Survey on Uniform Box Products

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A space is completely regular iff its topology is induced by a uniformity.

Using uniformities, CJ Knight (a student of CH Dowker) first proved:

THEOREM. The box product of completely regular spaces is completely regular.

This suggested the Uniform Box Product Topology.

A space is completely regular iff its topology is induced by a uniformity.

Induces means G⊆X is open iff ∀x∈G, iff ∃∈**D** with D(x) ⊆G

A completely regular space is induced by a *totally bounded uniformity* **-**

one such comes from any Cech-Stone compactification – let **F** be the set of finite open covers *R* of X.

∀*R*∈**F**, let D_R = ∪{R^2 : R∈*R*}.

Then **D** = {D_R : *R*∈**F**} is a uniformity inducing X.

DEFINITION.

Suppose $\mathbf{D} \subseteq X^2$ is a base for a diagonal uniformity on a space X.

Let $\prod$ represent the set product $\prod^{\kappa} X$.

For $D \in \mathbf{D}$, let $\underline{D} = \{<x,y> \in \prod^2 : \forall \alpha \in \kappa,\ <x(\alpha),y(\alpha)> \in D\}$.

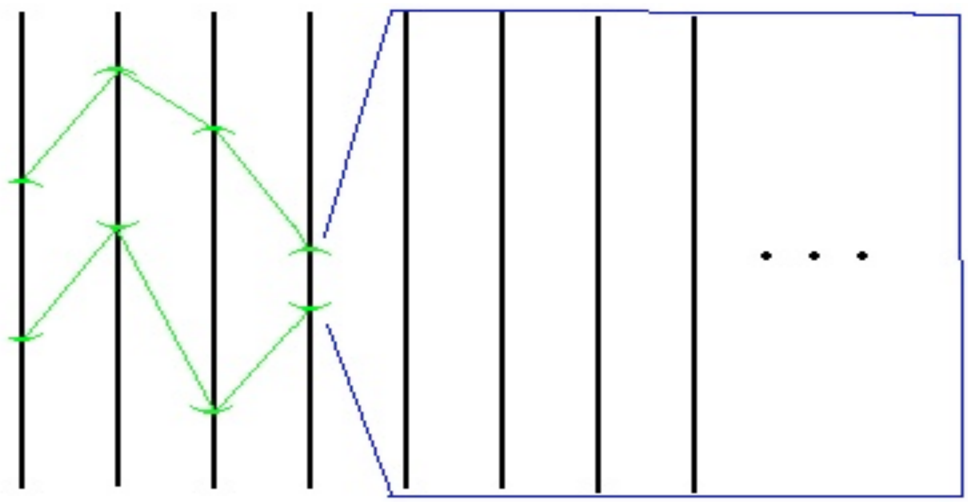
$\underline{\mathbf{D}} = \{\underline{D} : D \in \mathbf{D}\}$ is a uniformity on $\prod$ inducing the ***uniform box topology*** on $\prod$.

$\underline{D}$ -***balls*** are $\underline{D}(x) = \{y : <x,y> \in \underline{D}\}$.

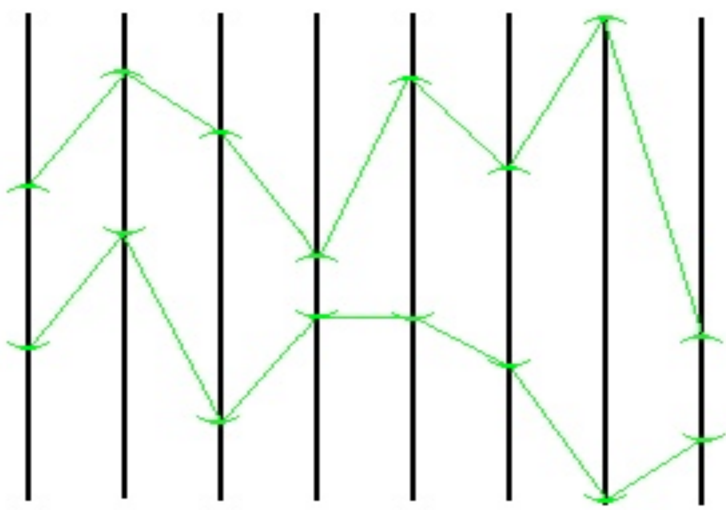
As each $\underline{D}(x) = \prod_{\alpha \in \kappa} D(x(\alpha))$,

Tychonov topology $\subseteq$ uniform box topology $\subseteq$ box topology.

