Approximations of Modal Logics: K and Beyond

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Abstract

Inspired by recent work on approximations of classical logic, we present a method that approximates several modal logics in a modular way. Our starting point is the limitation of the n-degree of introspection that is allowed, thus generating modal n-logics. Semantics for n-logics is presented, in which formulas are evaluated with respect to paths, and not possible worlds. A tableau-based proof system is presented, n-SST, and soundness and completeness is shown for the approximation of modal logics **K**, **T**, **D**, **S4** and **S5**.

Key words: Modal Logic, Approximated Inference, Single Step Tableaux

1 Introduction

George Boole defined logic as the science of "the fundamental laws of those operations of the mind by which reasoning is performed" [1]. Much of the later development emphasized the relations between logic and foundations of mathematics, leaving aside its use as a model for a thinking agent. With the development of Artificial Intelligence, however, the use of logic as the basis of the formalization of problems of Knowledge Representation has been shown, in many cases, to be successful, returning in a certain way to Boole's ideal.

Ideal agents know all the consequences of their beliefs. However, real agents are limited in their capabilities. The works of Cadoli and Schaerf [15], and Finger and Wassermann [5,6] developed a method for modelling limited and approximated reasoning for classical propositional logic. Their work considers a set of atoms — which is called *relevant propositions* — permitting a non-classical behavior of the negation operator on formulas containing atoms outside this set; this non-classical behaviour was later extended to any classical connective [7]. The set of relevant propositions is a *parameter* in the construction of a family of logics that "approximate" classical logic. Each logic in the family proves a subset of classical theorems, which increases monotonically with the set of relevant propositions; the approximation process consists of extending the parameter set when a certain approximated logic cannot prove a specific theorem, moving to a more powerful logic. This approximation process can be transformed into an anytime algorithm for theorem proving, as noted by Massacci [13].

In this paper, we concentrate on studying the approximation process for modal logics. There are at least two sources of complexity in modal logics: *logical omniscience* and *unbounded logical introspection* [1]. Logical omniscience means that the thinking agent knows all the consequences of his beliefs. The work of Finger and Wassermann, among others, give a method to control this feature in classical logic, and this method is easily translated to modal logic. This paper is concerned with the other source of complexity. Unbounded logical introspection allows an agent to reason about his beliefs of his beliefs ... of his beliefs. Limitation is accomplished by selecting a *maximum limit of introspection*, inside which the agent can reason classically about its beliefs; outside, the behavior is non-normal. By selecting arbitrarily large limits of introspection, we can recover classical normal modal logic. In this case, the parameter of approximation is this number, the maximum limit of introspection. It is this kind of limitation that we study in this paper.

In fact, we define a modal logic of limited introspection, called n-logic, where n is the maximum limit of introspection. The logic is defined semantically and, unlike usual presentations of modal logics, the semantics is evaluated with respect to a *path* in a frame (W, R), not a single *possible world*. As is usual in modal logics, by imposing several properties to the class of frames we obtain several approximated modal n-logics, such as **K**, **T**, **D**, **S4** and **S5**. Semantically, we define a relation $\models^n_{\mathbf{L}}$, where $\mathbf{L} \in \{\mathbf{K}, \mathbf{T}, \mathbf{D}, \mathbf{S4}, \mathbf{S5}\}$. The idea of approximation is expressed by the fact that:

$$\models^0_{\mathbf{L}} \;\subseteq\; \models^1_{\mathbf{L}} \;\subseteq\; \dots \;\subseteq\; \models^n_{\mathbf{L}} \;\subseteq\; \models^{n+1}_{\mathbf{L}} \;\subseteq\; \dots$$

As the number of introspection steps is finite for every formula, for every classically **L**-valid ψ, $\models_{\mathbf{L}} \psi$, there exists an n such that ψ is n-valid, $\models^n_{\mathbf{L}} \psi$.

On the proof theoretical side, we present a tableau-based proof theory in the context of Massacci's Single Step Tableaux (SST), giving rise to an approximation proof system which we call n-SST, represented by $\vdash^n_{\mathbf{L}}$. Due to the evaluation of formulas over paths, not on worlds, the proof theory has to be adapted, so n-SST's rules contain some fundamental modifications over those of of SST. Soundness and completeness of this approach is then shown for

[1] We observe that the use of the word "introspection" is usually restricted to the context of epistemic logic. Following [13], we extend its use to any modal logic.

several logics, ie, we show that $\models^n_{\mathbf{L}} \psi$ iff $\vdash^n_{\mathbf{L}} \psi$.

Related Works

This work extends and modifies considerably the work presented in [3]. That work dealt basically with n-approximations of modal logic $\mathbf{K}$. Here, by extending the approach to several other logics, we had to modify both the semantics and the proof theory. In fact, here formulas are evaluated with respect to paths, while in [3] they were evaluated with respect to possible worlds. The n-SST proof theory is also accordingly recast, so that the proof system presented has a more uniform and modular conception, and incorporates more fundamentally the idea of limiting the degree of introspection.

The current paradigm of approximating logics was proposed by Cadoli and Schaerf [15] in the context of clausal propositional logic. Annette ten Teije et al. [17] pioneered modal approximation, although they do not define a proof theory. Massacci [13] made the first complete attempt to extend Cadoli and Schaerf's approach to modal logics, inheriting both the good qualities of that approach, as well as some shortcomings. As a result, Massacci's approximation does not deal with *pure formulas*, but only with *signed formulas*, revealing a bias for dealing with tableau proof systems. A different approach to approximation was proposed by Finger and Wassermann [5,6], in which pure propositional formulas could be dealt with, and several approximated proof methods were studied, including tableaux. In [7] it is shown that Cadoli and Schaerf's approach is based on giving a non classical treatment to the negation connective, but there it is shown that an approximation can also be based in any other connective. In fact, in [4] it has been shown that approximation needs not be based on any connectives, and can be performed by limiting the use of the cut inference rule.

In comparison, the approach here is totally orthogonal to how the propositional approximation is done, and even if any such approximation is done at all. Unlike Massacci's modal approximation, that appears to be based on pragmatic considerations, our modal approximation is based on limiting the allowed degree of introspection, and we deal with pure formulas. In the context of Hilbert-style derivation systems, this kind of approximations is known in the work of Ghilardi [10]. The approach here can be later combined to any other propositional forms of approximation.

The rest of the paper develops as follows. Section 2 presents the notations used, some concepts of modal logics necessary for the following sections and what do labeled signed formulas (used in the tableau-based proof theory) mean. Section 3 presents the semantics of the approximating logics, examples and a theorem that states exactly what is needed to determine the truth of a formula in an approximating logic. Section 4 presents a modular tableau-based proof theory for several modal logics in the context of n-SST; it presents several examples and proofs of soundness and completeness of the tableaux rules with respect to the semantics of Section 3. In the conclusion, possible directions for future research are discussed.

2 Preliminaries

We assume familiarity with the basics of modal logics as presented, for instance, in [11], and Smullyan's tableaux [16]. Notations are as follows. The language is based on a denumerable set A of propositional variables, or *atoms*, and contains the logical symbols $\neg$, $\wedge$, $\vee$, $\supset$, $\Box$ and $\Diamond$. Atoms will be denoted by lower case latin letters in italics (as in p, q and r). Formulas will be denoted by lower case Greek letters (as in ϕ and ψ).

Consider a model $\mathcal{M} = \langle W, R, v\rangle$ for a modal logic. In this model, W is a set of *possible worlds*, $R \subseteq W \times W$ is the *accessibility relation* and $v : W \to (A \to \{0, 1\})$ is a function that assigns to each world $w \in W$ a *propositional valuation*. Different modal logics have different restrictions on the accessibility relation. For instance, in logic **T** it is reflexive ($\forall w\, wRw$); in logic **D** it is serial ($\forall w \exists w'\, wRw'$); in logic **K4** it is transitive ($\forall ww'w''(wRw' \wedge w'Rw'' \Rightarrow wRw''))$. In the logic **K** there are no restrictions on R.

