A Provisional Solution of Nyikos' Manifold Problem: Hereditarily Normal Manifolds of Dimension > 1 May All Be Metrizable

(Verification in progress: proofs by Todorcevic and by Dow need to be checked)

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July 23, 2013

History

- Wilder's problem
- Rudin's solution
- Nyikos' problem

Theorem 1 (Provisional)

If it is consistent there is a supercompact cardinal, it is consistent that every hereditarily normal manifold of dimension > 1 *is metrizable.*

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2. Construct a coherent Souslin tree S.
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We say *PFA*$(S)[S]$ *implies* Φ to mean that if Φ is a proposition, S is a coherent Souslin tree, then any model formed via (3) and (4) is a model of Φ.

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M-M: Compact, countably tight spaces are sequential.

Definition

A collection $\mathcal{I}$ of countable subsets of a set X is a **P-ideal** if each subset of a member of $\mathcal{I}$ is in $\mathcal{I}$, finite unions of members of $\mathcal{I}$ are in $\mathcal{I}$, and whenever $\{I_n : n \in \omega\} \subseteq \mathcal{I}$, there is a $J \in \mathcal{I}$ such that $I_n - J$ is finite for all n.

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$\mathcal{I}$ is $\aleph_1$-**generated** if there is $\{I_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{I}$ such that for each $I \in \mathcal{I}$, there is an α such that $I \subseteq I_\alpha$.

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1. there is a stationary $E \subseteq S$ such that every countable subset of E is in $\mathcal{I}$, or
2. there is a stationary $D \subseteq S$ such that for every countable subset D_1 of D, $D_1 \cap I$ is finite, for each $I \in \mathcal{I}$.

Theorem 2

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Theorem 3 (Provisional)
PFA$(S)[S]$ *implies* $\sum^{-}, \mathbf{CW}, \mathbf{PPI}$, *and* $\mathbf{P_{22}}(\aleph_1)$.

Proof of Theorem 2.
Assemble pieces from Balogh's and Nyikos' papers. □

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- **Steps remaining:** Check Dow's proof; Todorcevic fills gap in his proof that PFA$(S)[S]$ implies **M-M**.

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By doing a preliminary forcing, one can get a model in which also:

7. normal spaces which are either first countable or locally compact are collectionwise Hausdorff,
8. locally compact perfectly normal spaces are paracompact [LT10].

The topology

Definition

A locally compact space X is of **Type I** if it can be expressed as $X = \bigcup_{\alpha<\omega_1} M_\alpha$, where each M_α is open, $\overline{M}_\alpha$ is Lindelöf and included in $M_{\alpha+1}$, and for limit α, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.
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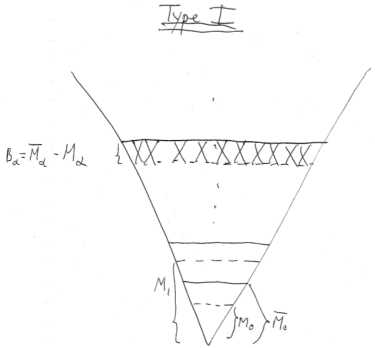

For a manifold, we may assume each M_α is Lindelöf.

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If it's consistent there is a supercompact, it's consistent that a locally compact T_5 space is paracompact iff it does not include a copy of ω_1.

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A selection of one point from each non-empty $B_\alpha = \overline{M}_\alpha - M_\alpha$, where $\{M_\alpha : \alpha < \omega_1\}$ is a canonical sequence for a Type I space is called a **bone-scan**.

Lemma 4

Suppose S is a stationary subset of ω_1 and $Y = \{y_\alpha : \alpha \in S\}$ is a subset of a bone-scan of a canonical sequence for a Type I, normal, first countable space M, such that countable subsets of Y have compact closure in M. Then $\overline{Y}$ is a perfect pre-image of ω_1.

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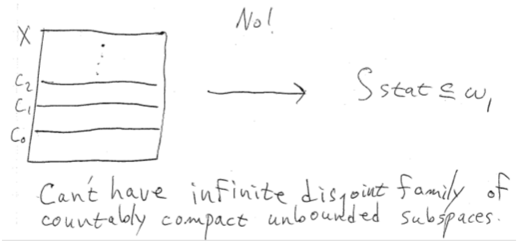

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Proof.

