The Change of Variable for the Riemann Integral on the Real Line

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Abstract.

We show a very elementary and practical version of the *Change of Variable Theorem* (for the unidimensional Riemann Integral) that is not a particular case of the influential H. Kestelman's version. Two examples are given.

Introduction.

The simplest *Change of Variable Theorem for the Riemann Integral on the Real Line* supposes that all the functions and derivatives involved are continuous on closed intervals, under the additional condition that the inverse map of the change of variable map is differentiable.

A comment based on T. M. Apostol *Análisis Matemático*, pp. 199–200. The probably most general version of the *Change of Variable Theorem* (for the unidimensional Riemann integral) does not require the continuity of the function involved or that the substitution map is invertible. Such version has the form

$$\int_{G(\alpha)}^{G(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\big(G(t)\big)g(t)dt,$$

with the function f integrable on the interval $G\big([\alpha,\beta]\big)$ and the substitution map

$$G(t) = G(\alpha) + \int_{\alpha}^{t} g(\tau)d\tau, \text{ where } t \in [\alpha,\beta],$$

for some function g integrable on $[\alpha,\beta]$. It is worth noting that the identity $G'(t) = g(t)$ is true on the points of continuity of g but is not assured on other points. The first proof of this general version is due to H. Kestelman (1961). In the same journal, same number, R. Davies simplifies the proof of Kestelman. Since then, many articles have been published about this general version.

Apostol's book does not prove the general version. Kestelman's proof employs the concept *measure zero*, from Lebesgue measure theory.

The version proven in this note presents the following four features.

- The enunciate is very simple and the proof is very elementary.
- It applies in some cases where the "general version" does not.
- It is stronger than the versions in *Basic Real Analysis*, A. W. Knapp, pp. 37–38 (the proof in this note has some similarities and many differences with respect to the proof in such book) and in *Principles of Mathematical Analysis*, W. Rudin, p. 133.
- It does not appear in the material cited in References.

Notation.

Along this proof we adopt the following type of partitions

$$\mathcal{X} = \{a = x_0 \leq \cdots \leq x_n = b\},$$

with repeated points, for the Riemann integration. Many authors (Knapp, Lang, Rudin, Spivak, etc.) adopt it. One advantage is that the integral of a function over a single point is automatically zero. Regarding the investigation of the existence and the value of the integral of a given function, it does not matter if the partition has repeated points or not.

Given a bounded function $f : [a,b] \to \mathbb{R}$, we indicate the inferior and the superior *Riemann sums* of f with respect to the partition $\mathcal{X}$ by, respectively, $s(f,\mathcal{X})$ and $S(f,\mathcal{X})$.

Just to recall, we have

$$s(f,\mathcal{X}) = \sum_{i=1}^{n} m_i \Delta x_i \text{ and } S(f,\mathcal{X}) = \sum_{i=1}^{n} M_i \Delta x_i,$$

where m_i and M_i are, respectively, the infimum and the supremum of f restricted to the sub-interval $[x_i, x_{i-1}]$, and $\Delta x_i = x_i - x_{i-1}$.

Theorem (Generalized Change of Variable Theorem). *Let us consider*

$$f : [a, b] \longrightarrow \mathbb{R} \text{ integrable and } \varphi : [\alpha, \beta] \longrightarrow [a, b]$$

surjective, increasing (not necessarily strictly increasing) and continuous. Suppose that φ is differentiable on the open interval (α, β). The following are true.

- *If φ' is integrable on $[\alpha, \beta]$, then the map $(f \circ \varphi)\varphi'$ is integrable on $[\alpha, \beta]$.*
- *If the product $(f \circ \varphi)\varphi'$ is integrable on $[\alpha, \beta]$, then we have the formula*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Proof.

◇ The hypothesis that φ is continuous is superfluous. Given $t \in [\alpha, \beta]$, we put $x = \varphi(t) \in [a, b]$. Let x' and x'' be such that $x \in [x', x''] \subset [a, b]$. Since the map φ is surjective and increasing, there exist t' and t'', both in $[\alpha, \beta]$ with $t' \le t''$, such that $\varphi(t') = x'$ and $\varphi(t'') = x''$. Since φ is increasing, we have $\varphi\big([t', t'']\big) \subset [x', x'']$. Hence, the map φ is continuous [we notice that if $x' < x$ then $t' < t$ and, analogously, if $x < x''$ then $t < t''$].