The *modal depth* in a formula ϕ, $\mathcal{N}(\phi)$, is defined as follows: if p is an atom, $\mathcal{N}(p) = 0$; $\mathcal{N}(\neg\phi) = \mathcal{N}(\phi)$; $\mathcal{N}(\phi \wedge \psi) = \mathcal{N}(\phi \vee \psi) = \mathcal{N}(\phi \supset \psi) = \max\{\mathcal{N}(\phi), \mathcal{N}(\psi)\}$; $\mathcal{N}(\Box\phi) = \mathcal{N}(\Diamond\phi) = \mathcal{N}(\phi)+1$. As an example, $\mathcal{N}(\Diamond\neg\Box(p\wedge \Diamond q)) = 3$.

We shall define truth with respect to paths over (W, R), and not worlds, which for technical reasons are better suited here. There follows the definitions and notations used.

Definition 2.1 *A* path ρ *from* w_1 *to* w_m *is a sequence* $w_1Rw_2, \ldots, w_{m-1}Rw_m$ *and can be denoted* $\rho = w_1Rw_2R\cdots Rw_{m-1}Rw_m$*. The elements* w_1 *and* w_m *of the path* ρ *are denoted by* $A(\rho)$ *and* $\Omega(\rho)$*, respectively. If* $m = 0$*, the path is empty (has no elements).*

Definition 2.2 *The* size *of the path* $\rho = w_1Rw_2R\cdots Rw_{m-1}Rw_m$*, denoted by* $|\rho|$*, is* $m - 1$*. Empty paths have size* -1*. A* minimal path *from world* w *to world* t *is a path with the shortest size.*

A world $w \in W$ can be identified with a path of size 0. Suppose that wRw_1.

We can designate the path $wRw_1Rw_2R\cdots Rw_n$ by $wR\rho$. The meaning of ρRt, $wR\rho Rt$ and similar expressions is defined analogously. Observe that, when ρ is empty, the world w can be identified with the size 0 path $w = \rho Rw = wR\rho$.

Definition 2.3 *An* immediate successor *of a path ρ is any path ρRt. The set of ρ's immediate successors is denoted by $S(\rho)$.*

Our intuition is that a path is associated with the degree of introspection needed to arrive at the truth of a modal formula. So, if we are worried about the truth of a formula an n-sized path, it means that we needed to make $n-1$ introspection steps to reason about another formula in the first world of the path. The use of paths is needed to make our approach work, avoiding the need to constantly worry about the number of introspection steps.

In Section 4, we use a variant of Massacci's Single Step Tableaux (SST) [14]. SST is itself a variant of Fitting's *prefixed tableaux* [8,9]. Tables 1 and 2 show the unifying notation in the presentation of signed tableau rules. *Signed formulas* are expressions $T\phi$ and $F\phi$ for a formula ϕ. Intuitively, $T\phi$ means "ϕ is true" and $F\phi$ means "ϕ is false"; so $T\phi$ has the same truth value as ϕ and $F\phi$ has the same truth value as $\neg\phi$. The components of SST are *prefixed formulas*, $\sigma : \theta$, where θ is a signed formula and σ is a prefix.

Definition 2.4 *A* prefix *is a finite sequence of positive integers that begins with 1. Different elements of this sequence are separated by dots.*

So 1, 1.2, 1.2.1 are all examples of prefixes. Intuitively, the prefix σ in $\sigma : \theta$ "names" a path where the formula θ is true, and every element of a path denotes a world. We assume that for every world there is associated a single number in a injective manner (so we are only dealing with countable models). The advantage of this representation is that it encodes all introspection steps needed in order to arrive from the starting world, denoted by 1, to the world denoted by the last element of σ (we "go" from one world w to another t when wRt by the accessibility relation R). For instance, the prefix 1.2.1.1 shows that, in order to arrive at the world 1 from the world 1, we can go from 1 to 2, them from 2 to 1, and finally from 1 to 1 again. The prefix $\sigma.m$ is said to be *accessible from* (the prefix) σ.

Definition 2.5 *The* size *of a prefix $1.n_1.n_2.\ldots.n_m$, denoted $|1.n_1.n_2\ldots n_m|$, is m.*

As in the case of paths, the meaning of such expressions as $\sigma.1$, $\sigma.n_1.n_2$ and the like is obvious. Every prefix $\sigma.n_1.n_2\ldots n_m$ is an *extension* of prefix σ. Notice the similarity of definitions 2.1, 2.2 on one side, and definitions 2.4, 2.5 on the other side. Those definitions will allow us to prove soundness and completeness of the proof theory, as well as providing appropriate counter models.

Our definition differs from Massacci's. Here, every number of a prefix denotes a world, but in Massacci's SST every prefix denotes a world. For Massacci, the prefix 1.1 says that world 1.1 is accessible from world 1 but gives no other information about it. For us, us prefix 1.1 says that the world 1 is accessible from itself, so our definition says a bit more.

α	α_1	α_2
$T\phi \wedge \psi$	$T\phi$	$T\psi$
$F\phi \vee \psi$	$F\phi$	$F\psi$
$F\phi \supset \psi$	$T\phi$	$F\psi$

neg	pos
$T\neg\phi$	$F\phi$
$F\neg\phi$	$T\phi$

β	β_1	β_2
$F\phi \wedge \psi$	$F\phi$	$F\psi$
$T\phi \vee \psi$	$T\phi$	$T\psi$
$T\phi \supset \psi$	$F\phi$	$T\psi$

Table 1
Smullyan's unifying notation

ν	ν_0
$T\Box\phi$	$T\phi$
$F\Diamond\phi$	$F\phi$

π	π_0
$F\Box\phi$	$F\phi$
$T\Diamond\phi$	$T\phi$

Table 2
Fitting's unifying notation

Suppose that we want to prove some unsigned formula ϕ. First, we sign ϕ as $F\phi$ and prefix it with 1, obtaining $1 : F\phi$ in the root of the tableau. Then we add nodes by the tableaux rules (Tables 3 and 4). A *branch* $\mathcal{B}$ is a path from the root to the leaf. Intuitively, each branch is a tentative model, and the tableau is a systematic attempt to obtain a countermodel to ϕ. A branch of the tableau *closes* when in contains a pair of formulas $\sigma.m : T\phi$ and $\sigma'.m : F\phi$; this means, intuitively, that the same world, irrespective of the path we take to arrive at it, cannot contain contradictions. The tableau *closes* when all its branches close. If the tableau closes, we succeeded in proving the original formula. If after the application of all tableaux rules it does not close, we say that it is *open* and *reduced* and we have the required counter model.

3 Semantics

The following semantics presents under what conditions a formula ψ is to be considered true in a path according to the truth of its subformulas. We have three parameters that can determine the truth: the model $\mathcal{M} = \langle W, R, v\rangle$, the path ρ (where all elements of the path are worlds in W) and a natural number n (so $n \in \{0, 1, 2, \ldots\}$), that is to be called *maximum limit of introspection.* The function $v : W \rightarrow (A \rightarrow \{0,1\})$ assigns to every world $w \in W$ a propositional valuation.

