Topologies determined by discrete subsets

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1. Alas, O. T. and Wilson, R. G., *Products of (weakly) discretely generated spaces*, Topology and its Applications, 160 (2013), 532-537.

2. Alas, O. T., Junqueira, L. R. and Wilson, R. G., *When is a P-space weakly discretely generated?*, To appear, Topology and its Applications, 2013.

3. Alas, O. T., Tkachuk, V. V. and Wilson, R. G., *Closures of discrete sets often reflect global properties*, Topology Proceedings, **25** (2000), 27-44.

4. Bella, A. and Simon, P., *Spaces which are generated by discrete sets*, Topology and its Applications, **135** (2004), 87-99.

5. Dow, A., Tkachenko, M. G., Tkachuk, V. V. and Wilson, R. G., *Topologies generated by discrete subspaces*, Glasnik Mat., Ser. III, **37:1** (2002), 187-210.

6. Ivanov, A. V. and Osipov, E. V., *Degree of discrete generation of compact sets,* Math. Notes, **87** No. 3 (2010), 367-371.

7. Tkachuk, V. V. and Wilson, R. G., *Box products are often discretely generated,* Topology and its Applications, **159** (2012), 272-278.

ALL SPACES ARE HAUSDORFF.

Definitions

A space (X, τ) is said to be *discretely generated* (or *DG*) if for each $A \subseteq X$ and $x \in \mathrm{cl}(A)$, there is a discrete set $D \subseteq A$ such that $x \in \mathrm{cl}(D)$.

A space (X, τ) is said to be *weakly discretely generated* (or *WDG*) if for each $A \subseteq X$ which is not closed, there is a discrete subset $D \subseteq A$ such that $\mathrm{cl}(D) \setminus A \neq \emptyset$.

These are obvious generalizations of the properties of being Fréchet and sequential in the class of Hausdorff spaces.

The following results are obvious or have very easy proofs:

(1) Every discretely generated space is weakly discretely generated;

(2) If X is Fréchet–Urysohn then it is discretely generated;

(3) If X is sequential then it is weakly discretely generated;

(4) If X is scattered then it is discretely generated;

(5) Discrete generability is hereditary;

(6) Weak discrete generability is closed-hereditary;

(7) A space is discretely generated iff it is hereditarily weakly discretely generated.

Some classes of (weakly) discretely generated spaces are:

1. Each space with a nested local base at each point is discretely generated.

2. Each monotonically normal topological space is discretely generated and hence so is each LOTS and any subspace of a LOTS.

3. Each Hausdorff sequential space is discretely generated.

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3. Each Hausdorff sequential space is discretely generated.

Bella and Simon generalized 1 by showing that

4. Each radial space is discretely generated, and

5. Each pseudoradial space is weakly discretely generated.

Most importantly,

Theorem

Each compact Hausdorff space is weakly discretely generated.

Proof.

Let X be a compact Hausdorff space. If $A \subseteq X$ and $A \neq \text{cl}(A)$ then A is not compact. Apply the following theorem of Tkachuk: If the closure of every discrete subset of a space is compact then the whole space is compact. There is a discrete $D \subseteq A$ such that $\text{cl}_A(D)$ is not compact. Since $\text{cl}_X(D)$ is compact we have $\text{cl}_X(D) \setminus A \neq \emptyset$. $\square$

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Proof.

Let $A \subseteq X$ and define $A = A_0$, and

$$A_{\alpha+1} = \bigcup\{\text{cl}(D) : D \subseteq A_\alpha, D \text{ discrete}\} \text{ and } A_\beta = \bigcup\{A_\alpha : \alpha < \beta\}$$

if β is a limit ordinal. Since X is WDG, there is some ordinal κ such that $A_\kappa = \text{cl}(A)$. However, if $x \in A_2$, then there is a countable discrete set $D = \{x_n : n \in \omega\} \subseteq A_1$ such that $x \in \text{cl}(D)$. The points of D can be separated by a family $\{U_n : n \in \omega\}$ of open sets and each point of D lies in the closure of a discrete set $D_n \subseteq U_n \cap A$. $D = \bigcup\{D_n : n \in \omega\}$ is discrete and $x \in \text{cl}(D)$ implying that $x \in A_1$. □

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Example

$\{0,1\}^{\omega_1}$ is not discretely generated. In fact, a dyadic compact space is DG if and only if it is metrizable. (This result depends on the existence of an L-space in ZFC.)

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Example

van Douwen's maximal space is not WDG, since each discrete subset is closed. Furthermore, no crowded submaximal space is weakly discretely generated.

