The Unrestricted Combination of Temporal Logic Systems

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Abstract

This paper generalises and complements the work on combining temporal logics started by Finger and Gabbay [11, 10]. We present proofs of transference of soundness, completeness and decidability for the temporalisation of logics $\mathsf{T}(\mathsf{L})$ for any flow of time, eliminating the original restriction that required linear time for the transference of those properties through logic combination. We also generalise such results to the external application of a multi-modal system containing any number of connectives with arbitrary arity, that respect normality.

This generalisation over generic flows of time propagates to other combinations of logics that can be interpreted in terms of temporalisations. In this way, the *independent combination* (also called *fusion*) of temporal logics is studied over generic flows of time. We show the transfer of soundness, completeness and decidability for independent combination of temporal logics. Finally, we also discuss the independent combination of any finite number of normal multi-modal logics.

Keywords: Temporal Logics, Combinations of Logical Systems, Completeness of Combination of Logical Systems, Decidability of Combination of Logical Systems.

1 Introduction

This paper is concerned with the study of methods for combining temporal logics. In its first part, we extend the study of the *temporalisation* of logic systems introduced by Finger and Gabbay in [11]. There, the temporalisation process was restricted to linear flows of time. Here, we aim to generalise it to any flow of time. We are interested in studying the *transference of properties* from the logic system L into its temporalised version $\mathsf{T}(\mathsf{L})$. In the case of linear flows of time, temporalisation was shown to be a useful building block in obtaining the *independent combination* of two temporal logics [10]. In the second part of this paper, we show that the same construction is applicable to any class of flows of time.

The logic system $\mathsf{T}(\mathsf{L})$ combines two logics: a temporal logic T, which is applied *externally* to a given logic system L. This combination process, called *temporalisation*, involves the combination of the languages, inference systems, and model structures of T and L into a language, inference system and model structure of $\mathsf{T}(\mathsf{L})$. We show that if the logic systems T and L are sound, complete or decidable, then $\mathsf{T}(\mathsf{L})$ is also sound, complete or decidable; no constraints are imposed on the nature of the flow of time.

To show the transference of completeness and decidability via temporalisation, we maintain the same general proof strategy of Finger and Gabbay [11]. However, because here we can no longer rely on the linearity of the flow of time in T, the underlying proof construction has to be almost fully reworked in Section 2.1. For that, we introduce a bound associated to the number of steps in "the past" and the number of steps in "the future" one must take to evaluate a given formula ψ in a temporalised model. This construction allows us to select the "relevant" time points in the evaluation of a formula. As is explained in Section 2.2, the set of "relevant" time points may be infinite, but each point can be reached in finitely many steps. This construction allows us to do without the original restriction of linearity. Our approach naturally leads us to decision procedures. In Section 2.3 we show that provided that L and T are decidable, so is $\mathsf{T}(\mathsf{L})$.

We then use these transference results as a building block in the transference of similar properties for the *independent combination* of two temporal logics (also called *fusion* of temporal logics) over any class of flows of time. Section 3 shows that the transference of completeness and decidability can be obtained in terms of unions of alternating temporalisation of two temporal logics; furthermore, we show that such transference occurs even in temporal logics containing the highly expressive binary temporal operators "until" and "since". The mere temporalisation of two US-logics gives us a very limited logic, $\mathsf{US}_1(\mathsf{US}_2)$, which does not allow the nesting of US_2-operators inside the temporal operators of US_1; the independent combination $\mathsf{US}_1 \oplus \mathsf{US}_2$ does allow for any nesting of temporal operators. We explore a property that was initially noted in [10], namely that the *independent combination* $\mathsf{US}_1 \oplus \mathsf{US}_2$ can be seen as the infinite union of several temporalisations $\mathsf{US}_1, \mathsf{US}_1(\mathsf{US}_2)$ and $\mathsf{US}_1(\mathsf{US}_2(\mathsf{US}_1)), \ldots$, and thus we show how the temporalisation results can be employed to obtain the transference of soundness, completeness and decidability for $\mathsf{US}_1 \oplus \mathsf{US}_2$ over generic flows of time.

Combination of logics have been previously analysed in the literature. The first property of independently combined modal logics, namely its conservativity, was presented by Thomason in [25]. Fine and Schurz [7] and Kracht and Wolter [22] have studied the transfer properties of systematically combining independently axiomatisable monomodal systems. The work of Fine and Schurz [7] is applicable to more than two independent normal modalities. A generalisation of such results for many-place multi-modal systems is presented by Wolter in [28]; we discuss in more detail some of Wolter's results in Section 3.

Finger and Gabbay [11, 10] were the first to address the issue of combining logics with two-place modalities, S ("since") and U ("until"), and with modalities that were not all independent, for "since" and "until" interact with each other. The results of [11, 10], however, are restricted to the case of linear flows of time and, because non-linear flows, e.g. over trees or over some other partially ordered sets, often appear in Mathematics as well as in Computer Science, our approach is needed.

1.1 Applications of combinations of temporal logics

Since it was initially proposed, temporalisation has been applied in several systems. Its initial application was the description of the evolution of a temporal database [8, 9], which needed two temporal references, one external (evolution) and one internal (the actual temporal database). The two-dimensional view of temporal database evolution is detailed in [15].

In temporal databases, time is generally considered to be linear, which explains the initial focusing on linear flows of time only. Also, linearity simplified the proofs of transference of completeness and decidability, for on a linear time one is allowed to express that "a formula A holds at all times".

Another application of linear temporalisation, involving two-dimensional time, is the work on temporal logic programming within the paradigm of *imperative future* [19, 3]. Such paradigm permitted both the specification of formally verifiable programs as well as the execution of such specifications as a temporal logic program. Its original formulation involved only one temporal dimension, which meant that no update on past states could be done, ie no temporal reasoning was carried on the program itself. To deal with temporal programs, the imperative dimension was applied externally to a system, generating a temporalised two-dimensional version of the imperative future in [12, 13].

Besides temporal databases and software specification, temporalisation was applied in the combination of grammar logics in the work of Blackburn *et al.* [4]. Here, however, the limitations of linearity started to show and the use of temporalisation for grammar formalisms lost preference in the face of other formalisms. Still in the realm of grammar formalisms, Blackburn's and Meyer-Viol's *tree logics* [5] is one possible formalism that can be externally applied to other logics with the results below, but not with the linearity restriction.

More recently, the work of Montanari and Franceschet [17, 16] on the study of structures representing time with multiple granularities has shown that temporalisation can be used to generate logics for several classes of granular structures, even with the linear restriction. It would be interesting to see if new logics would arise if we have more flexibility on the structure of the external flow of time.

As for the independent combination (or fusion) of two modal/temporal logics, several applications arise when combining two logics, the most common of which are the combinations of temporal and knowledge logics for the specification of algorithms and protocols, which are best described in [6]. Of course, the logics needed for practical purposes usually demand a stronger interaction between the component logics than that provided by the independent combination. So the logic obtained by the independent combination is in a sense a *minimal* logic and the addition of further properties and stronger interaction has to be analysed separately. This fact has been noted already in the first works of fusion of logics in [7, 22].

1.2 The organisation of this paper

This paper addresses several generalisations. As described above, we aim to generalise the notions of temporalisation and independent combination for generic flows of time. Once such generalisations are done, it is normal to ask if these methods also apply to multi-modal modal and temporal logics, where the connectives may have arbitrary arity. It is our aim here to show how the methods applied here can be extended to this generalised case.

The rest of this paper is organised as follows. Extended temporalisations are studied in Section 2. The basic notions are initially described in Section 2.1, and transference of soundness, completeness and decidability is proven in Sections 2.2 and 2.3. In Section 2.4 it is explained why and how those results are applicable to iterated temporalisations, as a prelude to the analysis of the independent combination in Section 3. Those results all concern temporal US-logics, and generalising them form multi-modal logics is the aim of Section 2.5.

The study of the independent combination of temporal logics starts with general definitions in 3 and the transference of basic properties in Sections 3.2, 3.3 and 3.4. Finally, the generalisation of the independent combination of any number of multi-modal logics is discussed in 3.5.

Section 4 concludes with a discussion on the relationship between the constructions of Sections 2.1 and 3, and an open problem is reported.

2 The temporalised system $\mathsf{T}(\mathsf{L})$

2.1 Definition of a temporalised system

In this section we describe the system $\mathsf{T}(\mathsf{L})$ introduced in [11]. By a *logic system* we mean a *quadruple* $S = (\mathcal{L}_S, \vdash_S, \mathcal{K}_S, \models_S)$, where $\mathcal{L}_S$ is the system's language, $\vdash_S$ is an inference system, $\mathcal{K}_S$ is a class of models for the system and $\models_S \subseteq \mathcal{K}_S \times \mathcal{L}_S$ is a semantic relation such that $M \models_S A$ means that the formula $A \in \mathcal{L}_S$ is satisfied in the model $M \in \mathcal{K}_S$.

The language of $\mathsf{T}(\mathsf{L})$

The language $\mathcal{L}_{\mathsf{US}}$ of the temporal system T is built from a denumerable set of atoms $\mathcal{A}$, applying the two-place modalities U (*until) and S, (*since), and the Boolean connectives $\neg$ (*negation*) and $\wedge$ (*conjunction*).

Very little is required of the internal logic L, except that its language is described from a denumerable set of *atoms* and that it has the classical Boolean connectives $\neg$ and $\wedge$. We also demand that the connectives of T and L be disjunct. Apart from that, any other type of modalities or predicates are accepted in the language.

Before we define the language of the *temporalised system* $\mathsf{T}(\mathsf{L})$ we need to introduce a few definitions.

The language $\mathcal{L}_{\mathsf{L}}$ of L is partitioned into the sets BC_{L} and ML_{L}, where:

- BC_{L}, the set of *Boolean combinations* consists of the formulas built up from *any* other formulas with the use of the Boolean connectives $\neg$ or $\wedge$;
- ML_{L}, the set of *monolithic formulas* is the complementary set of BC_{L} in $\mathcal{L}_{\mathsf{L}}$.

If the external logic L does not contain the classical connectives $\neg$ and $\wedge$, we assume that $ML_{\mathsf{L}} = \mathcal{L}_{\mathsf{L}}$ and $BC_{\mathsf{L}} = \emptyset$, so every formula in L is considered monolithic.