Semantics of the n-approximations

(1) $\mathcal{M}, \rho \models^n p$ iff $v(\Omega(\rho))(p) = 1$
(2) $\mathcal{M}, \rho \models^n \neg\phi$ iff $\mathcal{M}, \rho \not\models^n \phi$
(3) $\mathcal{M}, \rho \models^n \phi \wedge \psi$ iff $\mathcal{M}, \rho \models^n \phi$ and $\mathcal{M}, \rho \models^n \psi$
(4) $\mathcal{M}, \rho \models^n \phi \vee \psi$ iff either $\mathcal{M}, \rho \models^n \phi$ or $\mathcal{M}, \rho \models^n \psi$
(5) $\mathcal{M}, \rho \models^n \phi \supset \psi$ iff either $\mathcal{M}, \rho \not\models^n \phi$ or $\mathcal{M}, \rho \models^n \psi$
(6) for minimal paths ρ, if $|\rho| < n$, $\mathcal{M}, \rho \models^n \Box\phi$ iff for every path $\rho' \in S(\rho)$, $\mathcal{M}, \rho' \models^n \phi$
(7) for minimal paths ρ, if $|\rho| < n$, $\mathcal{M}, \rho \models^n \Diamond\phi$ iff there exists some path $\rho' \in S(\rho)$ such that $\mathcal{M}, \rho' \models^n \phi$
(8) if $|\rho| < n$, $|\rho'| < n$, $A(\rho) = A(\rho')$ and $\Omega(\rho) = \Omega(\rho')$, $\mathcal{M}, \rho \models^n \phi$ implies that $\mathcal{M}, \rho' \models^n \phi$

First, some intuitions on the semantics. Conditions 1–5 just say that the non-modal connectives work as expected. Conditions 6–7 say that the modal connectives work as expected when the path is sufficiently small, that is, smaller than the maximum limit of introspection (the requirementfor the path to be minimal in those conditions will be explained below). We notice that our semantics is *non-truth functional*: the truth of modal formulas is not determined by the function v. What we do with propositions beginning with modal operators for paths bigger than the maximum limit of introspection? We simply give to the last world of those paths arbitrary truth values, because our power of introspection does not allow for reasoning about them in any other way; such propositions work as atoms. This may be accomplished as follows. Define the set $A_{\Box\Diamond} = \{*\phi : \text{every formula } \phi \text{ and } * = \Box \text{ or } \Diamond\}$. Then extend v by defining it also in the set $A_{\Box\Diamond}$. The function $v^* : W \rightarrow (A \cup A_{\Box\Diamond} \rightarrow \{0,1\})$ thus obtained also assigns to every world $w \in W$ a valuation on formulas that begin with modal operators. For the truth of modal formulas $\phi' = \Box\phi$ or $\Diamond\phi$ in paths ρ with $|\rho| \geq n$, we simply define $\mathcal{M}, \rho \models^n \phi'$ iff $v^*(\Omega(\rho))(\phi') = 1$. It is important to stress that v^* is used only when conditions 6–8 cannot determine the truth value of a modal formula on a path.

This procedure has the following intuition. Whenever we reason about something which cannot be reached, we make an hypothesis about it. This is usually the case in most natural sciences, as in physics, where, for instance, we assume that the behaviour of distant galaxies still obey the laws which we can arrive from observation of our neighborhood (even though we cannot make experiments in those galaxies to substantiate this hypothesis) and draw conclusions about them. Condition 8 implies that, when we are inside the maximum limit of introspection, whenever we reason about some fact and we arrive at some conclusion following two different paths to the same world (introspect in two different ways to the same point), those conclusions must agree. This condition will be always used in logics that allow shotcuts in the paths, as logics with transitive relations. Using the example in physics cited earlier, it says that, if we know some facts about distant galaxies (say, facts obtained by infrared ray spectroscopy), we cannot simply make an arbitrary hypothesis about those galaxies without considering those facts first, that is, those facts give a shorter path to reason about distant galaxies.

There is also a technical reason for condition 8. Suppose that condition 8 does not hold and that conditions 6–7 were true in any path smaller than n. Now consider the following example. Suppose that $W = \{a, b\}$, $n = 3$, aRb, bRb, $v(x)(p) = 1$ for every $x \in W$ and $v^*(b)(\Box p) = 0$; as R is transitive and $\mathcal{N}(\Box p \supset \Box\Box p)$, we could expect $\Box p \supset \Box\Box p$ to be true in all paths of size smaller than 3. However, we have that $\mathcal{M}, aRbRb \models^3 \Box p$ and hence $\mathcal{M}, aRb \models^3 \Box\Box p$, but $\mathcal{M}, aRbRb \not\models^3 \Box\Box p$, even though $|aRbRb| < 3$, because $\mathcal{M}, aRbRbRb \not\models^3 \Box p$. This example shows that, if we want a non-contradictory semantics inside the maximum limit of introspection, a condition like 8 is needed, and cannot be just proved on the basis of other conditions. It forces us to accept $\mathcal{M}, aRbRb \models^3 \Box\Box p$ because for the minimal path aRb we have $\mathcal{M}, aRb \models^3 \Box\Box p$. Outside the limit, we are not really interested in contradictions.

The minimality of paths in conditions 6 and 7 is another technical requirement, because that is the only way we can unambiguously determine n-truths for all paths and formulas, as will be seen in the determination theorems ahead, where minimal paths play a central role. Intuitively, we must always reason with the smallest amount of introspection, because we lose information every time we introspect.

When we are not interested in restricting introspection, the truth parameters are usually $\mathcal{M}$ and a world $w \in W$. But the formulation given, using paths, is more convenient for our present purposes. There is, however, some connection with the usual semantics.

Theorem 3.1 *Suppose that $n = \infty$ (that is, n is unbounded). Associating every path ρ with its last member, $\Omega(\rho)$, the semantics of n-approximations transforms into classical modal logic semantics.*

Proof: The proof is a simple inspection. Non-classical evaluations, for which $|\rho| \geq n$, never occur because every path has a limited size, and n in our hypothesis is unbounded. $\Box$

We shall use $\mathcal{C}$ to denote a class of models associated with accessibility relations R, where we restrict R to be reflexive, transitive, serial, Euclidean or a combination of those conditions such as the modal logic may require. Now, some definitions:

Definition 3.2 *For a fixed n, when $\mathcal{M}, \rho \models^n \psi$ we say that ψ is n-*true *in the path ρ on the model $\mathcal{M}$, and that n-*false *if $\mathcal{M}, \rho \not\models^n \psi$. When ψ is n-true in every 0-sized path ρ on every model $\mathcal{M}$ in some class of models $\mathcal{C}$, we say simply that ψ is n-valid in $\mathcal{C}$, denoting this fact by $\models^n_{\mathcal{C}} \psi$. When ψ is n-valid in $\mathcal{C}$ for every $\mathcal{C}$, we simply say that ψ is n-valid, denoting this fact by $\models^n \psi$. The logic obtained by fixing some n is called n-logic.*

When the class of models $\mathcal{C}$ is associated with a logic **L**, we usually denote $\mathcal{C}$ simply by **L**. So, for instance, we denote the class of all models by **K** and the class of all reflexive models by **T**.

Let us see some examples of non-classical behaviour.

(1) The Rule of Necessitation, $\models^n \phi \Rightarrow \models^n \Box\phi$, does not hold in a n-approximation when $n = 0$. Let $\mathcal{M}$ be an arbitrary model and ρ some path in this model. Suppose that ϕ is n-valid (for instance, a tautology), so that $\mathcal{M}, \rho' \models^n \phi$ for every ρ' and n. Take $n = 0$. As $|\rho| \geq 0$, it is possible to choose the 0-truth of $\Box\phi$ in ρ (that is, to fix the function v^*). Assume that $v^*(\Omega(\rho))(\Box\phi) = 0$, so condition 8 implies that $\mathcal{M}, \rho \not\models^0 \Box\phi$.

(2) The principle K, $\Box(p \supset q) \supset (\Box p \supset \Box q)$, is not 0-valid. Let $\mathcal{M}$ be an arbitrary model and ρ some path in this model, and take $n = 0$. In the same way as the last example, we can choose the 0-truth of $\Box p$, $\Box q$ and $\Box(p \supset q)$ in ρ; assuming that $\mathcal{M}, \rho \models^0 \Box(p \supset q)$, $\mathcal{M}, \rho \models^0 \Box p$ and $\mathcal{M}, \rho \not\models^0 \Box q$, we obtain a counter model for K.