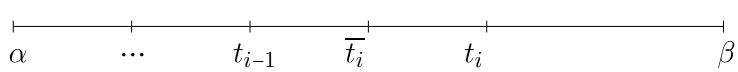
◇ The map φ is uniformly continuous. It is trivial.

◇ We have $\varphi' \ge 0$ on the open interval (α, β). It is obvious, since φ is increasing.

◇ To integrate φ', it is clear that we may define $\varphi'(\alpha)$ and $\varphi'(\beta)$ arbitrarily.

◇ Let $\mathcal{T} = \{\alpha = t_0 \le \cdots \le t_n = \beta\}$ be an arbitrary partition of the interval $[\alpha, \beta]$ and $\mathcal{X} = \{a = x_0 \le \cdots \le x_n = b\}$ be the partition of the interval $[a, b]$ given by $\mathcal{X} = \varphi(\mathcal{T})$. That is, suppose that $x_i = \varphi(t_i)$ for each $i = 0, \ldots, n$.

The mean-value theorem yields a point $\overline{t_i} \in [t_{i-1}, t_i]$ satisfying the condition $\Delta x_i = \varphi(t_i) - \varphi(t_{i-1}) = \varphi'(\overline{t_i})\Delta t_i$.



- *If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$.* It follows from the uniform continuity of φ, since given $\delta_1 > 0$ there exists $\delta_2 > 0$ such that $|\varphi(t) - \varphi(\tau)| \le \delta_1$ if $|t - \tau| \le \delta_2$.

- *If φ' is integrable, then $(f \circ \varphi)\varphi'$ is integrable.* Let τ_i be arbitrary in $[t_{i-1}, t_i]$. We notice that $\varphi(\tau_i) \in [x_{i-1}, x_i]$. Let us investigate the Riemann sum

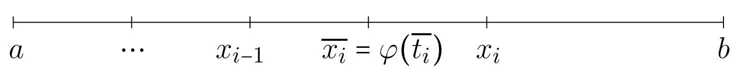
  $$\sum f(\varphi(\tau_i))\varphi'(\tau_i)\Delta t_i = \sum f(\varphi(\tau_i))\Delta x_i + \sum f(\varphi(\tau_i))\big[\varphi'(\tau_i)) - \varphi'(\overline{t_i})\big]\Delta t_i.$$

  If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the first sum on the right side goes to $\int_a^b f dx$. Let M be a constant such that $|f| \le M$ (obviously, f is bounded). We have

  $$\Big|\sum f(\varphi(\tau_i))\big[\varphi'(\tau_i)) - \varphi'(\overline{t_i})\big]\Delta t_i\Big| \le M\big[S(\varphi', \mathcal{T}) - s(\varphi', \mathcal{T})\big] \xrightarrow{|\mathcal{T}| \to 0} 0.$$

  Thus, $(f \circ \varphi)\varphi'$ is integrable (the value of its integral equals the one of f).

- *If $(f \circ \varphi)\varphi'$ is integrable, then the value of its integral equals the one of f.* With the above notation, we choose $\tau_i = \overline{t_i}$ and write $\overline{x_i} = \varphi(\overline{t_i})$.

  Hence, we have

  $$\sum f(\varphi(\overline{t_i}))\varphi'(\overline{t_i})\Delta t_i = \sum f(\overline{x_i})\Delta x_i.$$

  If $|\mathcal{T}| \to 0$, by definition the left hand side goes to the integral of $(f \circ \varphi)\varphi'$. If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the right hand side goes to the integral of f ♣

Before moving ahead, let us recall some usual definitions. Let us consider an arbitrary real map $\varphi : [\alpha, \beta] \to \mathbb{R}$. We say that φ is *monotone* if φ is increasing (not necessarily strictly), decreasing (not necessarily strictly), or constant. Moreover, we say that the map φ is *piecewise monotone* if there exists a finite sequence (or partition) $\{\alpha_0 = \alpha < \alpha_1 < \cdots < \alpha_N = \beta\}$ such that φ is monotone on each open sub-interval (α_j, α_{j+1}), where $j = 0, \dots, N-1$.

Corollary. *Keeping all the other hypotheses of the theorem, let us suppose that the surjective and continuous map $\varphi : [\alpha, \beta] \to [a, b]$ satisfies one of the following four conditions.*

(a) *φ is monotone.*

(b) *φ is piecewise monotone.*

(c) *φ' has a finite number of zeros.*

(d) *φ' has a finite number of zeros in $[\alpha + \epsilon, \beta]$, for each $0 < \epsilon < \beta - \alpha$.*

Then, the following claims are true.