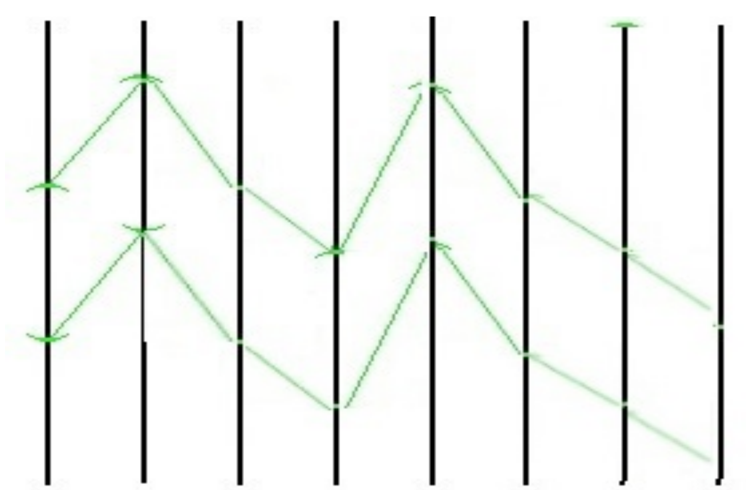
Henceforth, assume $\prod^{\kappa} X$ has the uniform box topology



Tychonov



Box



Uniform

PROPOSITION (SWilliams).

1. Suppose X is a metrizable space. Then $\prod^{\kappa}X$ is metrizable.

2. Suppose **D** is a complete uniformity on a space X. Then D is a complete uniformity on $\prod^{\kappa}X$.

EXAMPLE (S Williams). Is $\prod^{\omega}\mathbf{R}$ not connected.

Sketch. When points are in the same component iff the are finitely far apart.

Henceforth, assume the uniformity on X is totally bounded.

THEOREM1 (S Williams). Suppose (X,**D**) is a connected space X. Then $\prod^{\kappa}X$ is connected.

Sketch. For each finite partition P of $\prod$,

let $\prod_P$ = {x|P is constant : for each P∈P}.

$\prod_P$ is homeomorphic to the connected finite product $\prod^{|P|}X$.

As $\prod_P \cap \prod_{P'}$ contains the constants, so

Z = ∪{$\prod_{\mathbf{P}}$: **P** is a finite partition of I} is connected.

D is totally bounded, so Z is dense in $\prod^{\kappa}X$. So $\prod^{\kappa}X$ is connected.

As it is in box products, the chief questions we ask in uniform box products hover around normality.

DEFINITION.

A space X is ***pseudo-normal*** provided closed sets, one of which is countable, can be separated by disjoint open sets.

THEOREM (Van Douwen). The box product of compact spaces is pseudo-normal.

THEOREM2 (J Bell – S Williams). Suppose X is locally compact σ-compact *countable closure* space. Then $\prod^{\omega} X$ is pseudo-normal.

DEFINITION.

A space is *countable closure space* provided the closure of countable sets is countable. (Juhasz and Weiss called these *ω-fan* spaces.)

LEMMA (J Bell – S Williams).

1. Spaces with at most countably many limit points are countable closure spaces. (Ex. Fort Spaces)

2. Scattered linear ordered spaces. (Ex. Ordinal spaces)

3. countable closure is hereditary.

4. countable closure is finitely productive.

5. A locally compact σ-compact countable closure space is scattered and zero-dimensional.

I repeat.

THEOREM2 (J Bell – S Williams). Suppose X is locally compact σ-compact countable closure space. Then $\prod^{\omega}X$ is pseudo-normal.

Sketch proof of Theorem 2.