(3) The equivalence $\Diamond\phi \equiv \neg\Box\neg\phi$ is not 0-valid. Let $\mathcal{M}$ be an arbitrary model and ρ some path in this model, and take $n = 0$. As in the previous examples, we can choose the truth of $\Diamond\phi$ and $\Box\neg\phi$. Assume that $\mathcal{M}, \rho \not\models^0 \Box\neg\phi$ and that $\mathcal{M}, \rho \not\models^0 \Diamond\phi$. By the semantics of $\neg$, condition 2, $\mathcal{M}, \rho \not\models^0 \Box\neg\phi$ implies that $\mathcal{M}, \rho \models^0 \neg\Box\neg\phi$. Similarly, the other equivalences $\neg\Box\phi \equiv \Diamond\neg\phi$, $\neg\Box\neg\phi \equiv \Diamond\phi$ are not 0-valid.

Examples 1 and 2 above show that the logic we are treating here is not normal, because neither the Rule of Necessitation nor K are n-valid when $n = 0$.

The following three theorems look at what determines the n-truth of a formula at a path ρ between two worlds.

Theorem 3.3 *Let $\mathcal{M} = \langle W, R, v^* \rangle$ be a model and ρ a path with $|\rho| \geq n$. The n-truth of a formula ϕ for the path ρ is determined by v^*.*

Proof: Just notice that, when $|\rho| > n$, we need to use the function v to evaluate atoms and extend it to the function v^* to evaluate formulas beginning with modal operators. This is sufficient to evaluate the truth of any formula for such paths. $\Box$

Theorem 3.4 *Let $\mathcal{M} = \langle W, R, v^* \rangle$ be a model. The n-truth of a formula ϕ for a* minimal *path ρ, with $|\rho| < n$, is determined by v and by the n-truth of every subformula $\Box\xi$ and $\Diamond\xi$ of ϕ for all $\rho R \rho'$, where ρ' is some path and $|\rho R \rho'| \geq n$.*

Proof: The proof is by structural induction on ϕ. The cases where ϕ is atomic or a boolean combination are straightforward. Now suppose that $\phi = \Box\xi$ and that $|\rho| < n$. The n-truth of ϕ for ρ is determined by the n-truth of ξ for ρRt for every t such that $\Omega(\rho)Rt$ — this is equivalent to say that the n-truth of ϕ for ρ is determined by the n-truth of φ for ρ' for every $\rho' \in S(\rho)$ (condition 6 of the semantics), because the set of all $\rho' \in S(\rho)$ is the set of ρRt for all t such that $\Omega(\rho)Rt$.

By the inductive hypothesis, if ρRt is a minimal path, the n-truth of ξ for ρRt is determined by v and by the n-truth of every subformula $\Box\xi'$ and $\Diamond\xi'$ of ξ for all $\rho Rt R\rho'$ — subformulas of ϕ also —, where ρ' is some path and $|\rho Rt R \rho'| \geq n$. Therefore, the n-truth of ϕ in ρ is determined by v and by the n-truth of every subformula $\Box\xi'$ and $\Diamond\xi'$ of ξ for all $\rho Rt R \rho'$ for every t such that $\Omega(\rho)Rt$, where ρ' is some path and $|\rho Rt R\rho'| \geq n$. Denoting by ρ'' the path $tR\rho'$, we have that the n-truth of ϕ in w is determined by v and by the n-truth of every subformula $\Box\varphi'$ and $\Diamond\varphi''$ of ϕ for $\rho R\rho''$, where ρ'' is some path and $|\rho R \rho''| \geq n$.

If the path ρRt is not minimal, the n-truth of ξ in ρRt will be equal to that of its minimal path — let's call it μ —, and by the inductive hypothesis the n-truth of ξ in μ will be determined by that of subformulas $\Box\xi'$ and $\Diamond\xi'$ of ξ for all paths $\mu R\mu'$ with $|\mu R \mu'| \geq n$. To every path $\mu R\mu'$ corresponds another $\rho Rt R\mu'$. If $|\mu R\mu'| \geq n$, then obviously $|\rho R\mu'| \geq n$ and the truth of $\Box\xi'$ ($\Diamond\xi'$) for $\mu R\mu'$ and $\rho Rt R\mu'$ must coincide, for they are both equal to $v^*(\Omega(\mu'))(\Box\xi')$ ($v^*(\Omega(\mu'))(\Diamond\xi')$). If $|\mu R\mu'| < n$, the truth of $\mu R\mu'$ is determined by its minimal path, and we may repeat the procedure of this paragraph, which must eventually end since $|\mu| < |\rho Rt| \leq n$ form a decreasing sequence of positive integers. So, even in this case, the n-truth of ξ in ρRt is determined by v and by the n-truth of every subformula $\Box\xi'$ and $\Diamond\xi'$ of ξ for all $\rho Rt R\rho'$ — subformulas of ϕ also —, where ρ' is some path and $|\rho Rt R\rho'| \geq n$, so we may proceed as in the last paragraph.

For the connective $\Diamond$ when $|\rho| < n$ we proceed analogously, using condition 7 of the semantics. That completes the proof. Notice that to evaluate the truth of some sentence $\Box\xi$ or $\Diamond\phi$ in paths ρ where $|\rho| \geq n$ (such sentences may appear in the course of our evaluation) we must extend the function v to v^*, but we just need to define v^* to modal formulas that are actually used. $\Box$

Theorem 3.4 just concerns itself with minimal paths, but by condition 8 every other path smaller than the maximum limit of introspection must agree with it. In future discussions we never worry about the minimality of the path because of that theorem.

If there are two different minimal paths beginning and ending at the same worlds, do they agree in all formulas? The positive answer is given below:

Theorem 3.5 *Let $\mathcal{M} = \langle W, R, v^* \rangle$ be a model and ϕ a formula. If ρ, ρ' are two minimal paths and $A(\rho) = A(\rho')$, $\Omega(\rho) = \Omega(\rho')$, then $\mathcal{M}, \rho \models^n \phi$ iff $\mathcal{M}, \rho' \models^n \phi$.*

Proof: The proof is again by structural induction on ϕ, and the only non-trivial case is when $\phi = \Box\xi$ or $\Diamond\xi$. We take the case $\phi = \Box\xi$, the other case being similar. Suppose that $|\rho| = |\rho'| < n$. Then $\mathcal{M}, \rho \models^n \phi$ iff $\mathcal{M}, \mu \models^n \xi$ for every path $\mu \in S(\rho)$. It can happen that $S(\rho) \neq S(\rho')$, but every $\mu \in S(\rho)$ is of the form $\rho R t$, $t \in W$, and is associated with $\mu' = \rho' R t \in S(\rho')$, that begin and and at the same worlds, and vice-versa (in other words, the association $\mu - \mu'$ is a bijection). If $|\mu| = |\mu'| < n$, condition 8 of the semantics applies and the n-truth of ξ for μ and μ' must agree. If $|\mu| = |\mu'| = n$, the truth for ξ is, by Theorem 3.3, determined by v^* and also agree, because we only use v^* to determine truth or falsity of the atoms and modal sentences of ξ in the world $\Omega(\mu) = \Omega(\mu')$. Therefore, $\mathcal{M}, \mu \models^n \xi$ iff $\mathcal{M}, \mu' \models^n \xi$ for every $\mu \in S(\rho)$, $\mu' \in S(\rho')$, proving the theorem in this case. If $|\rho| = |\rho'| \geq n$, the formula ϕ must be true or false according to whether $v^*(\Omega(\rho))(\phi) = v^*(\Omega(\rho'))(\phi)$ is 1 or 0, so the truth of ϕ for both paths is the same in this case too. □

A very important question for our objectives is: when we increase n, is n-truth inherited? This must be so, otherwise our semantics would not approximate modal logic.