The set of temporalised formulas, $\mathcal{L}_{\mathsf{T}(\mathsf{L})}$, is defined as the smallest set closed under the rules

1. If $A \in ML_{\mathsf{L}}$, then $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$;
2. If $A, B \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, then $\neg A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$ and $A \wedge B \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$;
3. If $A, B \in ML_{\mathsf{L}}$, then $S(A, B) \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$ and $U(A, B) \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$.

We say that a formula in $\mathcal{L}_{\mathsf{T}(\mathsf{L})}$ is *monolithic* if it is a formula that is in the language of L that is monolithic in L.

Note that the atoms of $\mathcal{L}_{\mathsf{US}}$ are not elements of $\mathcal{L}_{\mathsf{T}(\mathsf{L})}$. As an example of a temporalised language, consider the atoms $p, q \in \mathcal{L}_{\mathsf{L}}$ and $\Box$ is a modal symbol in L, then $\Box p$ and $\Box(p \wedge q)$ are monolithic formulas whereas $\neg\Box p$ and $\Box p \wedge \Box q$ are two Boolean combinations.

The *mirror image* of a given formula is given by replacing U by S and vice-versa. We will use the connectives $\vee$ and $\rightarrow$ and the constants $\top$ and $\bot$ in its usual meaning. Also, the formulas PA, FA, GA and HA abbreviate respectively $S(A, \top)$, $U(A, \top)$, $\neg F(\neg A)$ and $\neg P(\neg A)$. The *complexity* of a formula A is the cardinality of its subformulas.

The semantics of $\mathsf{T}(\mathsf{L})$

A *flow of time* is a pair $(T, <)$ where T is a set of time points and $<$ is a binary relation on T. By imposing restrictions on $<$ we generate *classes of flows of time*, e.g. the class $\mathcal{K}_{lin}$ of all transitive, irreflexive and linear flows of time.

When dealing with a simple temporal logic, a model is a triple $(T, <, h)$, where $(T, <)$ is a flow of time and $h : T \rightarrow 2^{\mathcal{P}}$ is a mapping that associates every time point $t \in T$ with a set of propositions, namely with the set of propositions that are true at that point. If we restrict the class of flows of time to $\mathcal{K}'$, we also restrict the class of models; it is usual practice to also call this class of models $\mathcal{K}'$, leaving the context to disambiguate whether we mean the class of flows of time or the class of models.

This definition of temporal model indicates that every time point is mapped into a classical propositional model, and such a view will be generalised in the temporalised case.

For that, we have to specify some restrictions on the semantic relation $\models_L$ for the logic L, whose class of models will be called $\mathcal{K}_L$. The basic restriction imposes that, for each $\mathcal{M} \in \mathcal{K}_L$ and $A \in \mathcal{L}_L$ we have

$$\text{either } \mathcal{M} \models A \text{ or } \mathcal{M} \models \neg A. \quad (*)$$

This may need some adaptation on the notion of class of model. For instance, if L is modal logic $S5$, it is not the case that, for each Kripke frame (W, R) where R is an equivalence relation and every modal valuation V, either $W, R, V \models A$ or $W, R, V \models \neg A$. However, this problem is solved if we consider as the class of models the set of pairs $\langle \mathcal{M}, w \rangle$, where $\mathcal{M}$ is an $S5$ model (W, R, V) and $w \in W$. For that class of models the property $(*)$ above holds.

Let $(T, <)$ be a flow of time and let g be a mapping from T into $\mathcal{K}_L$, such that $(*)$ holds for $g(t)$, for all $t \in T$. A triple $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, g)$ is a *temporalised model* of $\mathsf{T}(\mathsf{L})$. We say that a temporal model $(T, <, g)$ belongs to a class $\mathcal{K}$ iff $(T, <) \in \mathcal{K}$. In general, we will use the term temporalised model to refer to a model of $\mathsf{T}(\mathsf{L})$ and temporal model to refer to a model of T.

The satisfaction relation $\models$ is defined recursively over structure of temporalised formulas:

1. $\mathcal{M}_{\mathsf{T(L)}}, t \models A$, $A \in ML_{\mathsf{L}}$, iff $g(t) = \mathcal{M}_L$ and $\mathcal{M}_L \models A$;
2. $\mathcal{M}_{\mathsf{T(L)}}, t \models \neg A$ iff $\mathcal{M}_{\mathsf{T(L)}}, t \not\models A$;
3. $\mathcal{M}_{\mathsf{T(L)}}, t \models A \wedge B$ iff $\mathcal{M}_{\mathsf{T(L)}}, t \models A$ and $\mathcal{M}_{\mathsf{T(L)}}, t \models B$;
4. $\mathcal{M}_{\mathsf{T(L)}}, t \models S(A, B)$ iff there exists $s < t$ such that $\mathcal{M}_{\mathsf{T(L)}}, s \models A$ and for all r, $s < r < t$, $\mathcal{M}_{\mathsf{T(L)}}, r \models B$;
5. $\mathcal{M}_{\mathsf{T(L)}}, t \models U(A, B)$ iff there exists $t < s$ such that $\mathcal{M}_{\mathsf{T(L)}}, s \models A$ and for all r, $t < r < s$, $\mathcal{M}_{\mathsf{T(L)}}, r \models B$.

A formula is *valid* in a class $\mathcal{K}$ if it is verified at all times at all models over that class.

The inference system of T(L)

We assume that an *inference system* for a generic logic system is a mechanism capable of recursively enumerating the set of all provable formulas of the system, here called *theorems* of the logic system.

An inference system is *sound* with respect to a class of models $\mathcal{C}$ if all its theorems are valid over $\mathcal{C}$. Conversely, it is *complete* if all valid formulas are theorems. We assume that L's inference system is sound and complete.

We will assume that the temporal logic T's inference system is given in an axiomatic form, consisting of a set of *axioms* and a set of inference rules. For example, consider the possible axiomatisations of US over several classes of flows of time presented in [29] or in [20]. When a temporal logic T is sound and complete over the class $\mathcal{K}$ of flows, we write $\mathsf{T}/\mathcal{K}$.

Given $\mathsf{T}/\mathcal{K}$, the inference system of T(L) is denoted by $\mathsf{T(L)}/\mathcal{K}$ and consists of the following elements:

- The axioms of $\mathsf{T}/\mathcal{K}$;
- The inference rules of $\mathsf{T}/\mathcal{K}$;
- The inference rule *Preserve*: For every formula φ in $\mathcal{L}_{\mathsf{L}}$, if $\vdash_L \varphi$ then $\vdash_{T(L)} \varphi$.

In [11] it is shown that if $\mathsf{T}/\mathcal{K}$ and L have a sound inference system, then the inference system of $\mathsf{T(L)}/\mathcal{K}$ is sound; no extra restrictions are made on the nature of $\mathcal{K}$. Also, in case L has a complete inference system and $\mathcal{K}_{lin}$ is a class of linear flows of time, then the inference system of if $\mathsf{T(L)}/\mathcal{K}_{lin}$ is complete. We want to eliminate this restriction on linearity.

2.2 Completeness of T(L)

To show the transference of completeness we maintain the same proof strategy of [11], but we introduce a new technique and rework its underlying constructions. In the presence of linearity, one can write a formula that expresses the fact that a formula A "is true everywhere" in a model. This simplifies life a lot, but cannot be reproduced in a generic model. So we introduce a technique that picks up the "relevant" worlds in a model for the evaluation of a given formula, and we construct a formula that forces A to be true over all such relevant worlds.

The strategy of the proof is illustrated in Figure 1. We start with a consistent $\mathcal{L}_{\mathsf{T(L)}}$-formula φ, translate it into a pure $\mathcal{L}_{\mathsf{US}}$-temporal logic consistent formula A; then

FIG. 1. Completeness proof strategy

completeness of $\mathcal{L}_{\mathsf{US}}/\mathcal{K}$ gives us a model for A; after some model manipulation using the completeness of L, we obtain a $\mathsf{T(L)}/\mathcal{K}$-model for φ, thus deriving the completeness for $\mathsf{T(L)}/\mathcal{K}$. The more sophisticated bit of the proof is the initial translation step, which in the generic case has to deal with the nesting of temporal operators in φ instead of the simpler translation used for the linear case. Such initial elaboration allows us later to do a straightforward model manipulation to construct a model for φ.

To deal with the nesting of temporal operators in a formula, we define the *operator nesting tree* of a temporal or temporalised formula ψ, D_ψ. A tree is represented here as a set of strings of 0's and 1's, with the symbol $*$ representing concatenation of strings; the empty string is represented by ε. The tree is closed under prefix formation of its strings, that is, if $101 \in D_\psi$, then $\varepsilon, 1, 10 \in D_\psi$ as well. The 0 represents a past operator (a step to the past) and the 1 represents a future operator (or a step to the future).

Notation 2.1 In the following we will use the Greek letters φ, ψ and χ to indicate T(L) formulas, and the letters A, B and C to indicate temporal US formulas. We use the Greek letters φ, ψ and χ also to refer to either a temporal or temporalised formula.

Definition 2.2 Given a formula $\psi \in \mathcal{L}_{\mathsf{US}} \cup \mathcal{L}_{\mathsf{T(L)}}$ we build its *operator nesting tree* D_ψ recursively over the structure of ψ:

1. If ψ is a literal or monolithic, then $D_\psi = \{\varepsilon\}$;
2. If $\psi = \varphi_1 \wedge \varphi_2$, then $D_\psi = D_{\varphi_1} \cup D_{\varphi_2}$;
3. If $\psi = \neg\varphi$, then $D_\psi = D_\varphi$;
4. If $\psi = S(\varphi_1, \varphi_2)$, then $D_\psi = \{\varepsilon\} \cup \{0 * s | s \in D_{\varphi_1} \cup D_{\varphi_2}\}$;
5. If $\psi = U(\varphi_1, \varphi_2)$, then $D_\psi = \{\varepsilon\} \cup \{1 * s | s \in D_{\varphi_1} \cup D_{\varphi_2}\}$.