Lemma

Suppose that (X, τ) *is a Hausdorff space,* $A \subseteq X$ *and* $p \in \mathrm{cl}(A)$*. If* $\psi(p, A \cup \{p\}) = \chi(p, A \cup \{p\})$*, then there is a discrete subset* $D \subseteq A$ *such that* $p \in \mathrm{cl}(D)$*.*

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Suppose that (X, τ) is a Hausdorff space, $A \subseteq X$ and $p \in \mathrm{cl}(A)$. If $\psi(p, A \cup \{p\}) = \chi(p, A \cup \{p\})$, then there is a discrete subset $D \subseteq A$ such that $p \in \mathrm{cl}(D)$.

Theorem

If X is countably compact regular space X and $\chi(X) \leq \omega_1$ then X is WDG.

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Proof.

Suppose $A \subseteq X$ is non-closed but $\mathrm{cl}(D) \subseteq A$ for all discrete $D \subseteq A$. A is countably compact. Take any $p \in \mathrm{cl}(A) \setminus A$. For any G_δ-set G and $p \in G$, we have $G \cap A \neq \emptyset$ for otherwise there is a decreasing sequence of closed non-empty subsets of A with empty intersection which contradicts countable compactness of A. Hence $\psi(p, \{p\} \cup A) > \omega$. Apply the lemma. □

One more result of this type is due to Bella and Simon:

Theorem (Bella and Simon, 2004)

If X is a countably compact regular space of countable tightness, then X is weakly discretely generated.

Regularity in the last two theorems is essential. Consistently, countably compact Hausdorff spaces of character ω_1 need not be WDG: Let V be van Douwen's maximal space and modify the topology of βV by declaring V to be open. The result is a countably compact Urysohn space of character $\mathfrak{c}$ which is not WDG. A more complicated modification gives a countably compact space of countable tightness which is not WDG.

As we have seen, the class of weakly discretely generated spaces is quite large; it includes for example, all compact, all sequential, all scattered and all monotonically normal spaces. Further, the construction of non weakly discretely generated (Tychonoff) spaces is not entirely trivial. Thus one might expect that this class behaves well under the taking of products and quotients. However the results in this area are few and far between.

Theorem

The product of a regular discretely generated space and a first countable regular space is discretely generated.

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Say that a space is *l-nested* if there is a nested local base at each point.

Theorem

A finite product of l-nested spaces is discretely generated.

Theorem

If X is WDG and κ is an ordinal, then $X \times \kappa$ is WDG.

Theorem

Let $\{X_n : n \in \omega\}$ be a countable family of regular discretely generated spaces. If each finite product $\Pi\{X_k : 0 \leq k \leq m\}$ is discretely generated, then $\Pi\{X_k : k \in \omega\}$ is discretely generated.

An uncountable product of non-trivial spaces contains a copy of $\{0, 1\}^{\omega_1}$ and hence cannot be discretely generated.

Corollary

A countable product of regular scattered spaces is DG.

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What about negative results?

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Theorem

Any box product of monotonically normal spaces is DG.

Corollary

Any box product of LOTS is discretely generated.

What about negative results? Only one!

Example (Ivanov and Osipov)

Under CH there is a compact DG space X such that X^2 is not DG.

Definition

A space is a *P-space* if every G_δ is open.

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Note that if a P-space X has character $\aleph_1$, then $\psi(p, A \cup \{p\}) = \chi(p, A \cup \{p\})$ for all $A \subseteq X$.

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Mimicking the result that each compact space is WDG, in their 2004 article, Bella and Simon conjectured that every Lindelöf P-space is WDG. The following partal results are known.

Theorem

A P-space of of character $\aleph_1$ is DG.

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Theorem

A P-space of of character $\aleph_1$ is DG.

The next theorem was first proved by Bella and Simon assuming $P_1 + 2^{\aleph_1} > \aleph_2$, but it is a ZFC result.

Theorem

A regular P-space of countable extent and character at most $\aleph_2$ is WDG.

The next two theorems are the best (known) partial solutions in ZFC to Bella and Simon's conjecture.

Theorem

Let X be a Lindelöf P-space; if
(a) $\psi(x, X) < \aleph_\omega$ for each $x \in X$, or
(b) $\chi(X) \leq \aleph_\omega$,
then X is weakly discretely generated.

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then X is weakly discretely generated.

Theorem

A Lindelöf P-space in which every linearly Lindelöf subspace is Lindelöf is WDG.

Assuming $\beth_\omega = \aleph_\omega$ we get the following result:

Theorem

Every Lindelöf P-space of tightness at most $\aleph_\omega$ is weakly discretely generated.

Without the Lindelöf property or limitation on the extent, the only other result is:

Theorem

If X is a regular P-space which is weakly discretely generated and $t(X) = \aleph_1$, then X is discretely generated.

Let $A \subseteq X$ and define $A = A_0$, and

$$A_{\alpha+1} = \bigcup\{\mathrm{cl}(D) : D \subseteq A_\alpha, D \text{ discrete}\} \text{ and } A_\beta = \bigcup\{A_\alpha : \alpha < \beta\}$$

if β is a limit ordinal.
If a space is weakly discretely generated, there is some ordinal κ such that $A_\kappa = \mathrm{cl}(A)$. How large (or small) is κ?