Theorem 3.6 *Let ϕ be some formula. If ϕ is n-valid, then ϕ is $(n+1)$-valid.*

Proof: Suppose that ϕ is n-valid. Fix some class $\mathcal{C}$ of models, arbitrary otherwise. Then, for any 0-sized path ρ on every model $\mathcal{M}$, we have that $\mathcal{M}, \rho \models^n_{\mathcal{C}} \phi$. This implies that ϕ is true in models of $\mathcal{C}$ for every function v^* defined on the atoms of ϕ and on the modal formulas of ϕ which, in the course of our analysis of the n-truth of ϕ, we arrive at by paths greater than n. Now assume that ϕ is not $n+1$-valid. This means that there is some model $\mathcal{M} = \langle W, R, v \rangle$ and some extension of the function v, denoted by v^*, in which, for some 0-sized path ρ, $\mathcal{M}, \rho \not\models^{n+1}_{\mathcal{C}} \phi$. But the modal formulas of ϕ in which v^* is defined is a subset of the modal formulas for which it has to be defined for n-truth (and, obviously, the atoms are the same). So, *a fortiori*, we should have $\mathcal{M}, \rho \not\models^n_{\mathcal{C}} \phi$, a contradiction. Intuitively, when we increase n, we are cutting some possible models, so it is "easier" for a formula to be valid; in particular, the n-valid formulas of the smaller n continue to be true, becoming $(n+1)$-valid. □

Another important property to inspect when dealing with approximations is the *uniform substitution property*, that is, whether the set of validities is closed under the substitution. We denote the uniform substitution os some atom p by a formula ψ as $\varphi[p := \psi]$.

Theorem 3.7 *If ϕ is n-valid, so is $\phi[p := \psi]$.*

Proof: It suffices to note that a formula ψ has the same or fewer possible valuations at a path ρ than an atom p, even if $|\rho| \geq n$. In an exact sense, if $\phi[p := \psi]$ was not n-valid, we could present a function v^* that in its atom p assumed the same truth value of ψ, and this would prove ϕ false in that model, a contradiction. Therefore, if $\mathcal{M}, \rho \models^n \phi$, we are garanteed that $\mathcal{M}, \rho \models^n \phi[p := \psi]$, so the n-validity of ϕ leads to the n-validity of $\phi[p := \psi]$. $\square$

Note that the results above hold even if a set of restrictions is imposed on the class of models, which means that they are valid for all possible approximations of modal logics. We now present a modular proof theory for modal n-logics that approximate several of the most well-known modal logics.

4 A Tableau-Based Proof Theory

Consider a fixed maximum limit of introspection n. We develop here a tableau-based proof theory corresponding to the semantics presented in the previous section, which we call n-Single Step Tableaux (n-SST). In the tables below, we use the notations explained on Tables 1 and 2.

$$\frac{\sigma : \alpha}{\begin{array}{l}\sigma : \alpha_1 \\ \sigma : \alpha_2\end{array}} \qquad \frac{\sigma : neg}{\sigma : pos} \qquad \frac{\sigma : \beta}{\sigma : \beta_1 \quad \sigma : \beta_2}$$

Table 3
Propositional n-SST rules

Table 3 gives the n-SST rules for propositional connectives, and is exactly the same of the SST rules for propositional connectives, given in [14].

Table 4 gives the n-SST rules for modal connectives. All modal logics rest on (π)-rule. However, different modal logics are obtained by different combinations of ν-rules, which makes n-SST rules modular. For instance, for logic n-**K** we just need rule (νK); for logic n-**T** we need (νK) and (νT); for logic n-**D**,

$$\frac{\sigma : \pi}{\sigma.m : \pi_0}\,(\pi) \quad \text{with } |\sigma| < n \text{ and } \sigma.m \text{ and } m \text{ new in the branch}$$

$$\frac{\sigma : \nu}{\sigma.m : \nu_0}\,(\nu K) \quad \text{with } \sigma \text{ and } \sigma.m \text{ already in the branch}$$

$$\frac{\sigma.m : \nu}{\sigma.m.m : \nu_0}\,(\nu T) \quad \text{with } |\sigma.m| < n \text{ and } \sigma.m \text{ already in the branch}$$

$$\frac{\sigma : \nu}{\sigma : \pi^\nu}\,(\nu D) \quad \text{with } |\sigma| < n \text{ and } \sigma \text{ already in the branch}$$

$$\frac{\sigma : \nu}{\sigma.m : \nu}\,(\nu 4) \quad \text{with } |\sigma| < n-1 \text{ and } \sigma,\, \sigma.m \text{ already in the branch}$$

$$\frac{\sigma.m : \nu}{\sigma : \nu}\,(\nu 4^R) \quad \text{with } |\sigma| < n-1 \text{ and } \sigma,\, \sigma.m \text{ already in the branch}$$

Table 4
Modal n-SST rules

(νK) and (νD); for logic n-**S4** we need (νT) and $(\nu 4)$; for logic n-**S5** we need (νT), $(\nu 4)$ and $(\nu 4^R)$ [14].

Notice that rule (νT) is a bit different from the formulation of Massacci, even if we consider n to be unbounded. The reason is that we understand $\sigma.m$ to be different from $\sigma.m.m$. As Massacci worked with worlds, instead of paths, his formulation of rule (νT) is: from $\sigma : \nu$ we may infer $\sigma : \nu_0$. This is because his intuition is that in the world σ we may accept ν_0 on the strenght of the acceptance of ν. For us σ is a path, and in logics where **T** is valid, we make one introspection step (from world w to w) to infer ϕ from $\Box\phi$, so this must be built into the proof theory.

The notation in rule (νD) needs some clarification. If $\nu = T\Box\phi$, then $\pi^\nu = T\Diamond\phi$; if $\nu = F\Diamond\phi$, then $\pi^\nu = F\Box\phi$. The first case states that if ϕ is necessary, then it is possible; the second states that if ϕ is impossible, then it is not necessary. In classical logic, as truth is always necessary and falsity is always impossible, infinite serial models arise. Not so in 0-SST, where both $\Box\phi$ and $\Diamond\phi$ can have arbitrary, independent truth values; thus, in 0-SST the rule (νD) can never be used, as its proviso is never satisfied.

Below we have two examples of proofs by n-SST.

Figure 1 gives a proof of the K-axiom, showing that it is not 0-SST provable, but it is n-SST provable for $n \geq 1$. We start with $n = 0$ on the left. Formulas

$n = 0$	$n \geq 1$
(1) $1 : F\Box(p \supset q) \supset (\Box p \supset \Box q)$	(1) $1 : F\Box(p \supset q) \supset (\Box p \supset \Box q)$
(2) $1 : T\Box(p \supset q)$	(2) $1 : T\Box(p \supset q)$
(3) $1 : F\Box p \supset \Box q$	(3) $1 : F\Box p \supset \Box q$
(4) $1 : T\Box p$	(4) $1 : T\Box p$
(5) $1 : F\Box q$	(5) $1 : F\Box q$
?	(6) $1.2 : Fq$
	(7) $1.2 : Tp \supset q$
	(8) $1.2 : Tp$
	(9) $1.2 : Fp$ (10) $1.2 : Tq$
	$\times$ $\times$

Fig. 1. Proof of K by n-SST

(2) and (3) of both sides are obtained from (1) by application of the α-rule; in the same way we obtain (4) and (5). Now, in the left side, there is nothing more that can be done, because we need to apply (π) in (5) to obtain a new prefix and this is only possible for $|\sigma| < n$. We then increment n and move to the right, where we can apply (π), obtaining (6). (7) and (8) are obtained from (2) and (4), respectively, by rule (νK). (9) and (10) come from (7) by the β-rule.

The left side of Figure 1 permits us to construct a counter model for K when $n = 0$. The counter model is seen in Example 2 of the previous section (using 1 in the place of w).

