Towards Polynomial Approximations of Full Propositional Logic

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Abstract. The aim of this paper is to study a family of logics that approximates classical inference, in which every step in the approximation can be decided in polynomial time. For clausal logic, this task has been shown to be possible by Dalal [4, 5]. However, Dalal's approach cannot be applied to full classical logic. In this paper we provide a family of logics, called *Limited Bivaluation Logics*, via a semantic approach to approximation that applies to full classical logic. Two approximation families are built on it. One is parametric and can be used in a depth-first approximation of classical logic. The other follows Dalal's spirit, and with a different technique we show that it performs at least as well as Dalal's polynomial approach over clausal logic.

1 Introduction

Logic has been used in several areas of Artificial Intelligence as a tool for modelling an intelligent agent reasoning capabilities. However, the computational costs associated with logical reasoning have always been a limitation. Even if we restrict ourselves to classical propositional logic, deciding whether a set of formulas logically implies a certain formula is a co-NP-complete problem [9].

To address this problem, researchers have proposed several ways of approximating classical reasoning. Cadoli and Schaerf have proposed the use of approximate entailment as a way of reaching at least partial results when solving a problem completely would be too expensive [13]. Their influential method is *parametric*, that is, a set S of atoms is the basis to define a logic. As we add more atoms to S, we get "closer" to classical logic, and eventually, when S contains all propositional symbols, we reach classical logic. In fact, Schaerf and Cadoli proposed two families of logic, intending to approximate classical entailment from two ends. The S_3 family approximates classical logic from below, in the following sense. Let $\mathcal{P}$ be a set of propositions and $\varnothing \subseteq S' \subseteq S'' \subseteq \ldots \subseteq \mathcal{P}$; let $\models^3_S$ indicate the set of the entailment relation of a logic in the family. Then:

$$\models^3_{\varnothing} \subseteq \models^3_{S'} \subseteq \models^3_{S''} \subseteq \ldots \subseteq \models^3_{\mathcal{P}} = \models_{\mathrm{CL}}$$

where CL is classical logic.

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Approximating a classical logic from below is useful for efficient theorem proving. Conversely, approximating classical logic from above is useful for *disproving theorems*, which is the satisfiability (SAT) problem and has a similar formulation. In this work we concentrate only in theorem proving and approximations from below.

The notion of approximation is also related with the notion of an *anytime decision procedure*, that is, an algorithm that, if stopped anytime during the computation, provides an approximate answer, that is, an answer of the form "up to logic L_i in the family, the result is/is not provable". This kind of anytime algorithms have been suggested by the proponents of the Knowledge Compilation approach [14, 15], in which a theory was transformed into a set of polynomially decidable Horn-clause theories. However, the compilation process is itself NP-complete.

Dalal's approximation method [4] was the first one designed such that each reasoner in an approximation family can be decided in polynomial time. Dalal's initial approach was algebraic only. A model-theoretic semantics was provided in [5]. However, this approach was restricted to clausal form logic only.

In this work, we generalize Dalal's approach. We create a family of logics of *Limited Bivalence (*LB*)* that approximates full propositional logic. We provide a model-theoretic semantics and two entailment relations based on it. The entailment $\models^{\mathsf{LB}}_{\Sigma}$ is a parametric approximation on the set of formulas Σ and follows Cadoli and Schaerf's approximation paradigm. The entailment $\models^{\mathsf{LB}}_{k}$ follows Dalal's approach and we show that for clausal form theories, the inference $\models^{\mathsf{LB}}_{k}$ is polynomially decidable and serves as a semantics for Dalal's inference $\vdash^{\text{BCP}}_{k}$.

This family of approximations is useful in defining families of efficiently decidable formulas with increasing complexity. In this way, we can define the set $\Gamma_k = \{\alpha | \models^{\mathsf{LB}}_{k} \alpha \text{ and } k\}$ of tractable theorems, such that $\Gamma_k \subseteq \Gamma_{k+1}$.

This paper proceeds as follows. Next section briefly presents Dalal's approximation strategy, its semantics and discuss its limitations. In Section 3 we present the family $\mathsf{LB}(\Sigma)$ of Limited Bivaluation Logics; the semantics for full propositional $\mathsf{LB}(\Sigma)$ is provided in Section 3.1; a parametric entailment $\models^{\mathsf{LB}}_{\Sigma}$ is presented in Section 3.2. The entailment $\models^{\mathsf{LB}}_{k}$ is presented in Section 3.4 and the soundness and completeness of Dalal's $\vdash^{\text{BCP}}_{k}$ with respect to $\models^{\mathsf{LB}}_{k}$ is Shown in Sections 3.3 and 3.4.

