Functorial constructions in paratopological groups reflecting separation axioms

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For the Sorgenfrey line $\mathbb{S}$, the topological group $\mathbb{S}^*$ is discrete.

Associated topological group

Theorem 1.1 (Alas–Sanchis, 2007).

Let G be a T_1 paratopological group. Then the diagonal $\Delta = \{(x,x) : x \in G\}$ is a closed subgroup of $G \times G'$ and, when considered with the topology induced from $G \times G'$, the diagonal Δ is a Hausdorff topological group topologically isomorphic to the group G^* associated to G.

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Corollary 1.3 (Reznichenko, 2005).

Every σ-compact Hausdorff (even T_1) paratopological group has countable cellularity.

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Idea of the proof: If G is a precompact paratopological group, then the non-empty open sets in G_* form a π-base for G.

Regularization of paratopological groups

Given a space X, let X_{sr} be the underlying set X endowed with the topology whose base is formed by the regular open sets in X:

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Theorem 1.6 (Ravsky, 2003).

For any paratopological group G, the semiregularization G_{sr} of G is a T_3 paratopological group. Hence the semiregularization of a Hausdorff paratopological group is a regular paratopological group.

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The group G_{sr} will be called the regularization of G and denoted by G_r.

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3) Use the Comfort–Ross theorem.
4) Note that a space X is feebly compact iff so is X_{sr}.

Discussion

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Question. *Is a similar construction possible in paratopological or semitopological groups?*

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The definition of a $\mathcal{P}$-reflection in the class of paratopological groups is the same (H must be a paratopological group).

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Theorem 2.2 (Tk., 2013).

For every $k = 0, 1, 2, 3, 3.5$*, there exists a covariant functor* T_k *in the category of semitopological groups such that* $T_k(G)$ *is the* T_k*-reflection of* G*, for an arbitrary semitopological group* G*.*

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$$T_1 \& T_3 \to \mathit{Reg} \text{ and } T_1 \& T_{3.5} \to \mathit{Tych}$$

Corollary 2.3.

For every semitopological (paratopological) group G and every $k \in \{0, 1, 2, 3, R\}$, there exists a continuous homomorphism $\varphi_{G,k}\colon G \to H$ onto a semitopological (paratopological) group H satisfying the T_k separation axiom such that for every continuous mapping $f\colon G \to X$ to a T_k-space X, one can find a continuous mapping $h\colon H \to X$ with $f = h \circ \varphi_{G,k}$. [$R$ stands for regularity.]

‘Internal’ description of the groups $T_0(G)$

The canonical homomorphism $\varphi_{G,k}\colon G \to T_k(G)$ is continuous, open, and surjective for $k = 0, 1, 2$ (Theorem 2.2). Hence $T_k(G)$ is a quotient group of G in this case.

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Conclusion: To describe the group $T_k(G)$ for $k = 0, 1, 2$ in 'internal' terms, it suffices to determine the kernel N_k of the homomorphism $\varphi_{G,k}$. Then $T_k(G) \cong G/N_k$ and $\varphi_{G,k}$ is simply the quotient homomorphism $\pi_k\colon G \to G/N_k$.

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Conclusion: To describe the group $T_k(G)$ for $k = 0, 1, 2$ in ‘internal’ terms, it suffices to determine the kernel N_k of the homomorphism $\varphi_{G,k}$. Then $T_k(G) \cong G/N_k$ and $\varphi_{G,k}$ is simply the quotient homomorphism $\pi_k \colon G \to G/N_k$.

Let us start with $k = 0$.

Theorem 3.1.
Let G be an arbitrary semitopological group and $\mathcal{N}(e)$ the family of open neighborhoods of the neutral element e in G. Then $N_0 = P \cap P^{-1}$, where $P = \bigcap \mathcal{N}(e)$. Hence $T_0(G) \cong G/N_0$.

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Warning: The subgroup N_0 of G is not necessarily closed in G.