Figure 2 gives a proof of $\Box p \supset \Box\Box p$ by n-SST in modal logic **S4**. We start on the left with $n = 0$. Formulas (2) and (3) of all sides are obtained from (1) by application of the α-rule. Formula (3) asks for application of the (π)-rule, and this cannot be done in the left side because, there, $n = 0$, and $0 = |1| \geq 0$. So we increment n and move to the middle, and line (4) is obtained by the (π)-rule, that can be applied because $|1| < 1$. Formula (5) in the middle and right sides is an application of rule (νK). Formula (6) is obtained by rule (νT), that can also be applied because $|1| < 1$. Formula (7) is another application of rule (νK). We cannot proceed anymore in the middle, so we increment n agaig and move to the left, where because $0 = |1| \leq 2 - 1$ we can apply the rule ($\nu 4$) in formula (2), obtaining (8).

$n = 0$	$n = 1$	$n \geq 2$
$(1) 1 : F\Box p \supset \Box\Box p$	$(1) 1 : F\Box p \supset \Box\Box p$	$(1) 1 : F\Box p \supset \Box\Box p$
$(2) 1 : T\Box p$	$(2) 1 : T\Box p$	$(2) 1 : T\Box p$
$(3) 1 : F\Box\Box p$	$(3) 1 : F\Box\Box p$	$(3) 1 : F\Box\Box p$
?	$(4) 1.2 : F\Box p$	$(4) 1.2 : F\Box p$
	$(5) 1.2 : Tp$	$(5) 1.2 : Tp$
	$(6) 1.1 : T\Box p$	$(6) 1.1 : T\Box p$
	$(7) 1.1 : Tp$	$(7) 1.1 : Tp$
	?	$(8) 1.2 : T\Box p$
		$\times$

Fig. 2. Proof of $\Box p \supset \Box\Box p$ by n-SST

The left side of Figure 2 permits us to construct a counter model $\mathcal{M} = \langle W, R, v^* \rangle$ for $\Box p \supset \Box\Box p$ when $n = 0$. We can take $W = \{1\}$ (the set of numbers that appear on prefixes in the branch), $R = \{1R1\}$ (that is reflexive and transitive), and $v(1)$ is a fixed valuation of the atoms of the language (any will work). Also choose $v^*(1)(\Box p) = 0$ and $v^*(1)(\Box\Box p) = 0$ and define v^* arbitrarily on other places. Because of Theorems 3.4–3.3, the 0-truth is completely determined by those conditions, so $\mathcal{M}, 1 \models^0 \Box p$ and $\mathcal{M}, 1 \not\models^0 \Box\Box p$.

The middle of Figure 2 permits us to construct a counter model for $\Box p \supset \Box\Box p$ when $n = 1$. Take $\mathcal{M} = \langle W, R, v^* \rangle$, where $W = \{1, 2\}$ (the set of numbers that appear on prefixes in the branch), $R = \{1R1, 1R2, 2R2\}$ (that is reflexive and transitive). Make $v(1)(p) = v(2)(p) = 1$ because of formulas (7) and (5). Also choose $v^*(2)(\Box p) = 0$ because of formula (4) (so $\mathcal{M}, 1R2 \not\models^1 \Box p$) and $v^*(1)(\Box p) = 1$ because of formula (6) (so $\mathcal{M}, 1R1 \models^1 \Box p$).

At this point, we should point out that our formulation is essentially bivalent. We assume that for distant worlds there is some truth value associated for every sentence, even if this leads to a contradiction outside the maximum limit of introspection. Massacci [13] develops a very interesting semantics with four truth values, separating atoms related to *interesting propositions* (true or false), to *incoherent propositions* (always truth), and finally *unknown propositions* (always false). But our approach gives better intuitions for the construction of a counter model in our proof theory, that is analytic from the very beginning, as seen in the examples above. We think that the introduction of another, "unknown", truth value at distant words would difficult the interpretation of the results of some tableau. And the intuition we intend for this theory is that, outside our introspection capabilities, we give truth or false values for propositions (this is certainly the case in analysing facts of a science like physics).

We must show soundness and completeness of the n-SST with respect to the semantics of the n-approximations presented in the previous section.

Tableaux rules are *sound* with respect to some semantics if that every inference that the tableaux rules do can be done by the semantics — that is, the tableaux does not infer facts that are not permitted already by the semantics. To show this, we must first explain what it means for a model to satisfy formulas in a branch of a tableau. Two definitions are needed.

Definition 4.1 *Let $\mathcal{M} = \langle W, R, v^* \rangle$ a model and $\mathcal{B}$ a branch of some n-SST tableau. An* n-SST interpretation *is a mapping from the prefixes $\sigma \in \mathcal{B}$ to paths $i(\sigma) \in \mathcal{M}$, defined as follows. For $m \in \sigma \in \mathcal{B}$, $i(m) \in W$ is a world (or a 0-sized path) such that if $\sigma'.m_1.m_2.\sigma'' \in \mathcal{B}$, then $i(m_1)Ri(m_2)$ in $\mathcal{M}$. Inductively define $i(\sigma.m) = i(\sigma)Ri(m)$.*

A simple induction shows that $|\sigma| = |i(\sigma)|$.

Definition 4.2 *A tableau branch $\mathcal{B}$ is* satisfiable *if there is a model $\mathcal{M} = \langle W, R, v^* \rangle$ and an n-SST interpretation i such that for every prefixed formula $\sigma : T\phi \in \mathcal{B}$, one has $\mathcal{M}, i(\sigma) \models^n \phi$, and for every prefixed formula $\sigma : F\phi \in \mathcal{B}$, one has $\mathcal{M}, i(\sigma) \not\models^n \phi$.*

In Definition 4.2, $\sigma : T\phi$ is to be considered *satisfiable* in a model $\mathcal{M}$ with an n-SST interpretation i if ϕ is n-true in the path $i(\sigma)$, and $\sigma : F\phi$ is to be considered *satisfiable* in the same model and interpretation if ϕ is n-false in the path $i(\sigma)$.

Now we prove soundness.

Theorem 4.3 (Soundness) *Suppose that a n-SST tableau for ψ closes using rule (νK) and possibly some of the rules (νT), (νD), $(\nu 4)$ and $(\nu 4^R)$. Then $\models^n_{\mathbf{L}} \psi$, where $\mathbf{L}$ is the logic corresponding to the choice of rules.*

Proof: We show that, if a model $\mathcal{M}$ with an n-SST interpretation i satisfies a partially expanded branch $\mathcal{B}$ of the tableau prior to the application of some n-SST rule, then one of the branches obtained by addition of the conclusions of the rule is also satisfied by $\mathcal{M}$ with some n-SST interpretation j.

If an α, β or *neg* formula is expanded, it is a classical propositional logic situation which preserves satisfiability [16]. We now analyse the modal expansion rules, one for each instance of it; other instances of each rule yield totally analogous proofs.

(π) Suppose that $\sigma : F\Box\phi \in \mathcal{B}$ such that $\mathcal{M}, i(\sigma) \not\models^n \Box\phi$ and the (π)-rule has been applied producing $\sigma.m : F\phi$ with m new on $\mathcal{B}$. Then $|\sigma| = |i(\sigma)| < n$, so that for some path $\rho = i(\sigma)Rw_m$, $\mathcal{M}, \rho \not\models^n \phi$.

Let j be an interpretation that extends i, such that $j(m) = w_m$ and $j(m') = i(m')$ for $m' \neq m$. Then $j(\sigma.m) = \rho$ and $\mathcal{M}$ and j satisfy $\sigma : F\Box\phi$. As j extends i and m is new, $\mathcal{M}$ and j satisfy every formula on the branch after the (π)-rule expansion.

(νK) Suppose that $\sigma : T\Box\phi \in \mathcal{B}$ such that $\mathcal{M}, i(\sigma) \models^n \Box\phi$ and there is some prefix $\sigma.m$ on the $\mathcal{B}$. The (π)-rule must have been used in a formula with prefix σ, so $|\sigma| < n$ and $|i(\sigma)| = |\sigma| < n$. Therefore, for every $i(\sigma)Rw$ accessible from $i(\sigma)$, $\mathcal{M}, i(\sigma)Rw \models^n \Box\phi$. As i is an interpretation, there must be an w_n such that $i(m) = w_m$ and $i(\sigma.m) = i(\sigma)Rw_m$ is a path in $\mathcal{M}$ accessible from $i(\sigma)$. So $\mathcal{M}$ and i satisfy $\sigma : T\phi$.

