Finite powers of countably compact free abelian groups

Artur Hideyuki Tomita

USP, Brasil

August 16, 2013

The author has received financial support from CNPq (Brazil) - Processo n. 305612/2010-7- Bolsa de Produtividade em Pesquisa and Auxílio á Pesquisa FAPESP Proc. 2012/01490-9.

Countably compact groups without non-trivial convergent sequences

Hajnál and Juhász [Gen. Topology Appl., 1976] showed under CH that there exists a countably compact group topology of order 2 without non-trivial convergent sequences.

E. van Douwen [Trans. Amer. Math. Soc.,1980] obtained from MA a countably compact group without non-trivial convergent sequences.

Garcia Ferreira, Tomita and Watson [Proc. Amer. Math. Soc., 2005] showed that if p is a selective ultrafilter then there exists a p-compact group (in particular countably compact) without non-trivial convergent sequences.

Szeptycki and Tomita [Topology Appl.,2009] showed that in the Random model there exists a countably compact group without non-trivial convergent sequences.

A question of Comfort

Comfort [Open Problems in Topology, 1990] asked for which cardinals $\kappa \leq 2^{\mathfrak{c}}$ there exists a topological group G such that G^α is countably compact for $\alpha < \kappa$ but G^κ is not countably compact.

Hart and van Mill [Trans. Amer. Math. Soc., 1991] showed from $\mathrm{MA}_{\mathrm{countable}}$ that 2 is such cardinal.

Tomita [CMUC, 1996] showed from $\mathrm{MA}_{\mathrm{countable}}$ that there are infinitely many natural numbers as in Comfort's question.

Tomita [Topology Appl., 1999] showed from $\mathrm{MA}_{\mathrm{countable}}$ that 3 is such cardinal. All the examples contain non-trivial convergent sequences.

A question of Comfort - cont.

Tomita [Topology Appl., 2005 b] showed $\mathrm{MA}_{\mathrm{countable}}$ that every finite cardinal is a cardinal as in Comfort's question.

Tomita [Fund. Math., 2005] showed that it is consistent that every cardinal is as in Comfort's question.

Sanchis and Tomita [Topology Appl., 2012] showed that if there exists a selective ultrafilter then every cardinal $\leq \omega_1$ is as in Comfort's question.

A theorem of van Douwen

E. van Douwen [Trans. Amer. Math. Soc., 1980] showed in ZFC that if there exists a countably compact group of order 2 without non-trivial convergent sequences then it contains two countably compact subgroups whose product is not countably compact.

Tomita [Topology Appl., 2005 a] showed that if there exists a countably compact abelian group without non-trivial convergent sequences then 2 is as in Comfort's question.

Tomita [in preparation] showed that if $\alpha \leq \omega$ and there exists a topological group without non-trivial convergent sequences whose αth power is countably compact then α^+ is a cardinal as in Comfort's question.

Countably compact free abelian groups

Fuchs showed that an infinite free abelian group does not admit a compact group topology.

Tkachenko [Izvestia VUZ, 1990] showed that the free abelian group of size $\mathfrak{c}$ can be endowed with a countably compact group topology under CH.

Tomita [CMUC] obtained such a topology from $\mathrm{MA}(\sigma\!-\!\mathrm{centered})$.

Koszmider, Tomita and Watson [Topology Proc., 2000] from $\mathrm{MA}_{\mathrm{countable}}$.

Madariaga-Garcia and Tomita [Topology Appl., 2007] established the same result assuming the existence of $\mathfrak{c}$ pairwise incomparable selective ultrafilters (according to the Rudin-Keisler ordering in ω^*).

Countably compact group topologies in abelian groups

Tkachenko and Yashenko [Topology Appl., 2002] showed from MA that almost torsion free abelian groups of cardinality $\mathfrak{c}$ admit a countably compact group topology without non-trivial convergent sequences.

Dikranjan and Tkachenko [Forum Math., 2003] obtained from MA the characterization of all abelian groups of cardinality $\mathfrak{c}$ that admit a countably compact group topology.

Castro Pereira and Tomita [Topology Appl., 2010] obtained the characterization for torsion abelian groups of cardinality $\mathfrak{c}$ assuming the existence of a selective ultrafilter.

Boero and Tomita [Houston J. Math., 2013] obtained the result of Tkachenko and Yashenko from the existence of $\mathfrak{c}$ incomparable selective ultrafilters.

Boero and Tomita [in preparation] obtained the result of Dikranjan and Tkachenko from the existence of $\mathfrak{c}$ incomparable selective ultrafilters.

Countably compact groups in larger abelian groups

Koszmider, Tomita and Watson [Topology Proc., 2000] obtained via forcing a countably compact group topology on the free abelian group of cardinality $\mathfrak{c}$.