- *If φ' is integrable, then $(f \circ \varphi)\varphi'$ also does.*
- *If $(f \circ \varphi)\varphi'$ is integrable, then*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

Proof.

(a) The case φ increasing is already proven. Let us suppose φ decreasing. Then

$$\psi(s) = \varphi(\alpha + \beta - s), \text{ where } s \in [\alpha, \beta],$$

is increasing, with $\psi(\alpha) = \varphi(\beta) = a$ and $\psi(\beta) = \varphi(\alpha) = b$.

Clearly, φ is integrable if and only if ψ also does. Analogously for the pair of derivatives φ' and ψ', and for the pair of functions $(f \circ \varphi)\varphi'$ and $(f \circ \psi)\psi'$.

By applying the formula on the theorem to the function $(f \circ \psi)\psi'$ we obtain

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f\big(\psi(s)\big)\psi'(s)ds = -\int_{\alpha}^{\beta} f\big(\varphi(\alpha + \beta - s)\big)\varphi'(\alpha + \beta - s)ds.$$

By returning to the variable $t = \alpha + \beta - s$, we conclude this case with

$$\int_{a}^{b} f(x)dx = -\int_{\beta}^{\alpha} f(\varphi(t))[-\varphi'(t)]dt = \int_{\beta}^{\alpha} f(\varphi(t))\varphi'(t)dt.$$

(b) In this case, there exists a finite sequence $\{\alpha_0 = \alpha < \alpha_1 < \cdots < \alpha_N = \beta\}$ such that φ is monotone on each closed sub-interval $[\alpha_j, \alpha_{j+1}]$, where $j = 0, \ldots, N-1$. Hence, for what we have already proven on (a), we have

$$\sum_{j=0}^{N-1} \int_{\varphi(\alpha_j)}^{\varphi(\alpha_{j+1})} f(x)\,dx = \sum_{j=0}^{N-1} \int_{\alpha_j}^{\alpha_{j+1}} f(\varphi(t))\varphi'(t)\,dt.$$

From which follows, and thus completing this case,

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt.$$

(c) Let $\{\alpha_1, \ldots, \alpha_{N-1}\}$, where $N \geq 2$, be the set of zeros of φ', if there are any. Let us put $\alpha_0 = \alpha$ and $\alpha_N = \beta$. By *The Intermediate Value Theorem for Derivatives (Darboux's Theorem)*, fixing an arbitrary sub-interval (α_j, α_{j+1}), where $j \in \{0, \ldots, N\}$, the derivative φ' assumes only one signal (either strictly positive or strictly negative) along such sub-interval. Hence, along this sub-interval, the map φ is either strictly increasing or strictly decreasing. This shows that the map φ is piecewise monotone. Then, the conclusion is immediate from the case (b).

(d) From the case (c) it follows that

$$\int_{\varphi(\alpha+\epsilon)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha+\epsilon}^{\beta} f(\varphi(t))\varphi'(t)\,dt, \text{ for each } 0 < \epsilon < \beta - \alpha.$$

Let us split the analysis of this identity integral into two sub-cases.

The sub-case $(f \circ \varphi)\varphi'$ integrable (on $[\alpha, \beta]$). From the continuity of an integral regarding the integration endpoints and the continuity of φ, we find

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt.$$

This sub-case is done. Let us go back to the identity integral under analysis.

The sub-case φ' integrable (on $[\alpha, \beta]$). Here we use the concept *measure zero*. Then, the well known *Lebesgue's Characterization Theorem* assures us that the set of discontinuities of $(f \circ \varphi)\varphi'$ in the sub-interval $[\alpha + \epsilon, \beta]$ has measure zero, for each allowed ϵ. Hence, it is not difficult to see, the set of discontinuities of the function $(f \circ \varphi)\varphi'$ in the interval $[\alpha, \beta]$ also has measure zero. By the other hand, since f and φ' are both integrable on their respective domains, we see that f and φ' are both bounded. Thus, by *Lebesgue's Characterization Theorem* it follows that $(f \circ \varphi)\varphi'$ is integrable. This sub-case is done.

The proof of the corollary is complete♣

Examples.