This definition implies that $\varepsilon \in D_\psi$ for any ψ and, as a consequence, the prefix of any string in D_ψ will also be a member of D_ψ. For example, consider the US formula

$$A = S(U(p, S(p, q)), S(p, p)) \wedge U(\neg U(p, q), S(p, q))$$

It's associated operator nesting tree will be:

$D_A = D_{S(U(p,S(p,q)),S(p,p))} \cup D_{U(\neg U(p,q),S(p,q))}$,
$D_A = \{\varepsilon\} \cup \{0 * s | s \in D_{U(p,S(p,q))} \cup D_{S(p,p)}\} \cup \{1 * r | r \in D_{U(p,q)} \cup D_{S(p,q)}\}$,
$D_A = \{\varepsilon, 0, 1\} \cup \{01 * s' | s' \in D_p \cup D_{S(p,q)}\} \cup \{00 * s'' | s'' \in D_p\} \cup$
$\{11 * r' | r' \in D_p \cup D_q\} \cup \{10 * r'' | r'' \in D_p \cup D_q\}$,
$D_A = \{\varepsilon, 0, 1, 01, 00, 11, 10\} \cup \{010 * s''' | s''' \in \cup D_p \cup D_q\}$,
$D_A = \{\varepsilon, 0, 1, 01, 00, 11, 10, 010\}$.

Let 1^m represent a string of m 1's, and similarly for 0^m. Let 0^0 and 1^0 represent the empty string. So each string in the nesting operator can be represented as $1^{m_1}0^{m_2}\ldots1^{m_{n-1}}0^{m_n}$, where all $m_i > 0$, except for m_1 and m_n, that can be 0. Note that n is always an even number.

Each such string is then associated to a temporal operator over H and G. Let $H^0\psi = G^0\psi = \psi$; let $G^{n+1}\psi = G(G^n\psi)$; and $H^{n+1}\psi = H(H^n\psi)$. So each string $1^{m_1}0^{m_2}\ldots1^{m_{n-1}}0^{m_n}$ is associated with the temporal operator $G^{m_1}H^{m_2}\ldots G^{m_{n-1}}H^{m_n}$, which we abbreviate as $\Box_{m_1,m_2,\ldots,m_{n-1},m_n}$.

As an example, $\Box_{0,2}(\Box_{0,3,1,0}\psi) \equiv \Box_{0,5,1,0}\psi$ instead of $\Box_{0,2,0,3,1,0}\psi$.

We can now start defining the translation of *consistent* formulas in $\mathsf{T(L)}$ into *consistent* formulas in US. The first step is the *correspondence mapping*.

Definition 2.3 Let $\{p_1, p_2, \ldots\}$ be an enumeration of the set of atoms of US, and let $\{\psi_1, \psi_2, \ldots\}$ be an enumeration of ML_L, the set of monolithic formulas of $\mathsf{T(L)}$. Define the *correspondence mapping* σ from $\mathcal{L}_{\mathsf{T(L)}}$ into $\mathcal{L}_{\mathsf{US}}$, inductively over a formula as:

$$\begin{array}{rcl}(\forall\psi_i \in ML_\mathsf{L})(\sigma(\psi_i)) &=& p_i \\ \sigma(\neg\chi) &=& \neg\sigma(\chi) \\ \sigma(\chi_1 \wedge \chi_2) &=& \sigma(\chi_1) \wedge \sigma(\chi_2) \\ \sigma(S(\chi_1, \chi_2)) &=& S(\sigma(\chi_1), \sigma(\chi_2)) \\ \sigma(U(\chi_1, \chi_2)) &=& U(\sigma(\chi_1), \sigma(\chi_2))\end{array}$$

The following two lemmas are shown in [11]:

Lemma 2.4 (The correspondence Lemma) The correspondence mapping σ is a bijection.

Lemma 2.5 For all $\mathsf{T(L)}$-consistent $\chi \in \mathcal{L}_{\mathsf{T(L)}}$, $\sigma(\chi)$ is US-consistent.

The reverse of Lemma 2.5 is not true, as we can see in this example:

Example 2.6 In a modal normal logic with the modality $\Box$, for all atoms φ, ψ,

$$\chi \equiv \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

is a theorem in L. The formulas $\Box(\varphi \to \psi)$, $\Box\varphi$ and $\Box\psi$ are monolithic, so they are mapped by σ into some atoms of US, say p_1, p_2 and p_3, respectively.

Thus, $\sigma(\chi) = p_1 \to (p_2 \to p_3)$, that is not a theorem in T.

For the model manipulation in the final part of the proof of completeness, we will need also the converse of Lemma 2.5, that is, $\mathsf{T(L)}$ theorems must be mapped into US theorems. To achieve that, we define the transformation $\eta(\psi)$, which makes use of the operator nesting tree D_ψ, and preserves ψ's consistency; such transformation will guarantee that $\mathsf{T(L)}$-theorems are mapped into US-theorems.

Definition 2.7 Given two formulas $\varphi, \psi \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, define:

1. $Mon(\varphi)$ is the set of monolithic subformulas of φ.
2. $Lit(\varphi) = Mon(\varphi) \cup \{\neg\psi | \psi \in Mon(\varphi)\}$;
3. $Inc(\varphi) = \{\bigwedge F | F \subseteq Lit(\varphi) \text{ and } F \vdash_{\mathsf{L}} \bot\}$; that is $Inc(\varphi)$ is the set of L-inconsistent formulas that can be built using the monolithic subformulas of φ;
4. $\Box_\varphi \psi$ is the conjunction of all formulas of the form $\Box_{m_1,\ldots,m_n}\psi$ such that $\Box_{m_1,\ldots,m_n}$ is a temporal operator associated to a string in the operator nesting tree D_φ;
5. $\eta(\varphi) = \bigwedge\{\Box_\varphi \neg\psi | \psi \in Inc(\varphi)\}$.

Example 2.8 If $\varphi = S(p, q)$, then $D_\varphi = \{\varepsilon, 0\}$. So, for any formula ψ, $\Box_\varphi \psi = \Box_{0,0}\psi \wedge \Box_{0,1}\psi = \psi \wedge H\psi$.

The terminology used in Definition 2.7 was introduced in [11]. The modification for the general case we had to make here is restricted to the definition of $\Box_\varphi \psi$ (used in the definitions of $\eta(\varphi)$).

The following Lemma is an adaptation of [11] for a the case of generic flows of time.

Lemma 2.9 $\vdash_{\mathsf{T}(\mathsf{L})} \eta(\psi)$.

PROOF. Every formula φ in $Inc(\psi)$ is a contradiction, and therefore its negation is a theorem of $\mathsf{T}(\mathsf{L})$. Now, if $\neg\varphi$ is a theorem, so are $H\neg\varphi$ and $G\neg\varphi$; by induction we get that $\Box_{m_1,\ldots,m_n}\neg\varphi$ is a theorem too, for any $m_1, \ldots, m_n$. ∎

Using Lemmas 2.5 and 2.9, we have that if ψ is $\mathsf{T}(\mathsf{L})$-consistent, then $\sigma(\psi \wedge \eta(\psi))$ is US-consistent. We can apply completeness of $\mathsf{US}/\mathcal{K}$ and obtain a US-model $\mathcal{M}_{\mathsf{US}}$ for $\sigma(\psi \wedge \eta(\psi))$ over $\mathcal{K}$. Furthermore, the theoremhood of the monolithic L-formulas in ψ is captured in $\eta(\psi)$ and will guarantee that its translation will be true in the "relevant part" of $\mathcal{M}_{\mathsf{US}}$. It is this notion of "relevant part" of a temporal model that we define next by associating subflows of time to binary trees (not very surprisingly). At this part of the proof we will be working at the US level.

Let $(T, <) \in \mathcal{K}$ be a flow of time, and let $t, s \in T$. We say that s is 1-related to t if $t < s$ (s is in the future of t); similarly, s is 0-related to t if $s < t$ (s is in the past of t). Let $t_1, \ldots, t_n \in T$ be a sequence of time points such that each pair t_i, t_{i+1} is 0- or 1-related. Such a sequence can then be associated to a string of 0's and 1's of length $n-1$, where the ith position is 1 if t_i and t_{i+1} are 1-related, and 0 otherwise; we represent it as $\mathsf{string}(t_1, \ldots, t_n)$.

The "relevant part" of a flow of time $(T, <)$, with respect to a temporal formula A at a point t, is formally defined as the *range of A at t over* $(T, <)$, $Rg(A, t)$:

$$Rg(A, t) = \{t\} \cup \{s \in T \mid \mathsf{string}(t, t_1, \ldots, t_n, s) \in D_A \text{ for some } t_1, \ldots, t_n \in T\}$$

Note that since $D_A = D_{\neg A}$, it follows that $Rg(A, t) = Rg(\neg A, t)$.

It is important to highlight that we are *not* constructing a submodel of a given model generated by $Rg(A, t)$. Our aim is to construct a model that belongs to a class $\mathcal{K}$. If we start in a model over $\mathcal{K}$ and generate a submodel based on $Rg(A, t)$, there is no way to guarantee that the generated submodel belongs to $\mathcal{K}$, and in general it does not. So $Rg(A, t)$ will be used to focus on a relevant part of the model. The satisfaction of a formula A at a point t in a temporal model depends only on the temporal valuation at points in $Rg(A, t)$, as shown below.

Lemma 2.10 Consider a temporal model $\mathcal{M} = (T, <, g)$, a formula $A \in \mathcal{L}_{\mathsf{US}}$, and a point $t \in T$. Then for any model $\mathcal{M}' = (T, <, g')$ such that $g'(s) = g(s)$ for every $s \in Rg(A, t)$,

$$\mathcal{M}, t \models A \textit{ iff } \mathcal{M}', t \models A.$$

Proof. Initially note that, both $\mathcal{M}$ and $\mathcal{M}'$ are based on the same flow of time, so for every subformula B of A and for every $s \in T$, $Rg(B, s)$ is the same set for both models. We proceed by structural induction over A.

- If A is atomic, then $g(t) = g'(t)$.
- If $A = \neg B$, then $Rg(A, t) = Rg(B, t)$, so the induction hypothesis directly gives us the result.
- If $A = B_1 \wedge B_2$, then $Rg(A, t) = Rg(B_1, t) \cup Rg(B_2, t)$, and therefore for every $s \in Rg(B_i, t)$, $g(t) = g'(t)$ $[i = 1, 2]$, so the induction hypothesis applies and gives us that $\mathcal{M}, t \models B_i$ *iff* $\mathcal{M}', t \models B_i$, from which the result follows immediately.
- If $A = S(B, C)$, then $\mathcal{M}, t \models A$ iff there exists a $t' < t$ with $\mathcal{M}, t' \models B$ and for every t'' such that $t' < t'' < t$, $\mathcal{M}, t'' \models C$. Note that both $t', t'' \in Rg(A, t)$. Furthermore, because the temporal nesting of B and C is smaller than that of A, we have $Rg(B, t') \subseteq Rg(A, t)$ and therefore $g(s) = g(s')$ for every $s \in Rg(B, t')$, so the induction hypothesis applies and yields $\mathcal{M}, t' \models B$ iff $\mathcal{M}', t' \models B$; analogously, we get that for every t'' such that $t' < t'' < t$, $\mathcal{M}, t'' \models C$ iff $\mathcal{M}', t' \models C$, and therefore the result follows.
- If $A = U(B, C)$ the reasoning is totally analogous to the previous case, finishing the proof.