Notation: Let $\mathcal{P}$ be a countable set of propositional letters. We concentrate on the classical propositional language $\mathcal{L}_C$ formed by the usual boolean connectives $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation).

Throughout the paper, we use lowercase Latin letters to denote propositional letters, α, β, γ denote formulas, φ, ψ denote clauses and λ denote a literal. Uppercase Greek letters denote sets of formulas. By $\text{atoms}(\alpha)$ we mean the set of all propositional letters in the formula α; if Σ is a set of formulas, $\text{atoms}(\Sigma) = \bigcup_{\alpha \in \Sigma} \text{atoms}(\alpha)$.

Due to space limitations, some proofs of lemmas have been omitted.

2 Dalal's Polynomial Approximation Strategy

Dalal specifies a family of anytime reasoners based on an equivalence relation between formulas [4]. The family is composed of a sequence of reasoners $\vdash_0, \vdash_1, \ldots$, such that each $\vdash_i$ is tractable, each $\vdash_{i+1}$ is at least as complete (with respect to classical logic) as $\vdash_i$, and for each theory there is a complete $\vdash_i$ to reason with it.

The equivalence relation that serves as a basis for the construction of a family has to obey several restrictions to be *admissible*, namely it has to be *sound*, *modular*, *independent*, *irredundand* and *simplifying* [4].

Dalal provides as an example a family of reasoners based on the classically sound but incomplete inference rule known as BCP (Boolean Constraint Propagation) [12], which is a variant of unit resolution [3]. For the initial presentation, no proof-theoretic or model-theoretic semantics were provided for BCP, but an algebraic presentation of an equivalence relation $=_{\text{BCP}}$ was provided. For that, consider a theory as a set of clauses, where a disjunction of zero literals is denoted by $\mathbf{f}$ and the conjunction of zero clauses is denoted $\mathbf{t}$. Let $\neg p$ denote the negation of the atom p, and let $\sim\psi$ be the *complement* of the formula ψ obtained by pushing the negation inside in the usual way using De Morgan's Laws until the atoms are reached, at which point $\sim p = \neg p$ and $\sim\neg p = p$.

The equivalence relation $=_{\text{BCP}}$ is then defined as:

$$\begin{aligned}\{\mathbf{f}\} \cup \Gamma &=_{\text{BCP}} \{\mathbf{f}\} \\ \{\lambda, \sim\lambda \vee \lambda_1 \vee \ldots \vee \lambda_n\} \cup \Gamma &=_{\text{BCP}} \{\lambda, \lambda_1 \vee \ldots \vee \lambda_n\} \cup \Gamma\end{aligned}$$

where λ, λ_i are literals.

The idea is to use an equivalence relation to generate an inference in which ψ can be inferred from Γ if $\Gamma \cup \{\sim\psi\}$ is equivalent to an inconsistency. In this way, the inference $\vdash_{\text{BCP}}$ is defined as $\Gamma \vdash_{\text{BCP}} \psi$ iff $\Gamma \cup \{\sim\psi\} =_{\text{BCP}} \{\mathbf{f}\}$.

Dalal presents an example[1] in which, for the theory $\Gamma_0 = \{p \vee q, p \vee \neg q, \neg p \vee s \vee t, \neg p \vee s \vee \neg t\}$ we both have $\Gamma_0 \vdash_{\text{BCP}} p$ and $\Gamma_0, p \vdash_{\text{BCP}} s$ but $\Gamma_0 \not\vdash_{\text{BCP}} s$.

This example shows that $\vdash_{\text{BCP}}$ is unable to use a previously inferred clause p to infer s. Based on this fact comes the proposal of an anytime family of reasoners.

2.1 The Family of Reasoners

Dalal defines a family of incomplete reasoners $\vdash_0^{\text{BCP}}, \vdash_1^{\text{BCP}}, \ldots$, where each $\vdash_k^{\text{BCP}}$ is given by the following:

$$1.\ \frac{\Gamma \vdash_{\text{BCP}} \varphi}{\Gamma \vdash_k^{\text{BCP}} \varphi} \qquad 2.\ \frac{\Gamma \vdash_k^{\text{BCP}} \psi \qquad \Gamma, \psi \vdash_k^{\text{BCP}} \varphi}{\Gamma \vdash_k^{\text{BCP}} \varphi} \text{ for } |\psi| \leq k$$

where the size of a clause ψ, $|\psi|$ is the number of literals it contains.

[1] This example is extracted from [5].

The first rule tells us that every $\vdash_{\text{BCP}}$-inference is also a $\vdash_k^{\text{BCP}}$-inference. The second rule tells us that if ψ was inferred from a theory and it can be used as a further hypothesis to infer φ, and the size of ψ is at most k, then φ is can also be inferred from the theory.