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If a space is weakly discretely generated, there is some ordinal κ such that $A_\kappa = \mathrm{cl}(A)$. How large (or small) is κ?

Theorem (Ivanov and Osipov, 2010)

If X is a compact space of character ω_α, then $\kappa \leq \alpha + 1$.

This can be generalized as follows:

Theorem

If X is a Čech complete space of character ω_α, then $\kappa \leq \alpha + 1$.

Example

The space $\beta\mathbb{R}$ contains remote points, that is to say, points which are not in the closure of any nowhere dense subset of $\mathbb{R}$. However, under CH, only two iterations of the “discrete closure" are needed to obtain the closure. It is not clear what happens if CH is not assumed.

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We have only one more result on this topic:

Theorem

If X is a regular P-space of countable extent and character at most $\aleph_2$, then at most two iterations are needed.

The main question here is the following:

If X is discretely generated, then which compactifications of X are discretely generated? (Of course, all are WDG.)

Example

The space of reals $\mathbb{R}$ is discretely generated (it is first countable), its one-point compactification (indeed any first countable compactification) is DG, but as we saw before, $\beta\mathbb{R}$ is not DG.

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Example

The space $\mathbb{N}$ of positive integers is discretely generated (of course), its one-point compactification (indeed any scattered compactification) is DG, but $\beta\mathbb{N}$ is not DG (it maps onto $[0, 1]^{\mathfrak{c}}$ and hence contains a copy of van Douwen's maximal space).

If X is locally compact, then the one-point compactification αX of X is its smallest compactification and hence the “most likely" to be discretely generated - only one point to check! So:

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If X is locally compact and DG, is αX discretely generated?

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First some positive results.

Theorem

If βX is DG, then X is pseudocompact.

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Question

If X is locally compact and DG, is αX discretely generated?

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Theorem

If βX is DG, then X is pseudocompact.

Proof.

If X is not pseudocompact, then it contains a C^*-embedded copy of $\mathbb{N}$ and hence βX contains a copy of $\beta\mathbb{N}$. □

Definition

A space is *isocompact* if every closed countably compact subspace is compact.

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Theorem

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If we assume metacompact (which implies isocompact) and Fréchet, we get a slightly stronger result.

Theorem

If X is a locally compact, metacompact Fréchet space, then αX is Fréchet.

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If X is a locally compact, metacompact Fréchet space, then αX is Fréchet.

There are however consistent examples of locally compact DG spaces whose one-point compactification is not DG.

Example

Let K denote the Cantor space less one point. K is locally compact, zero-dimensional but not compact and it then follows from a result of Plank that under CH, there is a point $p \in \beta K \setminus K$ which is both a P-point of $\beta K \setminus K$ and a remote point of βK. We can construct, a compactification γX of X and a map $q : \beta K \to \gamma K$ in a standard way so that $\gamma K \setminus K$ is (homeomorphic to) $\omega_1 + 1$, $q^{-1}[\{\omega_1\}] = \{p\}$ and $q|K$ is a homeomorphism. It follows that the trace of the neighbourhood filter of ω_1 on K in γK is identical to that of the trace of the neighbourhood filter of p on K in βK. Thus in the space γK, the point ω_1 does not lie in the closure of any discrete subset of K. Furthermore, $\gamma K \setminus \{\omega_1\}$ is first countable and hence is discretely generated. It now only remains to note that $X = \gamma K \setminus \{\omega_1\}$ is a discretely generated space whose one-point compactification αX is not discretely generated.

There are other solutions to this problem.

1. L. F. Aurichi (2009) has (incidentally) given an example assuming the existence of a Souslin line.
2. R. Hernández-Gutiérrez (2013) has given an example assuming that $\mathfrak{p} = \mathrm{cof}(\mathrm{nwd}(\omega \times 2^\omega))$.

[$\mathrm{cof}(\mathrm{nwd}(X))$ is the smallest cardinal κ such that there is a family $\mathcal{G}$ of size κ of nowhere dense sets in X which is cofinal in the family of all nowhere dense sets (ordered by inclusion).]

Problem

Is every countable product of Hausdorff scattered spaces discretely generated?

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Is there a ZFC example of a compact DG space whose square is not DG?

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Is there in ZFC a locally compact DG space whose one-point compactification is not DG?

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Problem

Is there in ZFC a locally compact DG space whose one-point compactification is not DG?

Problem

Is the continuous image of a compact DG space discretely generated? Is the perfect image of a DG (WDG) space DG (WDG)?

- Alas, O. T. and Wilson, R. G., *Products of (weakly) discretely generated spaces*, Topology and its Applications, 160 (2013), 532-537.

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Muito obrigado por sua atenção.