(νT) Rule (νT) is used in logics whose models have reflexive relations; so, if ρRw is a path, so is $\rho RwRw$. Suppose that $\sigma.m : T\Box\phi \in \mathcal{B}$ is satisfiable by a reflexive $\mathcal{M}$ and i, where $|\sigma.m| = |i(\sigma.m)| < n$. Then $\mathcal{M}, i(\sigma.m) \models^n \Box\phi$. By reflexivity, $\mathcal{M}, i(\sigma.m)Ri(m) \models^n \Box\phi$, which means that $\mathcal{M}$ and i satisfy all formulas in $\mathcal{B}$.

($\nu 4$) Rule ($\nu 4$) is used in logics whose models have transitive relations; so if ρRw and $\rho RwRt$ are paths, so is ρRt. Suppose that $\sigma : T\Box\phi$ is satisfiable by a transitive $\mathcal{M}$ and i, where $|\sigma| = |i(\sigma)| < n - 1 < n$. Then $\mathcal{M}, i(\sigma) \models^n \Box\phi$. As the rule ($\nu 4$) was applied on $\sigma : T\Box\phi$, there is a prefix $\sigma.m \in \mathcal{B}$, so $\mathcal{M}, i(\sigma)Ri(m) \models^n \phi$. Suppose, for contradiction, that $\mathcal{M}, i(\sigma)Ri(m) \not\models^n \Box\phi$, then there exists a $w \in W$ such that $\mathcal{M}, i(\sigma)Ri(m)Rw \not\models^n \phi$. By transitivity, $i(\sigma)Rw$ is also a path and $\mathcal{M}, i(\sigma)Rw \models^n \phi$, thus contradicting condition 8, that states that paths smaller than n with the same extremes must agree on all formulas. So $\mathcal{M}, i(\sigma)Ri(m) \models^n \Box\phi$, which means that $\mathcal{M}$ and i satisfy all formulas in $\mathcal{B}$.

We leave the cases of rules (νD) and ($\nu 4^R$), used in logics whose models have serial and transitive relations, respectively, for the reader; they are similar to the ones presented here. $\Box$

Tableaux rules are *complete* with respect to some semantics if every inference that the semantics does can also be done by the tableaux rules — that is, the tableau infers at least the facts permitted by the semantics. In a proof of completeness, the contrapositive is usually used. That is, we assume there is an open branch, and construct a countermodel for it.

Theorem 4.4 (Completeness) *Suppose that* $\models^n_{\mathbf{L}} \psi$*, for* $\mathbf{L} \in \{n\text{-}\mathbf{K}, n\text{-}\mathbf{T}, n\text{-}\mathbf{D}, n\text{-}\mathbf{S4}, n\text{-}\mathbf{S5}\}$. *Then the n-SST for* ψ *closes using the corresponding* ν*-rules.*

Proof: Let $\mathcal{B}$ be an reduced open branch of the tableau. Construct the model

$\mathcal{M} = \langle W, R, v^* \rangle$ as follows:

- $W = \{m : m \in \sigma$ for some $\sigma \in \mathcal{B}\}$.
- mRm' iff $\sigma.m.m' \in \mathcal{B}$ for some $\sigma \in \mathcal{B}$.
- For atomic formulas we build a (partial) valuation v as follows. $v(m)(p) = 1$ if $\sigma.m : Tp \in \mathcal{B}$; $v(m)(p) = 0$ if $\sigma.m : Fp \in \mathcal{B}$ for some $\sigma \in \mathcal{B}$. The values of v at other worlds and atoms can be chosen arbitrarily.

Associated with every prefix $\sigma = 1.n_1.\ldots.n_m$ there is a path $1Rn_1R\cdots Rn_m$. We denote this path by σ too. This leads to some ambiguity, but it is easier for discussion and avoids clumsy notations. Using this notation, we see that when σ' is accessible from σ as prefix, as a path $\sigma' \in S(\sigma)$. We notice that every prefix in a branch is an extension of the prefix 1.

- By Theorems 3.4–3.3, we need to define the n-truth of the formulas $\Box\phi$ and $\Diamond\phi$ for all paths ρ such that $|\rho| \geq n$. This we do by extending the valuation v to v^* on prefixes σ such that $|\sigma| \geq n$ and formulas ν and π as follows. $v^*(\Omega(\sigma))(\nu) = 1$ iff $\sigma : \nu \in \mathcal{B}$ and $v^*(\Omega(\sigma))(\pi) = 0$ iff $\sigma : \pi \in \mathcal{B}$. The values of v^* at other worlds and modal sentences can be chosen arbitrarily.

There is no inconsistency in the definition of the function v^*, because the branch does not close, that is, it does not happen simultaneously that $\sigma : T\phi$ and $\sigma' : F\phi \in \mathcal{B}$, where $\Omega(\sigma) = \Omega(\sigma')$ and ϕ is an atom or a modal sentence in which $|\sigma|$, $|\sigma'| \geq n$.

Now it is necessary to show that $\mathcal{B}$ is satisfiable by $\mathcal{M}$ so defined, with the n-SST interpretation that is the identity function ($i(\sigma) = \sigma$ for every $\sigma \in \mathcal{B}$). This is done by structural induction on formulas in the branch (the part on the right of the prefixes). First we shall prove completeness in the basic modal logic **K**; then, we will show the adaptations needed for the different modal logics **T**, **D**, **S4** and **S5**.

(1) $\mathcal{M}$ satisfies the atoms by definition.
(2) If there is a formula of the form $\sigma : F\phi \supset \psi$, then (because the branch is reduced, see page 6) we have that $\sigma : T\phi$, $\sigma : F\psi \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma : T\phi$ and $\sigma : F\psi$. So $\mathcal{M}, \sigma \models^n \phi$ and $\mathcal{M}, \sigma \not\models^n \psi$, therefore $\mathcal{M}, \sigma \not\models^n \phi \supset \psi$. Analogously, we prove that $\mathcal{M}$ satisfies every α-formula. The procedures needed for *neg* and β-formulas are similar and will not be given here.
(3) If there is a formula of the form $\sigma : F\Box\phi$, there are two cases to consider. First, if $|\sigma| \geq n$, the formula is satisfied by the branch by definition. Second, if $|\sigma| < n$ the (π)-rule can be applied. Because the branch is reduced, we have that $\sigma.m : F\phi$ for some $\sigma.m \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m : F\phi$ for some $\sigma.m \in \mathcal{B}$. This means that $\mathcal{M}, \sigma.m \not\models^n \phi$ for some prefix σ', accessible by σ. Therefore, $\mathcal{M}, \sigma \not\models^n \Box\phi$. Analogously, we prove that $\mathcal{M}$ satisfies every π-formula.

(4) If there is a formula of the form $\sigma : T\Box\phi$, there are also two cases to consider. When $|\sigma| \geq n$, the formula is satisfied by the branch by definition. Now suppose that $|\sigma| < n$. Because the branch is reduced we have that $\sigma.m : T\phi \in \mathcal{B}$ for every $\sigma.m \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m : T\phi$ for every $\sigma.m \in \mathcal{B}$. This means that $\mathcal{M}, \sigma' \models^n \phi$ for every prefix σ' accessible by σ. Therefore, $\mathcal{M}, \sigma \models^n \Box\phi$. Analogously, we prove that $\mathcal{M}$ satisfies every other ν-formula.