Dalal shows that this is indeed an anytime family of reasoners, that is, for each k, $\vdash_k^{\text{BCP}}$ is tractable, $\vdash_{k+1}^{\text{BCP}}$ is as complete as $\vdash_k^{\text{BCP}}$ and if you remove the restriction on the size of ψ in rule 2, then $\vdash_k^{\text{BCP}}$ becomes complete, that is, for each classically inferable $\Gamma \vdash \varphi$ there is a k such that $\Gamma \vdash_k^{\text{BCP}} \varphi$.

2.2 Semantics

In [5], Dalal proposed a semantics for $\vdash_k^{\text{BCP}}$ based on the notion of *k-valuations*, which we briefly present here.

Dalal's semantics is defined for sets of clauses. Given a clause ψ, the *support set* of ψ, $S(\psi)$ is defined as the set of all literals occurring in ψ. Support sets ignore multiple occurrences of the same literal and are used to extend valuations from atoms to clauses. According to Dalal's semantics, a propositional valuation is a function $v : \mathcal{P} \rightarrow [0, 1]$; note that the valuation maps atoms to real numbers. A valuation is then extended to literals and clauses in the following way:

1. $v(\neg p) = 1 - v(p)$ for any atom $p \in \mathcal{P}$.
2. $v(\psi) = \sum_{\lambda \in S(\psi)} v(\lambda)$, for any clause ψ.

Valuations of literals are real numbers in $[0, 1]$, but valuations of clauses are non-negative real numbers that can exceed 1. A valuation v is a *model* of ψ if $v(\psi) \geq 1$. A valuation is a countermodel of ψ if $v(\psi) = 0$. Therefore it is possible for a formula to have neither a model nor a countermodel. For instance, if $v(p) = v(q) = 0.2$, then $p \vee q$ has neither a model nor a countermodel. A valuation is a model of a theory (set of clauses) if it is a model of all clauses in it.

Define $\Gamma \models\!\!\approx \psi$ iff no model of the theory Γ is a countermodel of ψ.

Proposition 1 ([5]). *For every theory Γ and every clause ψ, $\Gamma \vdash_{\text{BCP}} \psi$ iff $\Gamma \models\!\!\approx \psi$.*

So $\vdash_{\text{BCP}}$ is sound and complete with respect to $\models\!\!\approx$. The next step is to generalize this approach to obtain a semantics of $\vdash_k^{\text{BCP}}$. For that, for any $k \geq 0$, a set V of valuations is a *k-valuation* iff for each clause ψ of size at most k, if V has a non-model of ψ then V has a countermodel of ψ. V is a *k-model* of ψ if each $v \in V$ is a model of ψ; this notion extends to theories as usual.

It is then possible to define $\Gamma \models\!\!\approx_k \psi$ iff there is no countermodel of ψ in any k-model of Γ.

Proposition 2 ([5]). *For every theory Γ and every clause ψ, $\Gamma \vdash_k^{\text{BCP}} \psi$ iff $\Gamma \models\!\!\approx_k \psi$.*

Thus the inference $\vdash_k^{\text{BCP}}$ is sound and complete with respect to $\models\!\!\approx_k$.

2.3 Analysis of $\vdash_k^{\text{BCP}}$

Dalal's notion of a family of anytime reasoners has very nice properties. First, every step in the approximation is sound and can be decided in polynomial time. Second, the approximation is guaranteed to converge to classical inference. Third, every step in the approximation has a sound and complete semantics, enabling an anytime approximation process.

However, the method based on $\vdash_k^{\text{BCP}}$-approximations also has its limitations:

1. It only applies to clausal form formulas. Although every propositional formula is classically equivalent to a set of clauses, this equivalence may not be preserved in any of the approximation steps. The conversion of a formula to clausal form is costly: one either has to add new propositional letters (increasing the complexity of the problem) or the number of clauses can be exponential in the size of the original formula. With regards to complexity, BCP is a form of resolution, and it is known that there are theorems that can be proven by resolution only in exponentially many steps [2].
2. Its non-standard semantics makes it hard to compare with other logics known in the literature, specially other approaches to approximation. Also, the semantics presented is based on support sets, which makes it impossible to generalize to non-clausal formulas.
3. The proof-theory for $\vdash_k^{\text{BCP}}$ is poor in computational terms. In fact, if we are trying to prove that $\Gamma \vdash_k^{\text{BCP}} \varphi$, and we have shown that $\Gamma \nvdash_{\text{BCP}} \varphi$, then we would have to guess a ψ with $|\psi| \leq k$, so that $\Gamma \vdash_k^{\text{BCP}} \psi$ and $\Gamma, \psi \vdash_k^{\text{BCP}} \varphi$. Since the BCP-approximations provide no method to guess the formula ψ, this means that a computation would have to generate and test all the $O((2n)^k)$ possible clauses, where n is the number of propositional symbols occurring in Γ and φ.