∎

The following lemma shows that the definition of $\eta(\psi)$ preserves ψ's truth value over that "relevant part" of a model.

Lemma 2.11 Let $\mathcal{M}_{\mathsf{US}} = (T, <, g)$ be a temporal model over $\mathcal{K}$ and $\varphi, \psi \in \mathcal{L}_{\mathsf{T(L)}}$. Let $t \in T$ so that $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\Box_\varphi \psi)$. Then for every $s \in Rg(\sigma(\varphi), t)$, $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$.

Proof. We know that

$$\Box_\varphi \psi = \bigwedge_{1^{m_1} \ldots 0^{m_n} \in D_\varphi} \Box_{m_1, \ldots, m_n} \psi.$$

A simple induction shows that $D_\varphi = D_{\sigma(\varphi)}$, and therefore

$$\sigma(\Box_\varphi \psi) = \bigwedge_{1^{m_1} \ldots 0^{m_n} \in D_{\sigma(\varphi)}} \Box_{m_1, \ldots, m_n} \sigma(\psi).$$

Consider $s \in Rg(\sigma(\varphi), t)$. Then either $s = t$ or there are $t_1, \ldots, t_n \in Rg(\sigma(\varphi), t)$ such that $\mathsf{string}(t, t_1, \ldots, t_n, s) \in D_{\sigma(\varphi)}$. If $s = t$, since $\varepsilon \in D_{\sigma(\varphi)}$, it follows that $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$. In the latter case, we show the result by induction on n.

For $n = 0$, we have that either $s < t$, in which case we have that $\mathcal{M}_{\mathsf{US}}, t \models H\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$, or $t < s$, in which case we have that $\mathcal{M}_{\mathsf{US}}, t \models G\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$.

For the inductive case, we have that $\mathsf{string}(t, t_1, \ldots, t_n, s) \in D_{\sigma(\varphi)}$. Again we have two possibilities. If $t_n < s$ then the rightmost operator in $\Box_{m_1,\ldots,m_n}$ is a G, and the induction hypothesis gives us that $\mathcal{M}_{\mathsf{US}}, t_n \models G\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$. If $s < t_n$ then the rightmost operator in $\Box_{m_1,\ldots,m_n}$ is an H, and the induction hypothesis gives us that $\mathcal{M}_{\mathsf{US}}, t_n \models H\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$.

This finishes the induction and the proof. ∎

We can now finally glue the pieces of the completeness proof.

Theorem 2.12 If the logical system L is complete and US is complete over a class of flows of time $\mathcal{K}$, then the logical system $\mathsf{T}(\mathsf{L})$ is complete over $\mathcal{K}$.

Proof. Let ψ be a $\mathsf{T}(\mathsf{L})/\mathcal{K}$-consistent formula. We will construct a $\mathsf{T}(\mathsf{L})$-model for ψ over the class $\mathcal{K}$.

By Lemma 2.9, $\psi \wedge \eta(\psi)$ is also a $\mathsf{T}(\mathsf{L})$-consistent formula. So, by Lemma 2.5, $\sigma(\psi \wedge \eta(\psi))$ is a T-consistent formula. As we assume that $\mathsf{US}/\mathcal{K}$ is complete, then there exists a temporal model $\mathcal{M}_{\mathsf{US}} = (T, <, g)$ with $(T, <) \in \mathcal{K}$ such that for some $t \in T$, $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi \wedge \eta(\psi))$. For every $s \in Rg(\psi, t)$, define:

$$G_\psi(s) = \{\varphi \in Lit(\psi) | \mathcal{M}_{\mathsf{US}}, s \models \sigma(\varphi)\}$$

Claim: For every $s \in Rg(\psi, t)$, $G_\psi(s)$ is finite and L-consistent.

Indeed, $G_\psi(s)$ is finite because $Lit(\psi)$ is finite. To prove consistency, suppose by absurd that for some $s \in T$, $G_\psi(s)$ is L-inconsistent. Then there exists a subset of $G_\psi(s)$, $\{\varphi_1, \ldots, \varphi_n\}$ such that $\vdash_{\mathsf{L}} \bigwedge_{1 \le i \le n} \varphi_i \to \bot$. Thus $\bigwedge_{1 \le i \le n} \varphi_i \in Inc(\psi)$.

Let $\xi = \Box_\psi \sigma(\neg \bigwedge_{1 \le i \le n} \varphi_i)$. By the definition of η and σ, it follows that ξ is a conjunct of $\sigma(\psi \wedge \eta(\psi))$. From $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi \wedge \eta(\psi))$ it follows $\mathcal{M}_{\mathsf{US}}, t \models \xi$, so by Lemma 2.11 $\mathcal{M}_{\mathsf{US}}, s \models \neg\sigma(\bigwedge_{1 \le i \le n} \varphi_i)$. However, by the definition of $G_\psi(s)$ we have that $\mathcal{M}_{\mathsf{US}}, s \models \bigwedge_{1 \le i \le n} \sigma(\varphi_i) = \sigma(\bigwedge_{1 \le i \le n} \varphi_i)$, which is clearly a contradiction.

Therefore $G_\psi(s)$ is always L-consistent, proving the claim.

This claim is then used to build a model for ψ in the following way. By Lemma 2.11, for each $s \in Rg(\psi, t)$, $\mathcal{M}_{\mathsf{US}}, s \models \sigma(G_\psi(t))$. By hypothesis, L is complete, so for each $s \in Rg(\psi, t)$ there exists a model for the L-consistent set $G_\psi(s)$, $\mathcal{M}^s_{\mathsf{L}}$. So, we can define a valuation h as:

$$h(s) = \mathcal{M}^s_{\mathsf{L}}$$

for every $s \in Rg(\psi, t)$; for $s \in T - Rg(\psi, t)$, $h(s)$ can be any model of L.

Consider $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, h)$. To obtain completeness, all we have to do is to prove that $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models \psi$. First, note that for every $s \in Rg(\psi, t)$, and every monolithic subformula B of ψ, $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models \varphi$ iff $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\varphi)$. Then a straightforward structural induction on φ generalises this to show that $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models \psi$ iff $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi)$; details omitted.

But since we have that $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi)$, it follows that $\mathcal{M}_{\mathsf{T}(\mathsf{L})}$ is a temporalised model for ψ over $\mathcal{K}$, finishing the proof. ∎

The following is a nice consequence of soundness and completeness. Let $Th(\mathsf{L})$ be the set of all theorems of logic L. A logic L' is an *extension* of logic L if $Th(\mathsf{L}) \subseteq Th(\mathsf{L}')$. Furthermore, such an extension is said to be *conservative* if the language of L' is a superset of the language of L and for every formula A in the language of L, $A \in Th(\mathsf{L}')$ only if $A \in Th(\mathsf{L})$.

Lemma 2.13 If the systems US and L are sound and complete, with disjoint sets of connectives, then the temporalised system $\mathsf{US}(\mathsf{L})$ is a conservative extension of both US and L.

PROOF. It is obvious from the definition of $\mathsf{US}(\mathsf{L})$ that it is an extension of both US and L.

For conservativeness, suppose A is a formula of US that is a theorem of $\mathsf{US}(\mathsf{L})$. Suppose for contradiction that A is not a theorem of US, $\nvdash_{US} A$. Then, since US extends classical logic, we have that $\neg A$ is a consistent formula, and by completeness, we have a model $\mathcal{M}$ for $\neg A$. We construct a temporalised model $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, g)$ such that $g(t) = \mathcal{M}$ for every $t \in T$. Such a model is indeed $\mathsf{T}(\mathsf{L})$-countermodel of A, contradicting the soundness of $\mathsf{T}(\mathsf{L})$. So A is a theorem of US.

If A is a formula of L, because the sets of connectives are disjoint, the only way it can be deduced is via the inference rule of *Preserve*, in which case it clearly is a theorem of L. Which finishes the proof. ∎

Remark 2.14 Note that the last step of the proof above relies on the fact that the languages of L and US are *disjoint*. If there is a connective of L that also appears in US, the result above would not hold. This seems a vacuous assertion, since we have always assume the disjunction of the languages in this section, but it will be particularly important when we discuss the decomposition of the independent composition in several temporalisations.

2.3 Decidability of $\mathsf{T}(\mathsf{L})$

Transference of decidability is shown in [11] conditioned to the underlying flow of time being linear. We extend here that result to any class of flows of time. Recall that a given system L is *decidable* if for any formula $\psi \in \mathsf{L}$, there exists a procedure that outputs "yes" if ψ is a theorem and "no" otherwise. So, if L is complete then L is decidable if for any formula $\psi \in \mathsf{L}$, it is possible to decide whether ψ is valid or not.

Let us suppose that both the temporal system T and the external system L are decidable. We assume that both T and L are sound and complete. Then, $\mathsf{T}(\mathsf{L})$ is also sound and complete, decidability is obtainable if we can decide the validity of a $\mathsf{T}(\mathsf{L})$ formula ψ in any temporalised model.

The transference of decidability is obtained through a construction similar to that used for completeness. The definitions of $\eta(\psi)$ and the mapping σ are the same. We have:

Lemma 2.15 Let L and T be sound and complete systems. A formula ψ is $\mathsf{T}(\mathsf{L})$-valid iff $\sigma(\eta(\psi) \to \psi)$ is US-valid.

PROOF. If $\sigma(\eta(\psi) \to \psi)$ US-valid, by US-completeness it is also a theorem, then we can simply mimic the US-proof at the temporalised level, since all US axioms and inference rules are present at $\mathsf{T}(\mathsf{L})$, so $\eta(\psi) \to \psi$ is also a $\mathsf{T}(\mathsf{L})$-theorem. And since, by construction, $\eta(\psi)$ is always a theorem, so is ψ. By derived soundness, it is also $\mathsf{T}(\mathsf{L})$-valid.