Example 1 (An example for which the theorem proven in this note applies but Kestelman's version does not). Let us consider the pair of functions

$$f(x) = x \text{ for all } x \in [0,1], \quad \text{and} \quad \varphi(t) = \sqrt{t} \text{ for all } t \in [0,1].$$

Evidently f is integrable. Moreover, $\varphi : [0,1] \to [0,1]$ is surjective, increasing, and continuous. Yet, the derivative φ' is defined on the open interval $(0,1)$ and

$$\varphi'(t) = \frac{1}{2\sqrt{t}}.$$

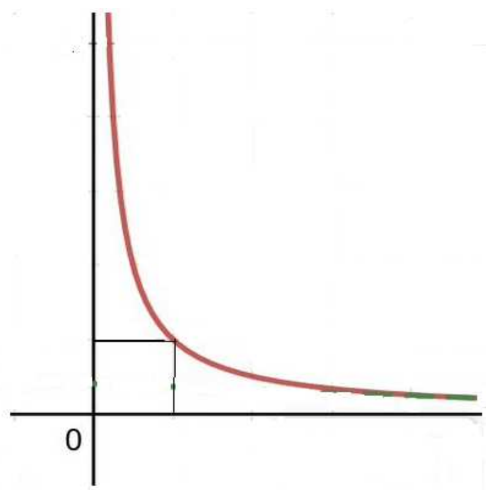


Figura 1: The graph of the function $t \mapsto 1/\sqrt{t}$, where $t \in (0,+\infty)$.

Evidently, φ' is not bounded and thus not integrable on $(0,1)$.

Clearly, the function

$$f(\varphi(t))\varphi'(t) = \frac{\sqrt{t}}{2\sqrt{t}} = \frac{1}{2}$$

is integrable. Hence, from the theorem proven in this note, we obtain

$$\int_0^1 x\,dx = \int_0^1 \frac{1}{2}dt.$$

On the other hand, φ' is not integrable ont $(0,1)$ and thus we cannot write

$$\sqrt{t} = \int_0^t \frac{1}{2\sqrt{\tau}}\,d\tau, \text{ for all } t \in [0,1].$$

Therefore, the "general version" (with $g = \varphi'$) does not apply in this case ♣

Example 2 (An example for which the Corollary (d) proven above applies, but Kestelman's version does not). Let us consider the pair

$$f(x) = x^3, \text{ if } x \in \left[0, \frac{2}{\pi}\right], \qquad \text{and} \qquad \varphi(t) = \begin{cases} 0, & \text{if } t = 0, \\ t\sin\frac{1}{t}, & \text{if } t \in \left(0, \frac{2}{\pi}\right]. \end{cases}$$

The function f is evidently integrable. The map φ is continuous and, as is well known, oscillates near the origin.

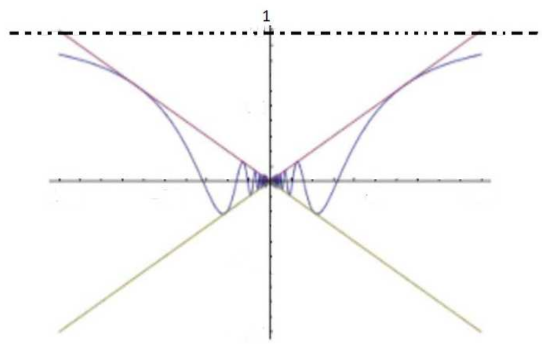


Figura 2: The graph of φ, supposing $t \in (-\infty, +\infty)$

Let us pick ϵ, with $0 < \epsilon < 2/\pi$. Then, in the sub-interval $[0, \epsilon]$, the map φ has an infinite number of points of local maximum, as well as an infinite number of points of minimum local. It then follows that the derivative φ' has an infinite number of zeros in the interval $[0, \epsilon]$ and, it is not difficult to see, the derivative φ' has a finite number of zeros in the interval $[\epsilon, 2/\pi]$.

The derivative φ' is defined on the open interval $(0, 2/\pi)$ and we have

$$\varphi'(t) = \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}.$$

Obviously φ' is unbounded and thus not integrable on $[0, 2/\pi]$.

By the other hand, it is not hard to show the integrability of the function

$$(f \circ \varphi)(t)\varphi'(t) = t^3\left(\sin^3\frac{1}{t}\right)\left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right).$$

By the Corollary, item (d), we find

$$\int_0^{\frac{\pi}{2}} x^3\, dx = \int_0^{\frac{2}{\pi}} [\varphi(t)]^3\varphi'(t)\, dt.$$

From which follows

$$\int_0^{\frac{\pi}{2}} x^3\, dx = \frac{\varphi^4(t)}{4}\Big|_0^{\frac{2}{\pi}} = \frac{\pi^4}{64}.$$

Since φ' is not integrable, we see that Kestelman's version does not apply♣

References.

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