In the rest of this paper, we address problems 1 and 2 above. That is, we are going to present a family of anytime reasoners for the full fragment of propositional logic, in which every approximation step has a semantics and can be decided in polynomial time. Problem 3 will be treated in further work.

3 The Family of Logics $\mathsf{LB}(\Sigma)$

We present here the family of logics of *Limited Bivalence*, $\mathsf{LB}(\Sigma)$. This is a parametric family that approximates classical logic, in which every approximation step can be decided in polynomial time. Unlike $\vdash_k^{\text{BCP}}$, $\mathsf{LB}(\Sigma)$ is parameterized by a set of formulas Σ; when Σ contains all formulas of size at most k, $\mathsf{LB}(\Sigma)$ can simulate an approximation step of $\vdash_k^{\text{BCP}}$.

The family $\mathsf{LB}(\Sigma)$ can be applied to the full language of propositional logic, and not only to clausal form formulas, with an alphabet consisting of a countable set of propositional letters (atoms) $\mathcal{P} = \{p_0, p_1, \ldots\}$, and the connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$, and the usual definition of well-formed propositional formulas; the set of all well-formed formulas is denoted by $\mathcal{L}$. The presentation of LB is made in terms of a model theoretic semantics.

3.1 Semantics of $\mathsf{LB}(\Sigma)$

The semantics of $\mathsf{LB}(\Sigma)$ is based of a three-level lattice, $(L, \sqcap, \sqcup, 0, 1)$, where L is a countable set of elements $L = \{0, 1, \varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots\}$, $\sqcup$ is the least upper bound, $\sqcap$ is the gratest lower bound, and $\sqsubseteq$ is defined, as usual, as $a \sqsubseteq b$ iff $a \sqcup b = b$ iff $a \sqcap b = a$; 1 is the $\sqsubseteq$-top and 0 is the $\sqsubseteq$-bottom. L is subject to the conditions: (i) $0 \sqsubseteq \varepsilon_i \sqsubseteq 1$, for every $i < \omega$; and (ii) $\varepsilon_i \not\sqsubseteq \varepsilon_j$ for $i \neq j$. This three-level lattice is illustrated in Figure 3.1(a).

(a) The 3-Level Lattice (b) The Converse Operation $\sim$

This lattice is enhanced with a *converse operation*, $\sim$, defined as: $\sim 0 = 1$, $\sim 1 = 0$ and $\sim \varepsilon_i = \varepsilon_i$ for all $i < \omega$. This is illustrated in Figure 3.1(b).

We next define the notion of an *unlimited valuation*, and then we present its limitations. An unlimited propositional valuation is a function $v_\Sigma : \mathcal{P} \to L$ that maps atoms to elements of the lattice. We extend v_Σ to all propositional formulas, $v_\Sigma : \mathcal{L} \to L$, in the following way:

$$
\begin{aligned}
&v_\Sigma(\neg\alpha) = \sim v_\Sigma(\alpha) \\
&v_\Sigma(\alpha \wedge \beta) = v_\Sigma(\alpha) \sqcap v_\Sigma(\beta) \\
&v_\Sigma(\alpha \vee \beta) = v_\Sigma(\alpha) \sqcup v_\Sigma(\beta) \\
&v_\Sigma(\alpha \to \beta) = (\sim v_\Sigma(\alpha)) \sqcup v_\Sigma(\beta)
\end{aligned}
$$

A formula can be mapped to any element of the lattice. However, the formulas that belong to the set Σ are bivalent, that is, they can only be mapped to the top or the bottom element of the lattice. Therefore, a *limited valuation* must satisfy the restriction of Limited Bivalence given by, for every $\alpha \in \mathcal{L}$:

$$v_\Sigma(\alpha) = 0 \text{ or } v_\Sigma(\alpha) = 1, \text{ if } \alpha \in \Sigma.$$

In the rest of this work, by a valuation v_Σ we mean a limited valuation subject to the condition above.

A valuation v_Σ *satisfies* α if $v_\Sigma(\alpha) = 1$, and α is said *satisfiable*; a set of formulas Γ is satisfied by v_Σ if all its formulas are satisfied by v_Σ. A valuation v_Σ *contradicts* α if $v_\Sigma(\alpha) = 0$; if α is neither satisfied nor contradicted by v_Σ, we say that v_Σ is *neutral* with respect to α. A valuation is *classical* if it assigns only 0 or 1 to all proposition symbols, and hence to all formulas.