Suppose by contradiction that ψ is $\mathsf{T}(\mathsf{L})$-valid and $\sigma(\eta(\psi) \to \psi)$ is not valid. From Lemmas 2.10 and 2.11 follows that if there was a countermodel for $\sigma(\eta(\psi) \to \psi)$, we would be able to construct a countermodel for $\eta(\psi) \to \psi$, and thus also a countermodel for ψ, which contradicts completeness. So $\sigma(\eta(\psi) \to \psi)$ must be US-valid. ∎

It is simple now to see the transference of decidability.

Theorem 2.16 If T and L are sound, complete and decidable, T(L) is decidable.

PROOF. From the definition of $\eta(\psi)$, if L is decidable then we have a direct way to construct $\eta(\psi)$. From Lemma 2.15, it follows that the decision of ψ is equivalent to the decision of $\sigma(\eta(\psi) \to \psi)$. Since such a formula is constructible, we can apply the decision procedure of T, thus deciding ψ. ∎

It is straightforward to show the following complexity result:

Lemma 2.17 Let N be the size of a formula, and let $\mathcal{O}(c_{\mathsf{L}}(N))$ and $\mathcal{O}(c_{\mathsf{US}}(N))$ be upper bounds of the complexity of the decision procedures of L and T, respectively. Then an upper bound of the complexity of the decision procedure for T(L) is $\mathcal{O}(c_{\mathsf{T}}(2^N) + 2^N \times c_{\mathsf{L}}(N))$.

PROOF. The complexity of the decision procedure for T(L) is divided in two parts. The first corresponds to the computation of $\eta(\psi)$, where N is the size of ψ. This this process consists of computing $Lit(\psi)$, which is $\mathcal{O}(N)$ and then testing all subset of it for inconsistency. Since there is $\mathcal{O}(2^N)$ potential subsets, this part has complexity $\mathcal{O}2^N \times c_{\mathsf{L}}(N))$.

The second part is to apply the decision procedure of T to $\sigma(\eta(\psi) \to \psi)$. The size of $\eta(\psi)$ is $\mathcal{O}(2^N)$, so this part has complexity $\mathcal{O}(c_{\mathsf{T}}(2^N))$, leading to the overall complexity $\mathcal{O}(c_{\mathsf{T}}(2^N) + 2^N \times c_{\mathsf{L}}(N))$.

∎

2.4 Iterated temporalisations

This section serves as a prelude to the independent combinations of logics to be presented in Section 3. Here we analyse what happens when we apply US to a logic L = T(US), where T may contain a renaming of U and S. This violates the initial assumption that the set of connectives from the external US and the internal L are disjoint, for U and S, without renaming, now appear in both systems.

To be a little bit more generic, let us consider a temporalisation $\mathsf{US}_1(\mathsf{L}_n)$, where $\mathsf{L}_n = \mathsf{US}_2(\mathsf{US}_1(\ldots))$ corresponds to n iterated temporalisations having US_2 as the outermost external application.

It is important to note, however, that the internal and external occurrence of the connectives obey the same logic rules (inference rules and semantics).

We now analyse how the results obtained so far can be brought to $\mathsf{US}_1(\mathsf{L}_n)$.

Completeness and decidability of $\mathsf{US}_1(\mathsf{L}_n)$

The main problem here is what is considered a monolithic subformula. For example, if we find a subformula of the format $G_1 p$ in a formula of $\mathsf{US}_1(\mathsf{L}_n)$, is it considered monolithic, for it is part of L_n, or is it considered part of US_1, and thus not monolithic?

This question affects the constructions of the proofs of completeness and decidability in that it affects is how to compute $\eta(\psi)$.

The answer to this problem is: for the purpose of computing $\eta(\psi)$ a monolithic subformula is any subformula that is a member of the language of L_n that is not a

Boolean combination. This is in the spirit of the way $\eta(\psi)$ was defined and preserves its properties for the case of $\mathsf{US_1(L_n)}$.

As defined for a normal temporalisation of $\mathsf{T(L)}$ where T and L have disjoint sets of operators,

$$\eta(\psi) = \bigwedge \{ \ \Box_\psi \neg\varphi | \varphi \in Inc(\psi) \ \}$$

where each φ consists of a Boolean combination of formulas of L only.

In $\mathsf{US_1(L_n)}$, φ may contain now subformulas that belong to the language of $\mathsf{US_1}$. This, however, does not pose a problem anywhere on the proofs. Let us clarify this point with an example.

Consider the following formula ψ in $\mathsf{US_1(US_2(US_1))}$[1].

$$\psi = G_1(p \wedge F_2H_2G_1\neg p)$$

Such a formula is inconsistent if $\mathsf{US_1}$ is complete in transitive flows of time with no end-points. In fact:

- $\vdash FHA \to A$ is a theorem of any temporal logic, so ψ implies $G_1(p \wedge G_1\neg p)$.
- By normality we get $G_1p \wedge G_1G_1\neg p$ and, by transitivity, we get $G_1G_1p \wedge G_1G_1\neg p$.
- By normality, this implies that $G_1G_1(p \wedge \neg p)$. which contradicts the no-endpoints property of $\mathsf{US_1}$.

One may, by mistake, construct a temporalised model for ψ, if one considers $Mon(\psi) = \{p, F_2H_2G_1\neg p, H_2G_1\neg p\}$, ignoring $G_1\neg p$. Indeed, in this case, $\eta(\psi) = \top$, so $\sigma(\eta(\psi) \wedge \psi)) = G_1(q_p \wedge q_{F_2H_2G_1\neg p})$, which clearly has a model. And since we can get models to $p \wedge F_2H_2G_1\neg p$, we have constructed a temporalised model to an inconsistent formula!

The problem with the construction above is that what we have to consider now as monolithic subformulas is the set.

$$Mon(\psi) = \{p, F_2H_2G_1\neg p, H_2G_1\neg p, G_1\neg p\}$$

With such $Mon(\psi)$, we see that $F_2H_2G_1\neg p$ and $\neg G_1\neg p$ contradict (because $F_2H_2A \to A$ is a theorem of any temporal logic). We see that a formula of $\mathsf{US_1}$, namely $G_1\neg p$ occurs in $\sigma(\eta(\psi) \wedge \psi))$. So among the conjuncts of $\sigma(\eta(\psi) \wedge \psi))$ we find:

$$G_1(q_p \wedge q_{F_2H_2G_1\neg p}), \qquad q_{F_2H_2G_1\neg p} \to G_1\neg q_p$$

The first is $\sigma(\psi)$ and the second is $\sigma(\Box^0(F_2H_2G_1\neg p \to G_1\neg p))$ These two formulas imply inconsistency, in the same pattern as we derived the inconsistency of ψ above, and by soundness of $\mathsf{US_1}$, no model can be built for it.

With this in mind, the proof of completeness of Section 2.2 applies immediately to $\mathsf{US_1(L_n)}$. This can be seen by an inspection on the proof. We see that nowhere else in that construction was it necessary to use the fact that the set of connectives of the external $\mathsf{US_1}$ and those of internal logic $\mathsf{L_n}$ is disjoint. In particular, any occurrence of a connective of $\mathsf{US_1}$ inside $\mathsf{L_n}$ always occurred within the scope of a $\mathsf{US_2}$ connective, avoiding any interaction between the external and internal occurrences of a $\mathsf{US_1}$ connective.

Similarly, the proof of decidability also applies to $\mathsf{US_1(L_n)}$.

[1] This example was suggested by Massimo Franceschet.

Conservativeness of $\mathsf{US_1(L_n)}$

The proof of Lemma 2.13, stating that US(L) is a conservative extension of both US and L uses the fact that the sets of connectives are disjoint.

However, in the case of $\mathsf{US_1(L_n)}$, because the internal and external occurrences of connectives of $\mathsf{US_1}$ respect the same semantic rules and inference rules, it is possible to adapt the of Lemma 2.13 to $\mathsf{US_1(L_n)}$. For that, we have to assume that $\mathsf{US_1}$ and $\mathsf{US_2}$ are sound and complete, which gives us the soundness of $\mathsf{L_n}$.

In fact, the interesting case arises when we prove A in $\mathsf{US_1(L_n)}$ and A is a formula of $\mathsf{L_n}$. Then we have to analyse two cases:

- If A is not in the language of $\mathsf{US_1}$, then the only way A could have been derived is by the use of *Preserve* rule. In which case A is also a theorem of $\mathsf{L_n}$.
- Suppose A is in the language of $\mathsf{US_1}$. This means that A is a pure $\mathsf{US_1}$ formula. As in Lemma 2.13, we show that A is a theorem of $\mathsf{US_1}$. Suppose for contradiction that it is not. Then $\neg A$ is a consistent formula, and by completeness of $\mathsf{US_1}$, $\neg A$ has a model $\mathcal{M}_A$. We construct a temporalised model $\mathcal{M}_{\mathsf{T(L)}} = (T, <, g)$ such that, for every $t \in T$, $g(t) = \mathcal{M}_A$, thus building a T(L)-countermodel for A, contradicting the soundness of T(L). So A is a theorem of US. But since the temporalisation is an extension of its components, any theorem of $\mathsf{US_1}$ is also a theorem of $\mathsf{L_n}$.

So if A is a theorem in $\mathsf{US_1(L_n)}$ that is in the language of $\mathsf{L_n}$, it is a theorem of $\mathsf{L_n}$. The second item above also showed that if A is in the language of $\mathsf{US_1}$, it is a theorem of $\mathsf{US_1}$. We have thus proved the following.

Lemma 2.18 If $\mathsf{US_1}$ and $\mathsf{US_2}$ are sound and complete, then $\mathsf{US_1(L_n)}$ is a conservative extension of both $\mathsf{US_1}$ and $\mathsf{L_n}$.

2.5 Extension to multimodal, non-temporal logics

Before we move to the independent combination of logics, we would like to discuss how the results presented so far generalise when in the external logic T the connectives may have any arity, and instead of only two (U and S) we may have n connectives $\triangle_1, \ldots, \triangle_n$, such that the arity of $\triangle_i$ is $r_i > 0$.