Tomita [Proc. Amer. Math. Soc., 2003] obtained via forcing a countably compact group of cardinality $\aleph_\omega$.

Castro Pereira and Tomita [Applied General Topology, 2004] obtained via forcing a countably compact free abelian group of cardinality $\aleph_\omega$.

Tomita [Topology Appl., 2005] obtained via forcing a countably compact group of cardinality $\aleph_\omega$ and weight greater than $\aleph_\omega$.

Countably compact groups in larger abelian groups - cont.

Dikranjan and Shakhmatov [Topology Appl., 2005] obtained via forcing the classification of all abelian groups of cardinality at most $2^{\mathfrak{c}}$ that admit a countably compact group topology.

Castro Pereira and Tomita [Topology Appl., 2010] obtained from the existence of a selective ultrafilter and a cardinal arithmetic weaker than GCH a characterization of all (without cardinality restrictions) torsion abelian groups that admit a countably compact group topology.

Powers of countably compact free Abelian groups

Tomita [CMUC, 1998] showed that if a non-trivial free abelian group is endowed with a group topology, then its ωth power cannot be countably compact. In the same paper it was shown that if follows from MA that there exists a countably compact free abelian group whose square is not countably compact.

Under $\mathfrak{p} = \mathfrak{c}$, Boero and Tomita [Fund. Math., 2011] proved that there exists a group topology on the free abelian group of size $\mathfrak{c}$ that makes its square countably compact. The natural generalization of the method in the paper above increased significantly the number of cases even for the cube.

To deal with finite powers we introduce stacks.

Stacks

We will call stacks a special finite family of sequences that will be the key to deal with countable compactness in finite powers of free abelian groups.

We will show that every sequence in a finite power of free abelian groups can be associated with a stack, so that, if the stack has an accumulation point, then the original sequence also has an accumulation point.

We will also construct, using the existence of $\mathfrak{c}$ incomparable selective ultrafilters, a group topology on the free abelian group of cardinality $\mathfrak{c}$ for which the associated stacks have an accumulation point.

Defining an integer stack

We give below a complete definition of integer stacks.
An *integer stack* $\mathcal{S}$ *on* A consists of

i) an infinite subset A of ω;

ii) natural numbers s, t, M, r_i and $r_{i,j}$ for each $0 \leq i < s$ and $0 \leq j < r_i$;

iii) functions $f_{i,j,k} \in (\mathbb{Z}^{(\mathfrak{c})})^A$ for each $0 \leq i < s$ and $0 \leq l < t$, $0 \leq j < r_i$ and $g_l \in (\mathbb{Z}^{(\mathfrak{c})})^A$ for each $0 \leq k < r_{i,j}$;

iv) sequences $\xi_i \in \mathfrak{c}^A$ for $0 \leq i < s$ and $\mu_l \in \mathfrak{c}^A$ for each $0 \leq l < t$ and

v) real numbers $\theta_{i,j,k}$ for each $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$ that satisfy

Defining an integer stack - cont.

1) $\mu_l(n) \in \operatorname{supp} g_l(n)$ for each $n \in A$;

2) $\mu_{l^*}(n) \notin \operatorname{supp} g_l(n)$ for each $n \in A$ and $0 \leq l^* < l < t$;

3) the elements of $\{\mu_l(n) : 0 \leq l < t \text{ and } n \in A\}$ are pairwise distinct;

4) $|g_l(n)| \leq M$ for each $n \in A$ and $0 \leq l < t$;

Defining an integer stack - cont.

5) $\{\theta_{i,j,k} : 0 \leq k < r_{i,j}\}$ is a linearly independent subset of $\mathbb{R}$ as a $\mathbb{Q}$-vector space for each $0 \leq i < s$ and $0 \leq j < r_i$;

6) $\lim_{n\in A} \frac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j,0}(n)(\xi_i(n))} \longrightarrow \theta_{i,j,k}$ for each $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$;

7) $\{|f_{i,j,k}(n)(\xi_i(n))| : n \in A\} \nearrow +\infty$ for each $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$;

8) $|f_{i,j,k}(n)(\xi_i(n))| > |f_{i,j,k^*}(n)(\xi_i(n))|$ for each $n \in A$, $i < s$, $j < r_i$ and $0 \leq k < k^* < r_{i,j}$;

Defining a stack - cont.

9) $\{\frac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j^*,k^*}(n)(\xi_i(n))} : n \in A\}$ converges monotonically to 0 for each $0 \leq i < s$, $0 \leq j^* < j < r_i$, $0 \leq k < r_{i,j}$ and $0 \leq k^* < r_{i,j^*}$ and

10) $\{f_{i,j,k}(n)(\xi_{i^*}(n)) : n \in A\} \subseteq [-M, M]$ for each $0 \leq i^* < i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$.