For example, consider the formula $p \to q$, and $\Sigma = \varnothing$. Then

– if $v_\Sigma(p) = 1$, then $v_\Sigma(p \to q) = v_\Sigma(q)$;

– if $v_\Sigma(p) = 0$, then $v_\Sigma(p \to q) = 1$;
– if $v_\Sigma(q) = 0$, then $v_\Sigma(p \to q) = v_\Sigma(p)$;
– if $v_\Sigma(q) = 1$, then $v_\Sigma(p \to q) = 1$;
– if $v_\Sigma(p) = \varepsilon_p$ and $v_\Sigma(q) = \varepsilon_q$, then $v_\Sigma(p \to q) = 1$;

The first four valuations coincide with a classical behavior. The last one shows that if p and q are mapped to distinct neutral values, then $p \to q$ will be satisfiable. Note that, in this case, $p \vee q$ will also be satisfiable, and that $p \wedge q$ will be contradicted.

3.2 LB-Entailment

The notion of a parameterized LB-Entailment, $\models^{\mathsf{LB}}_{\Sigma}$ follows the spirit of Dalal's entailment relation, namely $\Gamma \models^{\mathsf{LB}}_{\Sigma} \alpha$ if it is not possible to satisfy Γ and contradict α at the same time. More specifically, $\Gamma \models^{\mathsf{LB}}_{\Sigma} \alpha$ if no valuation v_Σ such that $v_\Sigma(\Gamma) = 1$ also makes $v_\Sigma(\alpha) = 0$. Note that since this logic is not classic, if $\Gamma \models^{\mathsf{LB}}_{\Sigma} \alpha$ and $v_\Sigma(\Gamma) = 1$ it is possible that α is either neutral or satisfied by v_Σ.

For example, we reconsider Dalal's example, where $\Gamma_0 = \{p \vee q, p \vee \neg q, \neg p \vee s \vee t, \neg p \vee s \vee \neg t\}$ and make $\Sigma = \varnothing$. We want to show that $\Gamma_0 \models^{\mathsf{LB}}_{\Sigma} p$, $\Gamma_0, p \models^{\mathsf{LB}}_{\Sigma} s$ but $\Gamma_0 \not\models^{\mathsf{LB}}_{\Sigma} s$.

To see that $\Gamma_0 \models^{\mathsf{LB}}_{\Sigma} p$, suppose there is a v_Σ such that $v_\Sigma(p) = 0$. Then we have $v_\Sigma(p \vee q) = v_\Sigma(q)$ and $v_\Sigma(p \vee \neg q) =\sim v_\Sigma(q)$. Since it is not possible to satisfy both, we cannot have $v_\Sigma(\Gamma_0) = 1$, so $\Gamma_0 \models^{\mathsf{LB}}_{\Sigma} p$.

To show that $\Gamma_0, p \models^{\mathsf{LB}}_{\Sigma} s$, suppose there is a v_Σ such that $v_\Sigma(s) = 0$ and $v_\Sigma(p) = 1$. Then $v_\Sigma(\neg p \vee s \vee t) = v_\Sigma(t)$ and $v_\Sigma(\neg p \vee s \vee \neg t) =\sim v_\Sigma(t)$. Again, it is not possible to satisfy both, so $\Gamma_0, p \models^{\mathsf{LB}}_{\Sigma} s$.

Finally, to see that $\Gamma_0 \not\models^{\mathsf{LB}}_{\Sigma} s$, take a valuation v_Σ such that $v_\Sigma(s) = 0, v_\Sigma(p) = \varepsilon_p, v_\Sigma(q) = \varepsilon_q, v_\Sigma(t) = \varepsilon_t$. Then $v_\Sigma(\Gamma_0) = 1$. However, if we make $\Sigma = \{p\}$ then we have only two possibilities for $v_\Sigma(p)$. If $v_\Sigma(p) = 1$, we have already seen that no valuation that contradicts s will satisfy Γ_0. If $v_\Sigma(p) = 0$, we have also seen that no valuation that contradicts s will satisfy Γ_0. So for $p \in \Sigma$, we obtain $\Gamma_0 \models^{\mathsf{LB}}_{\Sigma} s$.

This example indicates that $\models^{\mathsf{LB}}_{\varnothing}$ behave in a similar way to $\vdash_{\mathrm{BCP}}$, and that by adding an atom to Σ we have a behavior similar to $\vdash^{\mathrm{BCP}}_{1}$. We now have to demonstrate that this is not a mere coincidence.