On the semantical side, we assume that each connective $\triangle_i$ is associated with a binary relation R_i. The semantics of formulas is based on a multidimensional frame $(W, R_1, \ldots, R_n)$. We have, however, to impose certain semantic restrictions:

- The semantics of $\triangle_i(p_1, \ldots, p_{r_i})$ is a monadic first-order formula (or connective *truth table* in the sense of [20, Chapter 8]) build from predicates $P_1(\cdot), \ldots, P_{r_i}(\cdot)$, the relational symbols $R_1, \ldots, R_n$, and equality.
- For each relational symbol R_i, we must be able to express a derived connective $\Box_i$ such that $\Box_i p$ expresses $\forall x(t\, R_i\, x \Rightarrow P(x))$. Furthermore, the inference system must be able to derive that if $\vdash A$ then $\vdash \Box_i A$, $1 \leq i \leq n$.

This second restriction correspond to the notion of *normality*. Note, however, that the demand of normality made here is weaker than that made in [28], for there it is required that *every argument position* in $\triangle_i(p_1, \ldots, p_{r_i})$ be normal, a requirement that even U and S fail to keep, for they are only normal in the first position and not in the second. A weaker requirement, however, can be found in [2].

Let us call the resulting system a *generalised modal/temporal logic*. The process of applying it externally to a logic L will be called a generalised *modalisation* of L.

In the case of US-temporal logic, each connective has arity 2, U is associates with binary relation $R_1 = <$ and S with $R_2 = >$; also, $\Box_1 = G$ and $\Box_2 = H$ are derivable connectives. The fact that R_1 and R_2 are related is no limitation for this setting. This setting allows for many well-known modal logics, including the branching time modalities in CTL and CTL* and multi-arity connectives. The restriction (*) for the internal logics, however, remains.

We can then examine how our proof of completeness can be adapted. The definition of the *operator nesting tree* D_ψ of a formula ψ is simply extended to :

1. If ψ is a literal or monolithic, then $D_\psi = \{\varepsilon\}$;
2. If $\psi = \varphi_1 \wedge \varphi_2$, then $D_\psi = D_{\varphi_1} \cup D_{\varphi_2}$;
3. If $\psi = \neg\varphi$, then $D_\psi = D_\varphi$;
4. If $\psi = \triangle_i(\varphi_1, \ldots, \varphi_{r_i})$, then $D_\psi = \{\varepsilon\} \cup \{i * s | s \in D_{\varphi_i} \cup \ldots \cup D_{\varphi_{r_i}}\}$.

This implies that the strings that compose our strings take as atoms the elements of the interval $[1, i]$ and that each node in the tree can be at most i-branching. A temporal operator can then be associated with each string in a straightforward way, that is, each $j \in [1, i]$ is associated with the derived operator $\Box_j$ and a string $j_1 \cdots j_p$ is associated with the string of connectives $\Box_{j_1} \cdots \Box_{j_p}$.

The definition of $\Box_\varphi\psi$ remains the same as before, namely the conjunction of all formulas of the form $\Box_{m_1,\ldots,m_n}\psi$ such that $\Box_{m_1,\ldots,m_n}$ is a temporal operator associated to a string in the operator nesting tree D_φ.

Given a multi-dimensional frame $(W, R_1, \ldots, R_n)$ and $t_1, \ldots t_m \in W$, such that t_k is related with t_{k+1} by some R_i, we represent by $\mathsf{string}(t_1, \ldots t_m)$ the string of length $m - 1$ obtained by a path through all those points.

Finally, the correspondence mapping σ can be modified, remaining a homomorphism, so as to deal with generic modalities of the form $\triangle_i(\varphi_1,\ldots,\varphi_{r_i})$:

$$\sigma\left(\ \triangle_i(\varphi_1,\ldots,\varphi_{r_i})\ \right) = \triangle_i\left(\ \sigma(\varphi_1),\ldots,\sigma(\varphi_{r_i})\ \right).$$

The monolithic and Boolean cases remain the same.

Given those constructions all others constructions remain exactly the same. In particular, this way preserves the central notion of $Rg(A, t)$ as the "relevant part" of a multi-dimensional frame $(W, R_1, \ldots, R_n)$ with respect to a formula A at a point $t \in W$, which plays a crucial role in the proof of transference of completeness. With such generalised construction, all lemmas and theorems are straightforwardly generalised and the transference of completeness and decidability follows for the temporalisation/modalisation of a logic with n connectives of arbitrary arity that respect the semantical restrictions above. The reader is invited to verify the details.

What deserves note is the fact that in our construction the fact that U and S are mirror images is taken care by the definition of $\eta(\psi)$. In the same way, in the generalised modalisation, if there is any iteration between the connective and their respective semantical relations, this remains hidden in the construction of $\eta(\psi)$, and the proof generalises smoothly. We can then state the following result.

Theorem 2.19 The properties of completeness and decidability are transferred via generalised modalisation/temporalisation.

3 The independent combination of temporal systems

Once we have generalised the transference results for the unrestricted temporalisation of a logic system over any class of flows of time, the next obvious question is whether such results generalise for the *independent combination* of two temporal logics.

Such an investigation was pursued for the linear case in [10], in which the transference of completeness was obtained by the "unravelling" of the independent combination in a finite number of temporalisations. Here we investigate if such technique is still applicable for the unrestricted case.

The work of Frank Wolter [28] on independent combination of logics (there called *fusion* of logics) is perhaps the work in the literature that more closely relates to the goals of the present work. That work explores the fusion of any number of logics containing any number of operators, of arbitrary arity. One restriction of such work was that each modality had to respect a restriction of normality in every argument, and it turns out the U and S do not respect such condition. Such a restriction was only eliminated as a side effect in a later work [2].

The present work compares with Wolter's in the following ways:

- We present a proof of transfer of decidability for US over any class of flows of time.
- Wolter's presentation is algebraic, while ours is based on Kripke semantics.
- Our construction shows how the independent combination can be seen as an infinite union of alternating temporalisations.

3.1 Definitions

We now deal with the independent combination of two temporal logic systems, US_1 and US_2. If we temporalise US_1 with US_2, we obtain a very weakly expressive system; in such a system, if US_1 is the internal temporal logic (F_1 is a derived connective in US_1), and US_2 is the external one (F_2 is also derived in US_2), we cannot express that vertical and horizontal future operators commute,

$$F_1 F_2 A \leftrightarrow F_2 F_1 A.$$

In fact, the subformula $F_1 F_2 A$ is not even in the temporalised language of $\mathsf{US}_2(\mathsf{US}_1)$, nor is the whole formula. In other words, the interplay between the two-dimensions is not expressible in the language of the temporalised $\mathsf{US}_2(\mathsf{US}_1)$.

The idea is then to define a method for combining temporal logics that is symmetric. As usual, we combine the languages, inference systems and classes of models.

Definition 3.1 Let $Op(\mathsf{T})$ be the set of non-Boolean operators of a generic temporal logic T. Let T_1 and T_2 be two temporal logic systems such that $Op(\mathsf{T}_1) \cap Op(\mathsf{T}_2) = \emptyset$. The *fully combined language* of logic systems T_1 and T_2 over the set of atomic propositions $\mathcal{P}$ is obtained by the union of the respective set of connectives and the union of the formation rules of the languages of both logic systems.

Let the operators U_1 and S_1 be in the language of US_1 and U_2 and S_2 be in that of US_2. Their fully combined language over a set of atomic propositions $\mathcal{P}$ is given by

- every atomic proposition is in it;
- if A, B are in it, so are $\neg A$ and $A \wedge B$;
- if A, B are in it, so are $U_1(A, B)$ and $S_1(A, B)$.
- if A, B are in it, so are $U_2(A, B)$ and $S_2(A, B)$.

The two languages taken to be independent of each other and the set of axioms of the two systems are supposed to be disjoint. The following combination method is the *independent combination* of two temporal logics. An axiomatisation is given by a pair $(\Sigma, \mathcal{I})$, where Σ is a set of axioms and $\mathcal{I}$ is a set of inference rules.

We have very few limitations on the axiomatisations, namely:

- US_1 and US_2 are extensions of classical logic, so classical manipulations are admissible in the system; ie. if they are not primitive, they can be derived.
- Because the we are assuming a Kripke-style semantics, the logics have to be normal. This means that the axioms of normality (ie, the K-axioms) must be derivable for G_1, H_1, G_2 and H_2.
- The rule of necessitation has to be admissible: from $\vdash A$ derive $\vdash G_1 A$, $\vdash H_1 A$, $\vdash G_2 A$ and $\vdash H_2 A$.

Note that, since the set of operators of the two logics is disjoint, the set of axioms and inference rules referring to those operators will be disjoint.

Definition 3.2 Let US_1 and US_2 be two US-temporal logic systems defined over the same set $\mathcal{P}$ of propositional atoms such that their languages are independent. The *independent combination* $\mathsf{US}_1 \oplus \mathsf{US}_2$ is given by the following:

- The fully combined language of US_1 and US_2.
- If $(\Sigma_1, \mathcal{I}_1)$ is an axiomatisation for US_1 and $(\Sigma_2, \mathcal{I}_2)$ is an axiomatisation for US_2, then $(\Sigma_1 \cup \Sigma_2, \mathcal{I}_1 \cup \mathcal{I}_2)$ is an axiomatisation for $\mathsf{US}_1 \oplus \mathsf{US}_2$.
- The class of independently combined flows of time is $\mathcal{K}_1 \oplus \mathcal{K}_2$ composed of bi-ordered flows of the form $(T, <_1, <_2)$ where the connected components of $(T, <_1)$ are in $\mathcal{K}_1$ and the connected components of $(T, <_2)$ are in $\mathcal{K}_2$, and T is the (not necessarily disjoint) union of the sets of time points that constitute each connected component.

  A model structure for $\mathsf{US}_1 \oplus \mathsf{US}_2$ over the combined class $\mathcal{K}_1 \oplus \mathcal{K}_2$ is a 4-tuple $(T, <_1, <_2, g)$, where $(T, <_1, <_2) \in \mathcal{K}_1 \oplus \mathcal{K}_2$ and g is an assignment function $g : T \to 2^{\mathcal{P}}$.
- The semantics of a formula A in a model $\mathcal{M} = (T, <_1, <_2, g)$ is defined as the union of the rules defining the semantics of $\mathsf{US}_1/\mathcal{K}_1$ and $\mathsf{US}_2/\mathcal{K}_2$. The expression $\mathcal{M}, t \models A$ reads that the formula A is true in the (combined) model $\mathcal{M}$ at the point $t \in T$. The semantics of formulas is given by induction in the standard way:

$\mathcal{M}, t \models p$ iff $p \in g(t)$ and $p \in \mathcal{P}$.
$\mathcal{M}, t \models \neg A$ iff it is not the case that $\mathcal{M}, t \models A$.
$\mathcal{M}, t \models A \wedge B$ iff $\mathcal{M}, t \models A$ and $\mathcal{M}, t \models B$.
For $i = 1, 2$:
$\mathcal{M}, t \models S_i(A, B)$ iff there exists an $s \in T$ with $s <_i t$ and $\mathcal{M}, s \models A$ and for every $u \in T$, if $s <_i u <_i t$ then $\mathcal{M}, u \models B$.
$\mathcal{M}, t \models U_i(A, B)$ iff there exists an $s \in T$ with $t <_i s$ and $\mathcal{M}, s \models A$ and for every $u \in T$, if $t <_i u <_i s$ then $\mathcal{M}, u \models B$.