(5) For modal logic **T**, the model constructed must be reflexive. So, we add the condition mRm for all $m \in W$. The proof of completeness is as above for all formulas that are not ν. If there is a formula of the form $\sigma.m : T\Box\phi$, there are also two cases to consider. If $|\sigma.m| \geq n$, the formula is satisfied by the branch by definition. Now suppose that $|\sigma.m| < n$. Because the branch is reduced we have that $\sigma.m.l : T\phi \in \mathcal{B}$ for every $\sigma.m.l \in \mathcal{B}$. We notice here that l can be different from or equal to m, by application of rules (νK) or (νT), respectively. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m.l : T\phi$ for every $\sigma.m \in \mathcal{B}$. This means that $\mathcal{M}, \sigma' \models^n \phi$ for every prefix σ' accessible by $\sigma.m$. Therefore, $\mathcal{M}, \sigma.m \models^n \Box\phi$. Analogously, we prove that $\mathcal{M}$ satisfies every other ν-formula.

(6) For modal logic **D**, the model constructed must be serial. So, we add the condition mRm for all $m \in W$ that do not satisfy mRl for some l (essentially, we are adding reflexivity to the dead ends). The proof of completeness is as above for all formulas that are not ν. If there is a formula of the form $\sigma : T\Box\phi$, there are also two cases to consider. When $|\sigma| \geq n$ the formula is satisfied by the branch by definition. Now suppose that $|\sigma| < n$. Because the branch is reduced we have that $\sigma.m : T\phi \in \mathcal{B}$ for every $\sigma.m \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m : T\phi$ for every $\sigma.m \in \mathcal{B}$. This means that $\mathcal{M}, \sigma' \models^n \phi$ for every prefix σ' accessible by σ. Therefore, $\mathcal{M}, \sigma.m \models^n \Box\phi$. We also have that $\sigma : T\Diamond\phi \in \mathcal{B}$, by application of rule (D). This implies that the (π)-rule has been applied, because $|\sigma| < n$ so that $\sigma.m : T\phi \in \mathcal{B}$ for some $\sigma.m \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m : T\phi$. This means that $\mathcal{M}, \sigma' \models^n \phi$ for some prefix σ' accessible by σ. Therefore, $\mathcal{M}, \sigma \models^n \Diamond\phi$, and so $\mathcal{M}, \sigma \models^n \Box\phi$ is not contradicted. Analogously, we prove that $\mathcal{M}$ satisfies every other ν-formula.

(7) For modal logic **S4**, the model constructed must be reflexive and transitive. So, we add the condition mRm for all $m \in W$, and we add mRk whenever mRl and lRk (essentially, we are adding the transitive-reflexive closure to the model). The proof of completeness is as above for all formulas that are not ν. If there is a formula of the form $\sigma : T\Box\phi$, there are also two cases to consider. Suppose that $|\sigma| \geq n - 1$; in this case, $|\sigma.m| \geq n$, so the formula $\sigma.m : T\phi$ is satisfied by the branch by definition. Now suppose that $|\sigma| < n - 1$. Because the branch is reduced we have that $\sigma.m : T\phi \in \mathcal{B}$ for every $\sigma.m \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m : T\phi$ for every $\sigma.m \in \mathcal{B}$. This means that $\mathcal{M}, \sigma' \models^n \phi$ for every prefix σ' accessible by σ. Therefore, $\mathcal{M}, \sigma \models^n \Box\phi$. We also have

that $\sigma.m : T\Box\phi \in \mathcal{B}$ for every prefix $\sigma.m \in \mathcal{B}$, by application of rule ($\nu 4$). Now $|\sigma| < n-1$ and so $|\sigma| < n$. Because the branch is reduced, for every $\sigma.m.l$ in the branch, we must have that $\sigma.m.l : T\phi \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}, \sigma.m.l \models^n \phi$, which implies that $\mathcal{M}, \sigma.m \models^n \Box\phi$, and so $\mathcal{M}, \sigma \models^n \Box\phi$ is not contradicted. Analogously, we prove that $\mathcal{M}$ satisfies every other ν-formula.

(8) For modal logic **S5**, the model constructed must be reflexive, transitive and Euclidean. So, we add the condition mRm for all $m \in W$, and we add mRk whenever mRl and lRk whenever mRl and mRk (essentially, we are adding the reflexive, transitive and Euclidean closure to the model). The proof of completeness is as above for all formulas that are not ν. If there is a formula of the form $\sigma.m : T\Box\phi$, there are also two cases to consider. Suppose that $|\sigma| \geq n-1$; in this case, $|\sigma.m| \geq n$, so the formula $\sigma.m : T\phi$ is satisfied by the branch by definition. Now suppose that $|\sigma| < n-1$; therefore, $|\sigma.m| < n$. Because the branch is reduced we have that $\sigma.m.l : T\phi \in \mathcal{B}$ for every $\sigma.m.l \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m.l : T\phi$ for every $\sigma.m.l$ in the branch. This means that $\mathcal{M}, \sigma' \models^n \phi$ for every prefix σ' accessible by $\sigma.m$. Therefore, $\mathcal{M}, \sigma.m \models^n \Box\phi$. We also have that $\sigma : T\Box\phi \in \mathcal{B}$ for every prefix $\sigma \in \mathcal{B}$, by application of rule (4^R). Now $|\sigma| < n-1 < n$. Because the branch is reduced, for every $\sigma.l \in \mathcal{B}$ we must have that $\sigma.l : T\phi \in \mathcal{B}$. By the induction hypothesis, $\mathcal{M}, \sigma.l \models^n \phi$, which implies that $\mathcal{M}, \sigma \models^n \Box\phi$, and so $\mathcal{M}, \sigma.m \models^n \Box\phi$ is not contradicted. Analogously, we prove that $\mathcal{M}$ satisfies every other ν-formula.

That completes the proof by induction. □

We can be sure that if ϕ is a Theorem of a modal logic, then there is some n such that $\models^n \phi$ as follows. Consider the nesting of modal operators in a formula ϕ, $\mathcal{N}(\phi)$. We need at most $\mathcal{N}(\phi)$ introspection steps in order to prove ϕ, because this is the maximum number of times that the rules which create new prefixes can be applied to ever increasing prefixes. This is the essence of the approximation process. If we can prove some formula with some fixed n, then we are done; but if not, we can increase n and try to prove the same formula. If the formula ϕ can be proved classically, it will eventually be proved for some n-logic (it can be proved for the $\mathcal{N}(\phi)$-logic).

5 Conclusion

Extension of the semantics for temporal logics is promising, because it can cast light on the philosophical problem of determinism (because when we restrict introspection we control the ability to foresee future — and past —, so to say).

From the standpoint of theoretical logic we can also study the possibility of axiomatizing our semantics, as this can give new and valuable intuitions.

It would be interesting to investigate the relationship between the theory presented here with the awareness logic of Fagin and Halpern [2]. That logic presents us with an awareness set and a function that acts as a sieve, filtering the explicit beliefs from the bulk of implicit ones and has been recognized as a flexible and powerful logic [12]. In what way the maximum limit of introspection can act as this sieve?

One possible application of this procedure is the analysis of proofs in modal logics. For instance, "how many" proofs can we have with a given maximum limit of introspection, compared with another?

As another possible direction for future research, there is the question of the complexity of the family of n-logics. The provability of formulas for most modal logics is a PSPACE-complete problem. In classical logic, the provability is a NP-complete problem. In [4], it is presented a family of logics that approximates classical inference, in such a way that each step in the approximation can be decided in polynomial time (but not the full approximation). We know that every NP-complete problem is in PSPACE, but we don't know whether the reciprocal is true of false. Do the n-logics give rise to problems polynomially tractable? If so, as they approximate a PSPACE-complete problem, what light do they bring over the question whether PSPACE $\subseteq$ NP?

Acknowledgements

During the writing of this article, Guilherme de Souza Rabello was granted an institutional scholarship of CNPq by the IF-USP, grant 141038/2005-5. Marcelo Finger is partly supported by CNPq grant PQ 301294/2004-6 and FAPESP projects 04/14107-2 and 03/00312–0.

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