# Borel selectors for families of sets

Carlos Uzcátegui
Universidad de Los Andes, Mérida (Venezuela)
Brazilian Conference of General Topology and Set Theory
In honour of Ofelia Alas on the occasion of her 70th birthday
São Sebastião, August 12-16, 2013

## Borel Selectors

$\mathbb{N}^{[\infty]}$ collection of infinite subsets of $\mathbb{N}$ as a subspace of $\{0,1\}^{\mathbb{N}}$.

A collection $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ is cofinal if for all $A \in \mathbb{N}^{[\infty]}$ there is $B \subseteq A$ with $B \in \mathcal{C}$.

A selector for $\mathcal{C}$ is a function $\Phi : \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$

$$\Phi(A) \subseteq A \ \ \& \ \ \Phi(A) \in \mathcal{C}.$$

The problem:

Which cofinal families admit a Borel selector?

Example:

$$\mathcal{C} = \{A \in \mathbb{N}^{[\infty]} : \sum_{n \in A} \frac{1}{n+1} < +\infty\}$$

$\mathcal{C}$ is a cofinal family with a Borel selector.

$$\begin{aligned} a_0 &= \min A \\ a_{k+1} &= \min\{n \in A : n > a_k \ \& \ \tfrac{1}{n+1} \leq \tfrac{1}{k^2+1}\} \end{aligned}$$

$$\sum_k \frac{1}{a_k+1} \leq \sum_k \frac{1}{k^2+1}$$

$$\Phi(A) = \{a_k : k \in \mathbb{N}\}$$

## The classical uniformization problem

Let $X$ and $Y$ be Polish spaces and $R \subseteq X \times Y$ a Borel set. Does there exist a Borel function $f : X \to Y$ such that

$$\forall x( \ x \in proj_X(R) \to (x, f(x)) \in R)?$$

$f$ is called an uniformizing function for $R$.

A selector $\Phi$ for a cofinal family $\mathcal{C}$ is an uniformizing function for

$$R = \{(A, B) \in \mathbb{N}^{[\infty]} \times \mathbb{N}^{[\infty]} : \ \ B \subseteq A \ \ \& \ \ B \in \mathcal{C}\}$$

Theorem: (Jankov, Von Neumann) Every analytic relation $R \subseteq X \times Y$ has a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable uniformizing function.

Known fact: There is a closed $B \subseteq \mathbb{N}^{[\infty]} \times \mathbb{N}^{[\infty]}$ such that

(i) $proj_X(B) = \mathbb{N}^{[\infty]}$.

(ii) $B$ does not admit a Borel uniformization.

Theorem: There is a cofinal $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ without a perfect subset, therefore without a Borel selector.

Question: Is there a cofinal Borel family on $\mathbb{N}^{[\infty]}$ without a Borel selector?

# Convergent sequences in sequentially compact spaces

Let $(x_n)_n$ be a sequence in a sequentially compact space $X$.

$$\mathcal{C}(x_n)_n = \{A \in \mathbb{N}^{[\infty]} : (x_n)_{n\in A} \text{ is convergent}\}$$

$\mathcal{C}(x_n)_n$ is cofinal on $\mathbb{N}^{[\infty]}$.

Example: Let $(x_n)_n$ be any sequence in $[0,1]$. Then $\mathcal{C}(x_n)_n$ has a Borel selector.

$\Phi(A)$ selects in a Borel way a Cauchy subsequence of $(x_n)_{n\in A}$.

## Compact subsets of the first Baire Class

Let $P$ be a Polish space.

$\mathcal{B}_1(P)$= Baire class-1 functions from $P$ into $\mathbb{R}$ (i.e. pointwise limits of continuous functions).

$K$ is a Rosenthal compacta if it is homeomorphic to a compact subset of $\mathcal{B}_1(P) \subset \mathbb{R}^P$.

Examples: Compact metric spaces.

Helly space= $\{f : [0,1] \to [0,1] \mid f \text{ non decreasing}\}$.