The independent combination of two logics also appears in the literature under the names of *fusion* or *join*. The language of such a logic is referred to in the literature as a *two-dimensional* temporal language, even though its semantics is based on the evaluation of formulas at a single point (thus still one dimensional). The topic of two-dimensional modal/temporal languages and logics has been extensively discussed in the literature, e.g. [21, 23, 1, 26, 27, 20, 18].

We now proceed to examine the transference of properties through the independent combination.

3.2 Soundness of $\mathsf{T}_1 \oplus \mathsf{T}_2$

Before we show the transference of soundness, it is worth noting an early result by Thomason [25], which is indeed more general than the independent combination of two US-logics. This result is useful in the proof of both soundness and completeness.

Proposition 3.3 (Thomason [25]) With respect to the validity of formulas, the independent combination of two modal logics is a conservative extension of the original ones.

In algebraic presentations, Proposition 3.3 is considered a kind of soundness result. However, for our purposes, soundness has to do with the validity of all deductions. We present soundness as a consequence of Proposition 3.3, but it could also be obtained by verifying the validity of axioms and inference rules.

Theorem 3.4 (Soundness Transference) If $\mathsf{US}_1/\mathcal{K}_1$ and $\mathsf{US}_2/\mathcal{K}_2$ are sound logic systems, so is $\mathsf{US}_1 \oplus \mathsf{US}_2/\mathcal{K}_1 \oplus \mathcal{K}_2$.

PROOF. By induction of the length of a deduction. For the base case, we have to establish the validity of all axioms, which follows directly from the soundness of $\mathsf{US}_1/\mathcal{K}_1$ and $\mathsf{US}_2/\mathcal{K}_2$ and the fact that by Proposition 3.3, all $\mathsf{US}_1/\mathcal{K}_1$- and $\mathsf{US}_2/\mathcal{K}_2$-valid formulas are valid in the combined system (alternatively, their validity could be verified directly).

For the inductive case, all we are left to do is to verify that the inference rules transform valid formulas into valid formulas, which is a routine, straightforward task. ∎

3.3 Completeness

In the proof of completeness, just as in [10], we will use the temporalisation as an inductive step in the construction of a combined model. However, as discussed in the presentation of the semantics of temporalised logics in Section 2.1, the class of temporal models of the internal logic must also include the evaluation time point, so that a member of the class of models of US_1 or US_2 is a quadruple $(T, <, g, t)$, where $t \in T$.

Let us first define the *degree of alternation* of a $(\mathsf{US}_1 \oplus \mathsf{US}_2)$-formula A, $dg(A)$, as the maximum number of alternate times a connective of one of the temporal logics occurs inside a connective of the other temporal logic. In this way, formulas of US_1 and of US_2 all have degree of alternation 0. If we take a temporal formula of US_1, say $F_1 p$ and place it inside a connective of US_2, say H_2, the formula $H_2 F_1 p$ has degree 1; similarly, $U_1(H_2 F_1 p, q)$ has degree 2, and so on.

The main idea of the completeness proof is based on the fact that any formula A of $\mathsf{US}_1 \oplus \mathsf{US}_2$ can be seen as a formula of some finite number of alternating temporalisations of the form $\mathsf{US}_1(\mathsf{US}_2(\mathsf{US}_1(\ldots)))$; more precisely, A can be seen as a formula of $\mathsf{US}_1(\mathsf{L_n})$, where $dg(A) = n$, $\mathsf{US}_1(\mathsf{L_0}) = \mathsf{US}_1$, $\mathsf{US}_2(\mathsf{L_0}) = \mathsf{US}_2$, and $\mathsf{L}_{n-2i} = \mathsf{US}_2(\mathsf{L}_{n-2i-1})$, $\mathsf{L}_{n-2i-1} = \mathsf{US}_1(\mathsf{L}_{n-2i-2})$, for $i = 0, 1, \ldots, \lceil \frac{n}{2} \rceil - 1$.

The following Lemma actually allows us to obtain transference of completeness to the independent combination via finite number of alternating temporalisations of US_1 and US_2.

Lemma 3.5 Let US_1 and US_2 be sound and complete. A is a theorem of $\mathsf{US}_1 \oplus \mathsf{US}_2$ iff it is a theorem of $\mathsf{US}_1(\mathsf{L_n})$, where $dg(A) = n$.

Proof. If A is a theorem of $\mathsf{US}_1(\mathsf{L_n})$, all the inferences in its deduction can be repeated in $\mathsf{US}_1 \oplus \mathsf{US}_2$, so it is a theorem of $\mathsf{US}_1 \oplus \mathsf{US}_2$.

Suppose A is a theorem of $\mathsf{US}_1 \oplus \mathsf{US}_2$; let $B_1, \ldots, B_m = A$ be a deduction of A in $\mathsf{US}_1 \oplus \mathsf{US}_2$ and let $n' = max\{dg(B_i)\}$, $n' \geq n$. We claim that each B_i is a theorem of $\mathsf{US}_1(\mathsf{L_{n'}})$. In fact, by induction on m, if B_i is obtained in the deduction by substituting into an axiom, the same substitution can be done in $\mathsf{US}_1(\mathsf{L_{n'}})$; if B_i is obtained by some inference rule from $B_{j_1}, \ldots, B_{j_k}$, $j_1, \ldots, j_k < i$, then by the induction hypothesis, each B_{j_ℓ} is a theorem of $\mathsf{US}_1(\mathsf{L_{n'}})$ and so is B_i.

So A is a theorem of $\mathsf{US}_1(\mathsf{L_{n'}})$. It follows from the semantic definitions that the set of valid formulas in $\mathsf{US}_1(\mathsf{L_{n'}})$ is a subset of the valid formulas in $\mathsf{US}_1 \oplus \mathsf{US}_2$. Since US_1 and US_2 are two complete logic systems, by Theorem 2.12 we know that $\mathsf{US}_1(\mathsf{L_{n'}})$ is complete for each n'. So Lemma 2.18 yields that each of the alternating temporalisations in $\mathsf{US}_1(\mathsf{L_{n'}})$ is a conservative extension of $\mathsf{L_{n'}}$; it follows that A is a theorem of $\mathsf{US}_1(\mathsf{L_n})$, as desired. ∎

Theorem 3.6 (Completeness of $\mathsf{US}_1 \oplus \mathsf{US}_2$) Let $\mathsf{US}_1/\mathcal{K}_1$ and $\mathsf{US}_2/\mathcal{K}_2$ be two sound and complete logic systems. Then their independent combination $\mathsf{US}_1 \oplus \mathsf{US}_2$ is sound and complete over the class $\mathcal{K}_1 \oplus \mathcal{K}_2$.

Proof. Soundness is given by Theorem 3.4. For completeness, suppose that A is a consistent formula in $\mathsf{US}_1 \oplus \mathsf{US}_2$; by Lemma 3.5, A is consistent in $\mathsf{US}_1(\mathsf{L_n})$, so we construct a temporalised model for it, and we obtain a model $(T^1, <_1^1, g^1, o^1)$, where $o^1 \in T^1$ is the "current time" considered as part of a model to respect the restriction $(*)$ of Section 2.1. We show now how it can be transformed into a model over $\mathcal{K}_1 \oplus \mathcal{K}_2$.

Without loss of generality, suppose that $\mathsf{US_1}$ is the outermost logic system in the multi-layered temporalised system $\mathsf{US_1}(\mathsf{US_2}(\mathsf{US_1}(\ldots)))$, and let n be the number of alternations. The construction is recursive, starting with the outermost logic. Let $i \leq n$ denote the step of the construction; if i is odd, it is a $\mathsf{US_1}$-temporalisation, otherwise it is a $\mathsf{US_2}$-temporalisation. At every step i we construct the sets T^{i+1}, $<_1^{i+1}$ and $<_2^{i+1}$ and the function g^{i+1}.

We start the construction of the model at step $i = 0$ with the temporalised model $(T^1, <_1^1, g^1, o^1)$ such that $(T^1, <_1^1) \in \mathcal{K}_1$, and we take $<_2^1 = \emptyset$. At step $i < n$, consider the current set of time points T^i; according to the construction, each $t \in T^i$ is associated to:

- a temporalised model $g^i(t) = (T^{i+1}(t), <_1^{i+1}(t), g^{i+1}(t), o^{i+1}(t)) \in \mathcal{K}_1$ and take $<_2^{i+1}(t) = \emptyset$, if i is even; or
- a temporalised model $g^i(t) = (T^{i+1}(t), <_2^{i+1}(t), g^{i+1}(t), o^{i+1}(t)) \in \mathcal{K}_2$ and take $<_1^{i+1}(t) = \emptyset$, if i is odd.

The point t is made identical to $o^{i+1}(t) \in T^{i+1}(t)$, so as to add the new model to the current structure; note that this preserves the satisfiability of all formulae at t. Let T^{i+1} be the (possibly infinite) union of all $T^{i+1}(t)$ for $t \in T^i$; similarly, $<_1^{i+1}$ and $<_2^{i+1}$ are generated. And finally, for every $t \in T^{i+1}$, the function g^{i+1} is constructed as the union of all $g^{i+1}(t)$ for $t \in T^i$.

Repeating this construction n times, we obtain a combined model over $\mathcal{K}_1 \oplus \mathcal{K}_2$, $\mathcal{M} = (T^n, <_1^n, <_2^n, g^n)$, such that for all $t \in T^n$, $g^n(t) \subseteq \mathcal{P}$. Since satisfiability of formulae is preserved at each step, it follows that $\mathcal{M}$ is a model for A, and completeness is proved. ∎

3.4 Decidability

We are going to show the transference of decidability by a recursive application of the temporalisation, generalising the proof of decidability of $\mathsf{T(L)}$ in Section 2.3.