$CH \to \exists$ compact S-space: the Kunen Line [JKR]. $\sum^{-}$ implies there are no compact S-spaces, since S-spaces are countably tight. □

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Proof.

Non-metrizable T_5 Type I manifold M includes a perfect pre-image of ω_1, and hence a copy W of ω_1. W must meet stationarily many B_α, else it would be a closed subspace of a sum of Lindelöf spaces and so would be paracompact.

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$\sum^{-} + \mathbf{CW} + \mathbf{PPI} + \mathbf{P_{22}}(\aleph_1)$ *implies* T_5 *manifolds of dimension greater than* 1 *are metrizable.*

Proof.

Non-metrizable T_5 Type I manifold M includes a perfect pre-image of ω_1, and hence a copy W of ω_1. W must meet stationarily many B_α, else it would be a closed subspace of a sum of Lindelöf spaces and so would be paracompact. Pick $w_\alpha \in W \cap B_\alpha$, $\alpha \in$ some stationary $E_0 \subseteq \omega_1$.

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Proof continued.

A Type I space is paracompact iff for (some) every canonical sequence $\{M_\alpha\}_{\alpha<\omega_1}$, club many $(\overline{M}_\alpha - M_\alpha)$'s are empty.

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Some more proofs.

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Definition

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Lemma 5

In a normal space, the closure of a conditionally compact subspace is countably compact.

Proof.

Exercise. □

Lemma 4

Suppose S is a stationary subset of ω_1 and $Y = \{y_\alpha : \alpha \in S\}$ is a subset of a bone-scan of a canonical sequence for a Type I, normal, first countable space M, such that countable subsets of Y have compact closure in M. Then $\overline{Y}$ is a perfect pre-image of ω_1.

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Suppose S is a stationary subset of ω_1 and $Y = \{y_\alpha : \alpha \in S\}$ is a subset of a bone-scan of a canonical sequence for a Type I, normal, first countable space M, such that countable subsets of Y have compact closure in M. Then $\overline{Y}$ is a perfect pre-image of ω_1.

Proof.

Let $\{M_\alpha : \alpha < \omega_1\}$ be the canonical sequence. Let $B_\alpha = \overline{M}_\alpha - M_\alpha$. Let $Y_\alpha = \overline{Y} \cap B_\alpha$. Then $\overline{Y} = \bigcup_{\alpha<\omega_1} Y_\alpha$. Define

$$\pi : \overline{Y} \to \omega_1$$

by $\pi(y) = \alpha$ iff $y \in Y_\alpha$. First, claim $\pi^{-1}(\{\alpha\})$ is compact. Y_α is closed in B_α, which is closed and Lindelöf, so it is closed in $\overline{Y}$ and is Lindelöf. It suffices to show $\overline{Y}$ is countably compact. But Y is conditionally compact.

Proof continued.

Next, claim π is continuous. This is because if $y_{\alpha_n} \in Y_{\alpha_n}$ and $y_{\alpha_n} \to y_\alpha$, then $y_\alpha \in Y_\alpha$.

Proof continued.

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Note π is closed, since continuous images of countably compact spaces are countably compact.

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The range of π is unbounded, since S is uncountable and the B_α's are disjoint.

Proof continued.

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Note π is closed, since continuous images of countably compact spaces are countably compact.

The range of π is unbounded, since S is uncountable and the B_α's are disjoint. Range π is closed, since π is closed.

Proof continued.

Next, claim π is continuous. This is because if $y_{\alpha_n} \in Y_{\alpha_n}$ and $y_{\alpha_n} \to y_\alpha$, then $y_\alpha \in Y_\alpha$.

Note π is closed, since continuous images of countably compact spaces are countably compact.

The range of π is unbounded, since S is uncountable and the B_α's are disjoint. Range π is closed, since π is closed. Thus range π is club, so homeomorphic to ω_1. $\square$

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