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TRY IT! (A hint follows.)

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Open problem. Give an internal description of the kernel N_2 of the canonical homomorphism $\varphi_{G,2}\colon G \to T_2(G)$, for an arbitrary semitopological group G.

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We solve the problem for paratopological groups:

Theorem 3.3.
Let G be a paratopological group and $\mathcal{N}(e)$ the family of open neighborhoods of the neutral element e in G. Then

$$N_2 = \bigcap_{U \in \mathcal{N}(e)} \overline{U}$$

or, equivalently,

$$N_2 = \bigcap_{U \in \mathcal{N}(e)} UU^{-1}.$$

Hence $T_2(G) \cong G/N_2$.

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Again, we **do not know** any description of $T_3(G)$ or $Reg(G)$, for a semitopological group G.

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For every semitopological group G, the canonical homomorphism $\varphi_{G,3}\colon G \to T_3(G)$ is a continuous bijection. Hence the kernel N_3 of $\varphi_{G,3}$ is trivial.

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‘Internal’ description of the groups $T_3(G)$ and $Reg(G)$

Theorem 3.6.
$T_3(G)$ is the regularization of G, i.e., $T_3(G) \cong G_r$, for every paratopological group G.

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Here is a two-step description of the groups $Reg(G)$:

Theorem 3.7.
Let G be an arbitrary paratopological group. Then $Reg(G)$ is the regularization of the paratopological group $T_2(G)$. Therefore, $Reg(G) \cong (G/N_2)_r$.

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Theorem 3.7 admits a more general functorial form:

$$Reg \cong T_3 \circ T_2.$$

Properties of the functors T_k's

Regularity $= T_1 + T_3$. Does this imply that $Reg \cong T_3 \circ T_1$ or $Reg \cong T_1 \circ T_3$ in the category of paratopological groups?

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The functors Reg, $T_0 \circ T_3$, $T_1 \circ T_3$ and $T_2 \circ T_3$ are naturally equivalent in the category of semitopological groups.

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Open problem. *Do the functors T_2 and T_3 commute in the category of semitopological groups?*

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Similarly, $T_0 \circ T_3 \ncong T_3 \circ T_0$.

Products and functors

Let $\Pi = \prod_{i \in I} G_i$ be a product of semitopological (paratopological) groups. We wonder whether the 'equality'

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For each of the functors T_0, T_1, T_2, the proof of Theorem 5.1 is 'individual', depending on the form of $N_k = \ker \varphi_{G,k}$ for $k = 0, 1, 2$.

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Proof of Theorem 6.1. G is σ-compact $\implies T_2(G)$ is σ-compact. Hence, by Lemma 6.2, $c(G) = c(T_2(G)) \leq \omega$. $\square$

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Proof of Theorem 6.1. G is σ-compact $\Longrightarrow$ $T_2(G)$ is σ-compact. Hence, by Lemma 6.2, $c(G) = c(T_2(G)) \leq \omega$. $\square$

In fact, the conclusion of Theorem 6.1 can be strengthened: *Every σ-compact paratopological group has the Knaster property.*

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Theorem 6.3 (W.W. Comfort, J. Trigos–Arrieta, M. Sanchis ??).

The product of an arbitrary family of locally pseudocompact topological groups is a Moscow space.

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One can drop 'Hausdorff' in the above theorem!

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On the existence of T_k-reflections

Theorem 6.6 (Pontryagin, $\cong$ 1935).

For every continuous real-valued function f on a compact topological group G, one can find a continuous homomorphism $\pi\colon G \to H$ onto a compact metrizable topological group H and a continuous function g on H such that $f = g \circ \pi$.

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Pontryagin's idea: Given a continuous function f on G as above, consider the set

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Crucial step: Let us forget about both the compactness of G and topological group structure of G and then apply Pontryagin's formula directly to a continuous mapping $f\colon G \to X$ defined on a semitopological group G.

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