The idea of the recursive proof is to consider a formula ψ of the independent language $\mathsf{US_1} \oplus \mathsf{US_2}$ of alternation depth n as a temporalised formula $\mathsf{US(L_n)}$. By Lemma 3.5, ψ is a $\mathsf{US_1} \oplus \mathsf{US_2}$-theorem iff it is a $\mathsf{US(L_n)}$-theorem. Thus the decidability of ψ in $\mathsf{US_1} \oplus \mathsf{US_2}$ reduces to its decidability in $\mathsf{US(L_n)}$. The following is the basic result in the transference of decidability.

Lemma 3.7 Let $\mathsf{US_1}/\mathcal{K}_1$ and $\mathsf{US_2}/\mathcal{K}_2$ be two sound, complete and decidable temporal logics. Then for every formula ψ of $\mathsf{US_1} \oplus \mathsf{US_2}$, there exists a $\mathsf{US_1}$ formula A that is effectively constructible such that ψ is $\mathsf{US_1} \oplus \mathsf{US_2}$-valid iff A is $\mathsf{US_1}$-valid.

PROOF. Let ψ be a $\mathsf{US_1} \oplus \mathsf{US_2}$ formula of alternation depth n. We propose the following decision procedure, $\mathsf{US_1} \oplus \mathsf{US_2}$-Decide($\psi$):

> Let n be ψ's alternation degree. If $n = 0$, then ψ is a US-formula and we apply the $\mathsf{US_1}$- or $\mathsf{US_2}$-decision procedure to decide ψ, according to which language ψ belongs to.
>
> Otherwise, we construct the formula $\eta(\psi) \rightarrow \psi$ in the following way:
>
> - Let $Lit(\psi) = Mon(\psi) \cup \{\neg\phi | \phi \in Mon(\psi)\}$, where $Mon(\psi)$ is the set of monolithic subformulas of ψ.

- Let $Inc(\psi)$ be set of inconsistent conjunctions ϕ_i in $Lit(\psi)$; *this inconsistency is obtained by a recursive call to* $\mathsf{US_1} \oplus \mathsf{US_2}$-*Decide*($\phi_i$), where each ϕ_i now has alternation degree at most $n-1$.
- Build $\eta(\psi)$ from $Inc(\psi)$ as in Definition 2.7.

Apply $\mathsf{US_1}$-decision procedure to $\sigma(\eta(\psi) \to \psi)$ and return its output.

The recursive construction of $\eta(\psi)$ always terminates, for in each recursive call of the decision process, the degree of alternation decreases, and the procedure stops when it reaches a degree of alternation 0.

The correctness of the procedure is proven by induction on n. For $n = 0$ we simply apply the temporal decision procedure of the corresponding temporal logic.

For $n > 0$ we claim that deciding ϕ is equivalent to deciding $\sigma(\eta(\psi) \to \psi)$. In fact:

- ψ is $\mathsf{US_1} \oplus \mathsf{US_2}$-valid iff it is $\mathsf{US(L_n)}$-valid by completeness and Lemma 3.5.
- ψ is $\mathsf{US(L_n)}$-valid iff $\sigma(\eta(\psi) \to \psi)$ is $\mathsf{US_1}$-valid by Lemma 2.15 and $\eta(\psi)$ is constructed deciding the validity of a set of formulas with alternation degree at most $n-1$, so by induction hypothesis $\eta(\psi)$ is constructible.

Thus we have a correct, terminating decision procedure for $\mathsf{US_1} \oplus \mathsf{US_2}$. ∎

The transference of decidability directly follows from the previous Lemma.

Theorem 3.8 Let $\mathsf{US_1}/\mathcal{K}_1$ and $\mathsf{US_2}/\mathcal{K}_2$ be two sound, complete and decidable temporal logics. Then $\mathsf{US_1} \oplus \mathsf{US_2}$ is decidable.

PROOF. Let ψ be a $\mathsf{US_1} \oplus \mathsf{US_2}$-formula. By Lemma 3.7, we construct a $\mathsf{US_1}$-formula whose decision problem is equivalent to ψ and then apply $\mathsf{US_1}$'s decision procedure. ∎

With regards to the complexity of the decision problem, the algorithm outlined above does not give us a good starting point. However, a very detailed analysis of the complexity of such systems was done in [24].

3.5 Extension to an arbitrary number of multimodal logics

In Section 2.5 we showed how the process of applying a logic externally to another could be generalised to modal logics with n connectives of arbitrary arity. In the case of the independent combination, we can go even further. For the temporalisation (or the extended modalisation) is a combination process that involves only two logics: the external T and the internal L.

However, in the independent combination of logics, this limitation does not hold. For in a *generalised independent combination* any number of logics may be taken as input, each with any number of connectives of arbitrary arity.

Does the transference of properties hold is such a generalised form?

Let us first concentrate on the a combination of two generalised modal logics, $\mathsf{M_1}$ and $\mathsf{M_2}$. We start noting that the format of the combined model $(T, <_1, <_2)$ in the independent combination is basically the same of that of the generalised frame $(W, R_1, \ldots, R_n)$. As usual, we assume that the connectives of $\mathsf{M_1}$ are distinct from those of $\mathsf{M_2}$. Combining the languages and inferences systems poses no problems. In combining the two classes of frames, we would end up with frames of the form $(W, R^1_1, \ldots, R^1_{n_1}, R^2_1, \ldots, R^2_{n_2})$, which has the same format of a generalised frame.

This means that $\mathsf{M}_1 \oplus \mathsf{M}_2$ has the same format of a generalised modal logic, so the process can be iterated once more. That is we can independently combine $\mathsf{M}_1 \oplus \mathsf{M}_2$ with a generalised modal logic M_3 obtaining yet another generalised modal logic, $(\mathsf{M}_1 \oplus \mathsf{M}_2) \oplus \mathsf{M}_3$. Of course, this process can be iterated any number of time. And its not hard to see that the process, at least on the level of combining language and inference systems, is associative and commutative. On the semantic level, note that we do not distinguish, say, the frames (W, R_1, R_2) from (W, R_2, R_1), so the resulting frame will have a single set of points and the disjoint union of all relations involved which of course is associative and commutative.

This shows that if we can independently combine two generalised modal logics, we can easily independently combine any number of such logics. It remains to be shown that the generalised modalisation/temporalisation can still be used as a building block for the independent combination of M_1 and M_2.

To show that this is indeed the case, we will show how the construction above can be modified for multimodal logics.

The main thing to note here is that, no matter what the modal connectives are, the independent combination of two modal systems can be decomposed in a successive number of modalisations/temporalisations, for a formula of $\mathsf{M}_1 \oplus \mathsf{M}_2$ can always be seen as a formula of some finite number of temporalisations: $\mathsf{M}_1(\mathsf{M}_2(\mathsf{M}_1(\ldots)))$.

The notion of degree of alternation in this case is exactly the same as in the US case. The core of the completeness proof remains the same, namely the proof of the following lemma.

Lemma 3.9 Let M_1 and M_2 be two sound and complete generalised modal logics. The formula A is a theorem of $\mathsf{M}_1 \oplus \mathsf{M}_2$ iff it is a theorem of some modalised system $\mathsf{M}_1(\mathsf{M}_2(\mathsf{M}_1(\ldots)))$.

The depth of the temporalised system, as before, is bounded by the *degree of alternation* d in A of the nesting modal of M_1 inside M_2 operators, and vice-versa. Such a notion is exactly as it was in the US case. Also, the remarks made in Section 2.4 as to what should be a monolithic formula in $\mathsf{M}_1(\mathsf{M}_2(\mathsf{M}_1(\ldots)))$ also apply here.

With Lemma 3.9 all there is to do now is to mimick the construction of the model in Theorem 3.8. If A is a consistent $\mathsf{M}_1 \oplus \mathsf{M}_2$ formula, by Lemma 3.9 it is also consistent in some temporalised logic $\mathsf{M}_1(\mathsf{M}_2(\mathsf{M}_1(\ldots)))$ with at most d alternations. We apply the generalised modalisation transference of completeness to obtain a modalised model for A and then mimick the steps of Theorem 3.8 to transform such a model into a model of the independent combination. This is straightforward and we ommit the details. This shows that completeness is transferred through independent combination.

To obtain the transference of decidability we hardly have to make any changes to the proof in Section 3.4. There the decision procedure is based on the fact that a formula of $\mathsf{T}_1 \oplus \mathsf{T}_2$ is valid iff it is valid in some temporalised system. But the same result was generalised in Lemma 3.9. So the decision procedure for the generalised case is the same as the decision procedure for the US case, with barely any difference, for we have already shown in Section 2.5 how to extend the mappings σ and η to the generalised case, which are all that is needed in the decision procedure. So decidability is transferred.

And since soundness is transferred by the result of Thomason [25], we can then conclude the following.

Theorem 3.10 Let M_1 and M_2 be two sound, complete and decidable extended modal logics. Then $\mathsf{M}_1 \oplus \mathsf{M}_2$ is sound, complete and decidable.

4 Conclusion

We have extended the original results on temporalisation of [11] to any class of flows of time, extending the original result for linear classes only. This results was also extended to multi-modal logics with n-ary connectives.

Recursive temporalisations were used in [10] to show the transference of completeness and decidability for the independent combination of two linear US-temporal logics. Such construction was shown to generalise to the unrestricted case and was developed inside the traditional Kripke semantics for temporal logics. The same technique could also be applied to the independent combination of arbitrary number of multi-modal logics with n-ary connectives.

Recently, the work in [2] has generalized Wolter's algebraic results in [28] for the independent combination of US-logics in the algebraic tradition. That work was developed independently from ours, and did not have in mind US-logics, but was developed for Description Logics; The generalization of Wolter's result for decidability developed in [2] also applies to US-logic. So the relevant points of the results in here are the fact the independent combination was achieved using kripke-style semantics and that we can consistently see any kind of independent combinations as an iterations of modalisations/temporalisations. Note that in all such works, including ours, at least some form of normal behaviour was assumed from the connectives.

It remains an open problem whether the decidability of the logics with arbitrary operators (normal or non-normal) is transferred by their independent combination. The investigation of non-normal temporalisations/modalisations remains a viable way to explore such a question and is a path to be explored in the future.

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