An Approximation Process. As defined in [8], a family of logics, parameterized with a set, Σ, is said to be an approximation of classical logic "from below" if, for increasing size of the parameter set Σ we get closer to classical logic. That is, for $\varnothing \subseteq \Sigma' \subseteq \Sigma'' \subseteq \ldots \subseteq \mathcal{L}$ we have that,

$$\models^{\mathsf{LB}}_{\varnothing} \subseteq \models^{\mathsf{LB}}_{\Sigma'} \subseteq \models^{\mathsf{LB}}_{\Sigma''} \subseteq \ldots \subseteq \models^{\mathsf{LB}}_{\mathcal{L}} = \models_{\mathrm{CL}}$$

Lemma 1. *The family of logics $\mathsf{LB}(\Sigma)$ is an approximation of classical logic from below.*

Note that for a given pair (Γ, α) the approximation of $\Gamma \models \alpha$ can be done in a finite number of steps. In fact, if $\beta, \gamma \in \Sigma$ any formula made up of β and γ has the property of bivalence. In particular, if all atoms of Γ and α are in Σ, then only classical valuations are allowed.

An approximation method as above is not in the spirit of Dalal's approximation, but follows the paradigm of Cadoli and Schaerf [13, 1], also applied by Massacci [11, 10] and Finger and Wassermann [6, 7, 8].

We now show how Dalal's approximations can be obtained using LB.

3.3 Soundness and Completeness of $\vdash_{\text{BCP}}$ with respect to $\models^{\mathsf{LB}}_{\varnothing}$

For the sake of this section and the following, let Γ be a set of clauses and let ψ and φ denote clauses, and $\lambda, \lambda_1, \lambda_2, \ldots$ denote literals. We now show that, for $\Sigma = \varnothing$, $\Gamma \vdash_{\text{BCP}} \psi$ iff $\Gamma \models^{\mathsf{LB}}_{\varnothing} \psi$.

Lemma 2. *Suppose BCP transforms a set of clauses Γ into a set of clauses Δ, then $v_{\Sigma}(\Gamma) = 1$ iff $v_{\Sigma}(\Delta) = 1$.*

Lemma 3. *$\Gamma =_{\text{BCP}} \{\mathbf{f}\}$ iff for all valuations $v_{\varnothing}$, $v_{\varnothing}(\Gamma) \neq 1$.*

Theorem 1. *Let Γ be a set of clauses and ψ a clause. Then $\Gamma \vdash_{\text{BCP}} \psi$ iff $\Gamma \models^{\mathsf{LB}}_{\varnothing} \psi$.*

Proof. $\Gamma \models^{\mathsf{LB}}_{\varnothing} \psi$ iff for no $v_{\varnothing}$, $v_{\varnothing}(\Gamma) = 1$ and $v_{\varnothing}(\psi) = 0$ iff for no $v_{\varnothing}$, $v_{\varnothing}(\Gamma \cup \neg\psi) = 1$ iff, by Lemma 3, $\Gamma \cup \neg\psi =_{\text{BCP}} \{\mathbf{f}\}$ iff $\Gamma \vdash_{\text{BCP}} \psi$.

Lemma 4 (Deduction Theorem for $\vdash_{\text{BCP}}$). *Let Γ be a set of clauses, λ a literal and ψ a clause. Then the following are equivalent statements:*

$$\text{(a) } \Gamma, \lambda \vdash_{\text{BCP}} \psi; \quad \text{(b) } \Gamma \vdash_{\text{BCP}} \neg\lambda \vee \psi; \quad \text{(c) } \Gamma \vdash_{\text{BCP}} \lambda \to \psi.$$

3.4 Soundness and Completeness of $\vdash^{\text{BCP}}_{k}$

As mentioned before, the family of entailment relations $\models^{\mathsf{LB}}_{\Sigma}$ does not follow Dalal's approach to approximation, so in order to obtain a sound and complete semantics for $\vdash^{\text{BCP}}_{k}$ we need to provide another entailment relation based on $\models^{\mathsf{LB}}_{\Sigma}$, which we call $\models^{\mathsf{LB}}_{\mathbb{S}}$.

For that, let $\mathbb{S}$ be a set of sets of formulas and define $\Gamma \models^{\mathsf{LB}}_{\mathbb{S}} \psi$ iff there exists a set $\Sigma \in \mathbb{S}$ such that $\Gamma \models^{\mathsf{LB}}_{\Sigma} \psi$. We concentrate on the case where Γ is a set of clauses, ψ is a clause and each $\Sigma \in \mathbb{S}$ is a set of atoms. We define $\mathbb{S}_k = \{\Sigma \subseteq \mathcal{P} | \ |\Sigma| = k\}$.

