five Bernoulli numbers by employing the division algorithm.

Solution. First, let us write the power series expansion for the division

$$z\cot z = \frac{\cos z}{\frac{\sin z}{z}} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{(10)!} + \cdots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{(11)!} + \cdots}.$$

We have

z^0	z^2	z^4	z^6	z^8	z^{10}	$\cdots$
1	$-\frac{1}{2!}$	$+\frac{1}{4!}$	$-\frac{1}{6!}$	$+\frac{1}{8!}$	$-\frac{1}{(10)!}$	$\cdots$
$-$ 1	$-\frac{1}{3!}$	$+\frac{1}{5!}$	$-\frac{1}{7!}$	$+\frac{1}{9!}$	$-\frac{1}{(11)!}$	.
0	$-\frac{1}{3}$	$+\frac{4}{5!}$	$-\frac{6}{7!}$	$+\frac{8}{9!}$	$-\frac{10}{(11)!}$	$\cdots$
$-$	$-\frac{1}{3}$	$+\frac{1}{3!3}$	$-\frac{1}{5!3}$	$+\frac{1}{7!3}$	$-\frac{1}{9!3}$	$\cdots$
	0	$-\frac{8}{4!15}$	$+\frac{8}{6!7}$	$-\frac{16}{8!9}$	$+\frac{80}{(10)!33}$	$\cdots$
$-$		$-\frac{8}{4!15}$	$+\frac{8}{3!4!15}$	$-\frac{8}{5!4!15}$	$+\frac{8}{7!4!15}$	$\cdots$
		0	$-\frac{32}{6!21}$	$+\frac{256}{8!45}$	$-\frac{448}{(10)!33}$	$\cdots$
$-$			$-\frac{32}{6!21}$	$+\frac{32}{3!6!21}$	$-\frac{32}{5!6!21}$	$\cdots$
			0	$-\frac{128}{8!15}$	$+\frac{128\times 13}{(10)!33}$	$\cdots$
$-$				$-\frac{128}{8!15}$	$+\frac{128}{3!8!15}$	$\cdots$
				0	$-\frac{512\times 5}{(10)!33}$	$\cdots$
$-$					$-\frac{512\times 5}{(10)!33}$	$\cdots$
					0	$\cdots$

$$\begin{array}{l} 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{(11)!} + \cdots \\ \hline 1 - \frac{z^2}{3} - \frac{8z^4}{4!15} - \frac{32z^6}{6!21} - \frac{128z^8}{8!15} - \frac{2^9 5 z^{10}}{(10)!33} - \cdots \end{array}$$

By relating the coefficients for the power series for the function $(z/2)\cot(z/2)$ with the ones for the function $z\cot z$, we find

$$\begin{cases} -2^2\frac{B_1}{2!} = -\frac{1}{3} \\ -2^4\frac{B_2}{4!} = -\frac{2^3}{4!15} \\ -2^6\frac{B_3}{6!} = -\frac{2^5}{6!21} \\ -2^8\frac{B_4}{8!} = -\frac{2^7}{8!15} \\ -2^{10}\frac{B_5}{(10)!} = -\frac{2^9 5}{(10)!33} \end{cases} \Longrightarrow \begin{cases} B_1 = \frac{1}{6} \\ B_2 = \frac{1}{30} \\ B_3 = \frac{1}{42} \\ B_4 = \frac{1}{30} \\ B_5 = \frac{5}{66} \end{cases} \clubsuit$$

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Departamento de Matemática, Universidade de São Paulo
Rua do Matão 1010 - CEP 05508-090
São Paulo, SP - Brasil
e-mail: oliveira@ime.usp.br