Theorem (Rosenthal, 1977) Every Rosenthal compacta is sequentially compact.

# Separable Rosenthal compacta

Let $(f_n)_n$ be a dense set in a Rosenthal compacta $\mathcal{K} \subseteq \mathcal{B}_1(P)$.

$$\mathcal{C}(f_n)_n = \{A \in \mathbb{N}^{[\infty]} : \ (f_n)_{n\in A} \text{ is pointwise convergent}\}$$

Since $\mathcal{K}$ is sequentially compact, then $\mathcal{C}(f_n)_n$ is cofinal.

Theorem (G. Debs, 1987) $\mathcal{C}(f_n)_n$ has a Borel selector.

Theorem (P. Dodos, 2006)

(i) $\mathcal{C}(f_n)_n$ is coanalytic. If $\mathcal{K}$ is not first countable, then $\mathcal{C}(f_n)_n$ is non Borel.

(ii) There is a Borel $G \subseteq \mathcal{C}(f_n)_n$ cofinal. $G$ is used for coding $\mathcal{K}$.

# Ramsey's theorem

Ramsey's theorem: Let $A \subseteq \mathbb{N}$ be infinity and $\varphi : A^{[2]} \to \{0,1\}$. There is $H \subseteq A$ infinite such that $\varphi$ is constant on $H^{[2]}$.

$H$ is said to be $\varphi$-homogeneous.

$$hom(\varphi) = \{H \in \mathbb{N}^{[\infty]} : \ H \text{ is } \varphi\text{-homogeneous}\}$$

Theorem: $hom(\varphi)$ admits a Borel selector.

Example: Let $(x_n)_n$ be a sequence in a compact metric space. There is $\varphi : \mathbb{N}^{[2]} \to 2$ such that

$$hom(\varphi) \subseteq \{A \in \mathbb{N}^{[\infty]} : \ (x_n)_{n\in A} \text{ is convergent}\} = \mathcal{C}(x_n)_n$$

In particular, $\mathcal{C}(x_n)_n$ has a Borel selector.

## Cofinal $p$-ideals

$\mathcal{I}$ is a $p$-ideal, if for all $A_n \in \mathcal{I}$, $n \in \mathbb{N}$, there is $A \in \mathcal{I}$ such that

$$A_n \subseteq^* A \quad \text{for all } n \in \mathbb{N}$$

Theorem: If $\mathcal{I}$ is an analytic cofinal (i.e. dense) $p$-ideal, then there is $\varphi : \mathbb{N}^{[2]} \to 2$ such that

$$hom(\varphi) \subseteq \mathcal{I}$$

In particular, $\mathcal{I}$ has a Borel selector.

Example:

$$\mathcal{I} = \{A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n+1} < +\infty\}$$

## Galvin's Lemma

Theorem: (Galvin, 1968) Let $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ be an open set and $A \subseteq \mathbb{N}$ infinite. There exists $B \subseteq A$ infinite such that either

$$B^{[\omega]} \cap \mathcal{O} = \emptyset \text{ or } B^{[\omega]} \subseteq \mathcal{O}.$$

Such sets $B$ are called homogeneous for $\mathcal{O}$.

Question: Does $hom(\mathcal{O})$ have a Borel selector for every open $\mathcal{O}$?

For $\mathcal{F} \subseteq \mathrm{FIN}$, let

$$\mathcal{O}_{\mathcal{F}} = \bigcup_{s \in \mathcal{F}} \{A \in \mathbb{N}^{[\infty]} : \ s \sqsubset A\}$$

Theorem: Let $\mathcal{B}$ be a front over $\mathbb{N}$. Let $\mathcal{F} \subseteq \mathcal{B}$ then $hom(\mathcal{O}_{\mathcal{F}})$ has a Borel selector.

An $\sqsubset$-antichain $\mathcal{B} \subseteq \mathrm{FIN}$ is a front on $\mathbb{N}$, if for all infinite $B \subseteq \mathbb{N}$ there is $s \in \mathcal{B}$ such that $s \sqsubseteq B$.