That is, $\mathbb{S}_k$ is a set of sets of atoms of size k. Note that if we restrict our attention to n atoms, $|\mathbb{S}_k| = \binom{n}{k} = O(n^k)$ sets of k atoms. For a fixed k, we only have to consider a polynomial number of sets of k atoms.

We then write $\models^{\mathsf{LB}}_{k}$ to mean $\models^{\mathsf{LB}}_{\mathbb{S}_k}$.

Theorem 2. *Let Γ be a set of clauses and ψ a clause. Then $\Gamma \vdash^{\text{BCP}}_{k} \psi$ iff $\Gamma \models^{\mathsf{LB}}_{k} \psi$.*

Proof. ($\Rightarrow$) By induction on the number of uses of rule 2 in the definition of $\vdash_k^{\mathrm{BCP}}$. For the base case, Theorem 1 gives us the result. Assume that $\Gamma \vdash_k^{\mathrm{BCP}} \psi$ due to $\Gamma \vdash_k^{\mathrm{BCP}} \varphi$ and $\Gamma, \varphi \vdash_k^{\mathrm{BCP}} \varphi$. Suppose for contradiction that $\Gamma \not\models_k^{\mathsf{LB}} \psi$, then for all $\Sigma \subseteq \mathcal{P}$, $|\Sigma| \leq k$, there exists v_Σ such that $v_\Sigma(\Gamma) = 1$ and $v_\Sigma(\psi) = 0$. By the induction hypothesis, $\Gamma \models_k^{\mathsf{LB}} \varphi$, which implies $v_\Sigma(\varphi) \neq 0$, and $\Gamma, \varphi \models_k^{\mathsf{LB}} \varphi$, which implies $v_\Sigma(\varphi) \neq 1$. So $v_\sigma(\varphi) = \varepsilon_i$, for some $i < \omega$, which implies that $\mathrm{atoms}(\varphi) \cap \Sigma = \varnothing$, but this cannot hold for all Σ, a contradiction. So $\Gamma \models_k^{\mathsf{LB}} \psi$.

($\Leftarrow$) Suppose $\Gamma \models_k^{\mathsf{LB}} \psi$. Then for some Σ with $|\Sigma| \leq k$, $\Gamma \models_\Sigma^{\mathsf{LB}} \psi$ and suppose that Σ is a smallest set with such property. Therefore, for all with v_Σ with $v_\Sigma(\Gamma) = 1$ we have $v_\Sigma(\psi) \neq 0$. Choose one such v_Σ and define the set of literals $\Lambda = \{\lambda \text{ is a literal whose atom is in } \Sigma | v_\Sigma(\lambda) = 1\}$.

We first show that $\Gamma \models_\Sigma^{\mathsf{LB}} \lambda$ for every $\lambda \in \Lambda$. Suppose for contradiction that for some $\lambda \in \Lambda$, $\Gamma \not\models_\Sigma^{\mathsf{LB}} \lambda$, then there is a v'_Σ with $v'_\Sigma(\Gamma) = 1$ and $v'_\Sigma(\psi) \neq 0$ but $v'_\Sigma(\lambda) = 0$. Let $\mathrm{atoms}(\lambda) = \{p\}$. If p does not occur in ψ, then $\Gamma \models_{\Sigma-\{p\}}^{\mathsf{LB}} \psi$, which contradicts the minimality of Σ. So $\psi = p \vee \chi'$ or $\psi = \neg p \vee \chi''$. Consider a $v_{\Sigma-\{p\}}$ such that $v_{\Sigma-\{p\}}(\Gamma) = 1$; if $v_{\Sigma-\{p\}}$ maps p to 0 or 1 it is a v_Σ, so $v_{\Sigma-\{p\}}(\psi) \neq 0$; if $v_{\Sigma-\{p\}}(p) = \varepsilon_i$ for some i, then clearly we have that $v_{\Sigma-\{p\}}(\psi) \neq 0$, so $\Gamma \models_{\Sigma-\{p\}}^{\mathsf{LB}} \psi$, which contradicts the minimality of Σ. It follows that $\Gamma \models_\Sigma^{\mathsf{LB}} \Lambda$.

We now show that $\Gamma \cup \Lambda \vdash_{\mathrm{BCP}} \psi$. Suppose for contradiction that $\Gamma \cup \Lambda \nvdash_{\mathrm{BCP}} \psi$. Then, by Theorem 1, $\Gamma \cup \Lambda \not\models_\varnothing^{\mathsf{LB}} \psi$, that is, there exists $v_\varnothing$ such that $v_\varnothing(\Gamma \cup \Lambda) = 1$ and $v_\varnothing(\psi) = 0$. However, such $v_\varnothing$ maps all atoms of Σ to 0 or 1, so it is actually a v_Σ that contradicts $\Gamma \models_\Sigma^{\mathsf{LB}} \psi$. So $\Gamma \cup \Lambda \vdash_{\mathrm{BCP}} \psi$.

