Discontinuity of multiplication and left traslations in βG

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Given a discrete group G, the operation naturally extends to the Stone-Čech compactification βG of G so that for each $a \in G$, the left translation

$$\beta G \ni x \mapsto ax \in \beta G$$

is continuous, and for each $q \in \beta G$, the right translation

$$\beta G \ni x \mapsto xq \in \beta G$$

is continuous.

We take the points of βG to be the ultrafilters on G, the principal ultrafilters being identified with the points of G, and $G^* = \beta G \setminus G$. The topology of βG is generated by taking as a base the subsets

$$\overline{A} = \{p \in \beta G : A \in p\}$$

where $A \subseteq G$. For $p, q \in \beta G$, the ultrafilter pq has a base consisting of subsets of the form

$$\bigcup_{x \in A} x B_x$$

where $A \in p$ and $B_x \in q$.

For every $p \in G^*$,

$$\lambda_p : \beta G \ni x \mapsto px \in \beta G$$

is discontinuous, and moreover,

$$\lambda_p^* : G^* \ni x \mapsto px \in G^*$$

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Question 1. Are there $p, q \in G^*$ such that λ_p (or λ_p^*) is continuous at q?

Recall that a nonprincipal ultrafilter p on ω is a P-point if the intersection of countably many neighborhoods of $p \in \omega^*$ is again a neighborhood of p. MA implies the existence of P-points [Rudin 1956]. However, it is consistent with ZFC that there is no P-point [Shelah 1982].

Recall that a nonprincipal ultrafilter p on ω is a P-point if the intersection of countably many neighborhoods of $p \in \omega^*$ is again a neighborhood of p. MA implies the existence of P-points [Rudin 1956]. However, it is consistent with ZFC that there is no P-point [Shelah 1982].

Example 1. Let G be a countable group and let $q \in G^*$ be a P-point. Then for every $p \in G^*$, λ_p^* is continuous at q, and if G is Abelian, then λ_q is continuous at q.

Question 2. **Are there p, $q \in G^*$ such that**

$$\mu : \beta G \times \beta G \ni (x, y) \mapsto xy \in \beta G$$

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Notice that μ is continuous at (p, q) if and only if

$$\{AB : A \in p, B \in q\}$$

is an ultrafilter base.

For every countable Abelian group G with finitely many elements of order 2 (in particular, for $G = \mathbb{Z}$) and for every $p, q \in G^*$, μ is discontinuous at (p, q) [Protasov 1996].

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Example 2. Assume MA. Let $G = \bigoplus_\omega \mathbb{Z}_2$. There is a group topology on G with exactly two nonprincipal ultrafilters p, q on G converging to 0 such that $p+p = p$ and $q+q = p+q = q+p = q$. It then follows that μ is continuous at (p, q).

Theorem 1. **Let G be an Abelian group with finitely many elements of order 2 such that $|G|$ is not Ulam-measurable and let $p, q \in G^*$. Then μ is discontinuous at (p, q).**

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Equivalently, for every Abelian group G with finitely many elements of order 2 such that $|G|$ is not Ulam-measurable (in particular, for $G = \mathbb{R}$) and for every nonprincipal ultrafilters p, q on G,

$$\{A + B : A \in p, B \in q\}$$

is not an ultrafilter base.

Theorem 2. Let G be an Abelian group such that $|G|$ is not Ulam-measurable and let $p, q \in G^*$. If λ_p^* is continuous at q, then there is $f : G \to \omega$ such that $\overline{f}(q)$ is a P-point.

Theorem 2. Let G be an Abelian group such that $|G|$ is not Ulam-measurable and let $p, q \in G^*$. If λ_p^* is continuous at q, then there is $f : G \to \omega$ such that $\overline{f}(q)$ is a P-point.

Corollary. It is consistent with ZFC that for every Abelian group G and for every $p, q \in G^*$, λ_p^* is discontinuous at q.