Subdividing the integer stack

We will subdivide the stack in bricks, blocks and substacks to give an overview of the construction

Definition

Let $\mathcal{S}$ be an integer stack on A. We will call $B_{i,j} = \{f_{i,j,k} : 0 \leq k < r_{i,j}\}$ a ξ_i-brick for each $0 \leq i < s$ and $0 \leq j < r_i$, $B_i = \bigcup\{B_{i,j} : 0 \leq j < r_i\}$ the ξ_i-block for each $0 \leq i < s$, $\mathcal{S}_{l.i.} = \bigcup\{B_i : 0 \leq i < s\}$ the l.i.-substack of $\mathcal{S}$ and $\mathcal{S}_d = \{g_l : 0 \leq l < t\}$ the d-substack of $\mathcal{S}$.

Properties 1)-4) are for the d-substack, properties 5)-8) are for a ξ_i-brick, property 9) is for the relation between the ξ_i-bricks inside the ξ_i-block and property 10) is for the relation between different blocks in the l.i.-substack.

Definition

Given an integer stack $\mathcal{S}$ and a natural number N, the Nth root of $\mathcal{S}$, $\frac{1}{N} - \mathcal{S}$ is obtained by keeping the same structure of $\mathcal{S}$ but replacing the functions by $f^*_{i,j,k} = \frac{1}{N}.f_{i,j,k}$ for each $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$ and $g^*_l = \frac{1}{N}.g_l$ for each $0 \leq l < t$.
A stack will be the Nth root of an integer stack for some positive integer N.

Note that a stack may be related to a set of sequences that are not in $\mathbb{Z}^{(\mathfrak{c})}$.

An overview of the construction

We first show that each sequence in a finite power of a free abelian group can be associated with a stack and an ultrafilter $\mathcal{U}$ so that if the stack has a $\mathcal{U}$-limit then the original sequence has an accumulation point. Then we construct a group topology in which enough stacks have accumulation points.

For the first task, start with a finite family $\{f_0, \ldots, f_{a-1}\}$ and take a selective ultrafilter $\mathcal{U}$. Consider $\mathbb{H}$ the group generated by the classes $\{[f_0]_\mathcal{U}, \ldots, [f_{a-1}]_\mathcal{U}\}$. Show that $\mathbb{H}$ can be written as a direct sum where the subgroup of the classes of constant sequences is a direct summand. Let $\mathcal{B}$ be a basis for the other direct summand. Let B be a finite family in the group generated by $\{f_0, \ldots, f_{a-1}\}$ such that $\mathcal{B} = \{[g]_\mathcal{U} : g \in B\}$. The set B is not adequate to produce the homomorphisms we need. Thus, we will take rational combinations of B to produce an integer stack whose equivalent classes produce the same subspace generated by $\{[g]_\mathcal{U} : g \in B\}$ as $\mathbb{Q}$-vector spaces. Here it is important that no non-trivial combination gives a class of constants since we may need to divide them by integers.

An overview of the construction -cont.

For the second task, we construct homomorphisms that will generate the group topology. To make these homomorphisms preserve the pre-assigned accumulation points for the stacks, it is necessary that there is some freedom between the functions in the stack. The order of the construction is first the d-substack and then the l.i.-substack. Basically the elements of the d-substack require the full circle, and the first brick in each ξ_i-blocks start with an arc of similar size. Within the ξ_i-block, the first brick treated uses a larger arc which will be shrinking during the construction of the other bricks. The technical part consists in solving equations involving arcs. The equations related to a brick use that large enough segments will have some density property using Kronecker's Lemma and linearity property. The length of the segments depend on $|f_{i,j,0}(n)(\xi_i(n))|$.

Selective ultrafilters

We recall some properties of selective ultrafilters.

Definition

A free ultrafilter $\mathcal{U}$ *is selective if for every partition* $\{A_i : i \in \omega\}$ *of* ω*, either there exists* $i \in \omega$ *such that* $A_i \in \mathcal{U}$ *or there exists* $B \in \mathcal{U}$ *such that* $|B \cap A_i| \leq 1$ *for each* $i \in \omega$*.*

Selective ultrafilters

We will use that the following properties are equivalent for a free ultrafilter:
a) If $\{a_i : i \in \omega\}$ is a sequence then there is $A \in \mathcal{U}$ such that $\{a_i : i \in \omega\}$ is either a constant sequence or a 1-1 sequence.
b) If $P_0 \cup P_1 = [D]^2$ and $D \in \mathcal{U}$ then there exists $C \in \mathcal{U}$, $C \subseteq D$ such that $[C]^2 \subseteq P_s$ for some $s < 2$.
c) $\mathcal{U}$ is a selective ultrafilter.
These equivalences were used in Tomita and Watson [Topology Appl., 2004] to obtain the first countably compact groups with special properties using selective ultrafilters.