Suppose C = $\{c_n : n\in\omega\}$ is closed in $\prod^{\omega}X$, G is open in $\prod^{\omega}X$ and $C\subseteq G$.

We construct, by recursion, a cellular family Q of clopen sets in $\prod^{\omega}X$ such that $C\subseteq\cup Q\subseteq cl(\cup Q)\subseteq G$.

$\exists\ P_1\in\mathbf{D}$ such that $P_1(c_1)\subseteq G$.

Let $Q_1 = \{P_1(c) : c\in C \text{ and } P_1(c)\subseteq G\}$.

Let $R_1 = \{ P_1(c) : \text{both } c\in C \text{ and } P_1(c)\cap G \neq \emptyset \text{ holds}\}$.

As $\{P_1(x) : x\in\prod^{\omega}X \}$ is a clopen partition, is $\cup Q_1$ clopen and contained in G.

Do it again for each $P_1(x)\in R_1$.

Let n be smallest integer such that $P_1(c_n)\in R_1$. Find $P_2\in\mathbf{D}$, P_2 refines P_1 and $P_2(c_n)\subseteq G$.

Let $\mathcal{Q}_2 = \{P_2(x) : x \in C, P_1(x) \in \mathcal{R}_1, P_2(x) \subseteq G\}$.

Let $\mathcal{R}_2 = \{P_2(x) : x \in C, P_1(x) \in \mathcal{R}_1, P_2(x) \cap G \neq \emptyset\}$.

As $\{P_2(x) : x \in \prod^{\omega} X\}$ is a clopen partition, $\cup(\mathcal{Q}_1 \cup \mathcal{Q}_2)$ is clopen and contained in G.

So we continue in this fashion building a tree

$$(\cup\{\mathcal{Q}_n : n \in \omega\}) \cup (\cup\{\mathcal{R}_n : n \in \omega\})$$

Let $H = \cup\{\cup\mathcal{Q}_n : n \in \omega\}$. Clearly, $C = \{c_n : n \in \omega\} \subseteq H$.

We claim H is closed.

Suppose y is a limit point of H not in H.

Then $\forall n \in \omega, \exists s_n \in C, P_n(s_n) \in \mathcal{R}_n$.

Let $t_n \in P_n(s_n) \setminus G$.

As P_n is a partition of $\prod^{\omega} X$, $y \in P_n(b_n)$.

Therefore, $y \in \cap_{n \in \omega} P_n(b_n)$.

Now we apply …

LEMMA. Suppose $<a_n : n\in\omega>$ is a sequence in $\prod^\omega X$ and

$\{D_n : n\in\omega\}\subseteq\mathbf{D}$ are such that

1. $a_{n+1}\in D_n(a_n)$.

2. D_{n+1} refines D_n.

Then either $\cap_{n\in\omega}D_n(a_n) = \emptyset$ or

the sequence $<a_n : n\in\omega>$ converges to $z\in\cap_{n\in\omega}D_n(a_n)$.

COROLLARY (J Bell – S Williams). Suppose X is an ordinal space. Then $\prod^\omega X$ is pseudo-normal.

COROLLARY (J Bell – S Williams). $\prod^\omega(\omega_1+1)$ is pseudo-normal.

OPEN PROBLEM. (S Williams-2001) Is $\prod^\omega(\omega_1+1)$ normal?

THEOREM (K Kunen). [**CH**] The box product of compact spaces scattered spaces is paracompact.

THEOREM3 (J Bell). Suppose P is the Tychonov product of countably many Fort spaces. Then $\prod^{\omega}P$ is a *proximal space*, (and hence is collectionwise-normal and countably paracompact).

We consider the *proximal game* (J Bell) played on a uniform space. (I present the simpler zero-dimensional version. For the full version Player 1 has $2D_n \subseteq D_{n+1}$ and Player 2 has $x(n+1) \in 4D_{n-1}(x(n))$)

Suppose (X,**D**) is a uniform space.

Round1. Player A chooses $D_1 \in \mathbf{D}$.

Player B chooses $x(1) \in X$.

Round(n+1). Player A chooses $D_n \subseteq D_{n-1}$.