If $\Gamma \vdash_{\mathrm{BCP}} \psi$ then clearly $\Gamma \vdash_k^{\mathrm{BCP}} \psi$. So suppose $\Gamma \nvdash_{\mathrm{BCP}} \psi$. In this case, we show that $\Gamma \vdash_k^{\mathrm{BCP}} \bigwedge \Lambda$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$, we prove by induction that for $1 \leq i \leq m, \Gamma, \lambda_1, \ldots, \lambda_{i-1} \vdash_{\mathrm{BCP}} \lambda_i$. From $\Gamma \nvdash_{\mathrm{BCP}} \psi$ and Theorem 1 we know that there is a valuation $v_\varnothing$ such that $v_\varnothing(\Gamma) = 1$ and $v_\varnothing(\psi) = 0$. From $\Gamma \cup \Lambda \vdash_{\mathrm{BCP}} \psi$ we infer that there must exist a $\lambda \in \Lambda$ such that $v_\varnothing(\lambda) \neq 1$; without loss of generality, let $\lambda = \lambda_m$. Suppose for contradiction that $\Gamma, \lambda_1, \ldots, \lambda_{m-1} \nvdash_{\mathrm{BCP}} \lambda_m$. Then there exists a valuation $v'_\varnothing$ such that $v'_\varnothing(\Gamma, \lambda_1, \ldots, \lambda_{m-1}) = 1$ but $v'_\varnothing(\lambda_m) = 0$, which contradicts $\Gamma \models_\Sigma^{\mathsf{LB}} \Lambda$. So $\Gamma, \lambda_1, \ldots, \lambda_{m-1} \vdash_{\mathrm{BCP}} \lambda_m$.

Now note that for $1 < i \leq m$, $\Gamma, \lambda_i, \ldots, \lambda_m \nvdash_{\mathrm{BCP}} \psi$, otherwise the minimality of Σ would be violated. From Theorem 1 we know that there is a valuation $v_\varnothing$ such that $v_\varnothing(\Gamma, \lambda_i, \ldots, \lambda_m) = 1$ and $v_\varnothing(\psi) = 0$. From $\Gamma \cup \Lambda \vdash_{\mathrm{BCP}} \psi$ we infer that there must exist a $\lambda \in \{\lambda_1, \ldots, \lambda_{i-1}\}$ such that $v_\varnothing(\lambda) \neq 1$; without loss of generality, let $\lambda = \lambda_{i-1}$. Suppose for contradiction that $\Gamma, \lambda_1, \ldots, \lambda_{i-2} \nvdash_{\mathrm{BCP}} \lambda_{i-1}$. Then there exists a valuation $v'_\varnothing$ such that $v'_\varnothing(\Gamma, \lambda_1, \ldots, \lambda_{i-2}) = 1$ but $v'_\varnothing(\lambda_{i-1}) = 0$, but this contradicts $\Gamma \models_\Sigma^{\mathsf{LB}} \Lambda$. So $\Gamma, \lambda_1, \ldots, \lambda_{i-2} \vdash_{\mathrm{BCP}} \lambda_{i-1}$.

Thus we have that $\Gamma \vdash_{\mathrm{BCP}} \lambda_1; \Gamma, \lambda_1 \vdash_{\mathrm{BCP}} \lambda_2; \ldots; \Gamma, \lambda_1, \ldots, \lambda_{m-1} \vdash_{\mathrm{BCP}} \lambda_m$. It follows that $\Gamma \vdash_k^{\mathrm{BCP}} \bigwedge \Lambda$ as desired. Finally, from $\Gamma \cup \Lambda \vdash_{\mathrm{BCP}} \psi$ and $\Gamma \vdash_k^{\mathrm{BCP}} \bigwedge \Lambda$ we obtain that $\Gamma \vdash_k^{\mathrm{BCP}} \psi$, and the result is proved.

The technique above differs considerably from Dalal's use of the notion of *vividness*. It follows from Dalal's result that each approximation step $\models_k^{\mathsf{LB}}$ is decidable in polynomial time.

4 Conclusions and Future Work

In this paper we presented the family of logics $\mathsf{LB}(\Sigma)$ and provided it with a lattice-based semantics. We showed that it can be a basis for both a parametric and a polynomial clausal approximation of classical logic. This semantics is sound and complete with respect to Dalal's polynomial approximations $\vdash_k^{\text{BCP}}$.

Future work should extend polynomial approximations to non-clausal logics. It should also provide a proof-theory for these approximations.

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