Player B chooses $x(n+1) \in D_{n-1}(x(n))$

Player A wins if the sequence <x(n) : n∈ω> converges; otherwise *Player B wins*.

DEFINITION. Uniform space (X,**D**).

D is a *proximal* (or X is a *proximal space*) provided player A always has a winning strategy.

PROPOSITION.

1. (J Bell) A metric space is proximal (in its canonincal uniformity) iff it is complete.
2. (J Bell) A Fort space (the one point compactification of a discrete space) is proximal.
3. (S Williams) The Long Line is proximal (use the totally bounded uniformity from its 1-point compactification).
4. (J Bell) ω_1+1 is not proximal.

LEMMA (J Bell).

1. Proximal is closed hereditary.

2. Proximal is preserved by countable Tychonov products and ∑-products.

3. Proximal spaces are Gruenhage W-spaces (and so Frechet).

4. Proximal spaces are countably paracompact, and collectionwise normal.

Sketch proof of LEMMA (3).

Here we assume the uniformity base **D** for X is symmetric and open. F(X) denotes finite sequences of X, k∈**N**. We view proximal strategy as a function w: F(X) → **D** where $x_{n+1} \in kw(x_1, x_2, \ldots, x_{n-1})[x_n]$ is where $x_1, x_2, \ldots, x_n$ are the first n choices of player B. σ denotes the W-space game.

Fix the point x.

Round 1.

(proximal) Player A chooses $w(\emptyset) = X^2$.

(proximal) Player B chooses x.

(W-space) Player 1 choose $\sigma(\emptyset) = w(x)[x] = X$.

(W-space) Player 2 choose some $y_1 \in \sigma(\emptyset)$.

Round 2.

(proximal) Player A gets $w(x,y_1)$.

(proximal) Player B also chooses, y_1.

The pair $<x,y_1> \in w(x)$ so symmetricity finds $x \in w(x)[y_1]$. So now Player B can choose x legally.

Player A gets $w(x,y_1,x)$ from the winning strategy.

(W-space) Player 1 chooses $\sigma(x) = w(x,y_1,x)[x]$.

(W-space) Player 2 choose some $y_2 \in w(x,y_1,x)[x]$.

General rounds.

$\sigma(x, y_1, y_2, \ldots, y_n) = w(x,y_1,x, y_2,x, y_3, \ldots,x, y_n,x)[x]$. Because of thewinning strategy $x,y_1,x, y_2,x, y_3, \ldots,x, y_n,x, \ldots$ converges, obviously to x. So to y_n x. Thus, X is a W-space.

Sketch proof of normal in zero-dimensional LEMMA (4).

So we can assume the members of the uniformity are squares of members of clopen partitions.

Let C_0 and C_1 be disjoint open sets. Choose $x_i \in C_i$.

Choose D_1 so that $x_i \in C_i$ and $D_1(x_i) \cap C_{1-i} = \emptyset$.

Let $\mathcal{H}_1 = \{D_1(x) : x \in C_0 \cup C_1\}$.

Let $\mathcal{Q}_1 = \{D_1(x) : i \in 2, x \in C_i, D_1(x) \cap C_{1-i} = \emptyset\}$.

Let $\mathcal{R}_1 = \mathcal{H}_1 \setminus \mathcal{Q}_1$. Remember $\{D_1(x) : x \in X\}$.

For $R \in \mathcal{R}_1$, choose $x_i \in C_i$ such that $D_1(x_i) = D_1(x_{1-i})$.

Choose $D_{2,R} \subseteq D_1$ so that $x_i \in C_i$ and $D_{2,R}(x_i) \cap C_{1-i} = \emptyset$.

Let $\mathcal{H}_2 = \mathcal{Q}_1 \cup \{D_{2,R}(x) : R \in \mathcal{R}_1, x \in R \cap (C_0 \cup C_1)\}$.

Let $\mathcal{Q}_2 = \mathcal{Q}_1 \cup \{D_{2,R}(x) : i \in 2, x \in C_i, R \in \mathcal{R}_1, D_{2,R}(x) \cap C_{1-i} = \emptyset\}$.

Let $\mathcal{R}_2 = \mathcal{H}_2 \setminus \mathcal{Q}_2$.

Continue in this fashion.

Let $\mathcal{H} = \cup_{n \in \omega} \mathcal{H}_n$. Then H is the union of pairwise-disjoint clopen sets.

Let $G_i = \{H \in \mathcal{H} : H \cap C_i \neq \emptyset\}$. Then $G_0 \cap G_1 = \emptyset$.

Suppose $z \in C_i \setminus G_i$. Then $\exists \forall n \in \omega, D_n(y_n) \in \mathcal{R}_n$ and $y_n \in C_{1-i}$ such that

$z \in \cap_{n \in \omega} D_n(y_n)$.

Define the sequence $<z_n : n \in \omega>$, by $z_{2n} = z$ and $z_{2n+1} = y_{2n+1}$.

As X is proximal, $<z_n : n \in \omega>$ converges to some point. As both are C_0 and C_1

closed. Its limit point belongs to $C_0 \cap C_1$ - a contradiction. Therefore, $C_0 \subseteq G_0$ and $C_1 \subseteq G_1$.

Applications to Uniform Box Products

THEOREM3 (J Bell). Suppose X is the Tychonov product of countably many Fort spaces. Then $\prod^{\omega} X$ is a proximal space.

COROLLARY (J Bell - to appear in the Proc. AMS). Suppose X is the Fort spaces. Then $\prod^{\omega} X$ is collectionwise Hausdorff, countably paracompact, normal).

Applications to Uniform Box Products3

THEOREM4 (S Williams). Suppose L is the long line. Then

$\prod^{\omega}$L is proximal.

Sketch proof of THEOREM4.

For simplicity we observe L is proximal.

Given a finite non-decreasing sequence P = <$\alpha_1,\alpha_2, \dots \alpha_n$> in $[0, \omega_1)$, we consider

finite open coverings *R* of L of the form

$R = [0, \alpha_1.2^{-n}),(\alpha_1, \alpha_1.2\cdot 2^{-n}), \dots , (\alpha_1.(2^{-n} -1)\cdot 2^{-n}, (\alpha_1+1).2^{-n}), ((\alpha_1+1), \alpha_2.2^{-n}),$

$(\alpha_2, \alpha_2.2\cdot 2^{-n}), \dots, (\alpha_n,\omega_1).$

$D_R = \cup\{R^2 : R\in R \}$. The set of all such D_R form a base for a uniformity on L.

The strategy for Player A is as follows:

If Player B chooses the point α.r P, then we add α to that sequence in the "right place" to get an (n+1)-term sequence (repeating is okay). So if the terms Player B chooses increase, then the sequence converges to the sup. If they don't then the tems will be stuck inside some $[\alpha_i, \alpha_i+2]$ where the sequence will proved to be Cauchy. As it must have a cluster point it converges

This is essentially the tool in proving $\prod^{\omega}L$ is a proximal space.

Suppose in Round k, Player A has chosen the non-decreasing n-sequence

$P = <\alpha_1,\alpha_2, \ldots \alpha_n>$ in $[0, \omega_1)$, on L, $D_{\mathcal{R}}$ and on $\prod^{\omega}L$, $\underline{D_{\mathcal{R}}}$.

Now Player B chooses some point $x \in \prod^{\omega}L$.

Say $x(n) = \alpha(n).r(n)$.

If we are at Round k+1, we add to the sequence P, the first m $\alpha(n)$'s of the first m choices of x. Now this forces B's choices to converge coordinate-wise in a copy of the connected $\prod^{\omega}[0,1]$ (see Theorem 1).

COROLLARY4. (P Nyikos). $\prod^{\omega}\omega_1$ is collectionwise-normal.

EXAMPLE (J. Hankins). Suppose X is the Fort space of cardinality ω_1. Then $\prod^{\omega}X$ is not paracompact.

PROBLEM1. (Williams-2001) Is $\prod^{\omega}(\omega_1+1)$ normal?

PROBLEM2. (Williams-2001) Suppose X is compact first countable is $\prod^{\omega}X$ normal?

~ THE END ~