(a)-spaces and selectively (a)-spaces from almost disjoint families

Samuel Gomes da Silva

Federal University of Bahia
Salvador – Bahia – Brazil

Maresias – Brazil

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A star selection principle

This paper was accepted for publication in **Acta Mathematica Hungarica**, and it is dedicated to Ofelia Alas - who, in her kind and generous way, made several comments and suggestions on previous versions of the paper.

Thanks again, Ofelia !

Throughout the paper, we work with a **star covering property** and with a **selective version** of it.

(After the submission of this paper, I learned that selective versions of star covering properties, as well as other similar notions, are becoming to be known as **star selection principles**.)

The definitions

Matveev, 94/97

X has **property** (a) (or is said to be an (a)-**space**) if for every open cover $\mathcal{U}$ of X and for every dense set $D \subseteq X$ there is $F \subseteq D$ such that F is a closed discrete subset of X and $St(F, \mathcal{U}) = X$.

Caserta, Di Maio, Kočinac, 2011

A topological space X is said to be a **selectively** (a)-**space** if for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers and for every dense set $D \subseteq X$ there is a sequence $\langle A_n : n < \omega \rangle$ of subsets of D which are closed and discrete in X and such that $\{St(A_n, \mathcal{U}_n) : n < \omega\}$ is a open cover of X.

Spaces from almost disjoint families

It is natural, given a class of topological spaces, to wonder under which conditions these notions are equivalent – or not – when restricted to such class.

We consider such question for spaces constructed from **almost disjoint families** - the well-known Mrówka-Isbell spaces of the form $\Psi(\mathcal{A})$, where $\mathcal{A}$ denotes an almost disjoint family of infinite subsets of ω.

As probably expected, both properties under investigation, when restricted to Ψ-spaces, have nice combinatorial characterizations.

Combinatorial characterizations

Szeptycki and Vaughan, 1998

Given an almost disjoint family $\mathcal{A}$, the corresponding Ψ-space satisfies property (a) if, and only if,

$(\forall f : \mathcal{A} \to \omega)\ (\exists P \subseteq \omega)\ (\forall A \in \mathcal{A})\ [0 < |P \cap (A \setminus f(A))| < \omega]$.

da Silva, 2013

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an a. d. family. The corresponding space $\Psi(\mathcal{A})$ is selectively (a) if, and only if, the following property holds:

For every sequence $\langle f_n : n < \omega \rangle$ of functions such that $f_n \in {}^{\mathcal{A}}\omega$ for every $n < \omega$, there is a sequence $\langle P_n : n < \omega \rangle$ of subsets of ω satisfying both following clauses:

(i) $(\forall n < \omega)(\forall A \in \mathcal{A})[|P_n \cap A| < \omega]$
(ii) $(\forall A \in \mathcal{A})(\exists n < \omega)[P_n \cap (A \setminus f_n(A)) \neq \emptyset]$

Matveev, in 1997, showed that separable, (a)-spaces cannot include closed discrete subsets of size $\mathfrak{c}$.

Such result is usually referred as **Matveev's** (a)-**Jones' Lemma**. His proof was done for the separable case, but is straightforward to give a general proof (for $d(X) = \kappa$).

Now we give the selective version of such result (also in the general case).

The separable case of the following proposition was already remarked, without a proof, by Caserta, di Maio and Kočinac.

On the extent of selectively (a)-spaces

da Silva, 2013

If X is a selectively (a)-space and H is a closed discrete subset of X, then $|H| < 2^{d(X)}$.

Sketch of the proof: The proof is by contraposition. Let D be a dense set, $|D| = d(X)$, and $|H| \geqslant 2^{d(X)}$. W.l.g., $H \cap D = \emptyset$.

$(2^{d(X)})^{\aleph_0} = 2^{d(X)} \leqslant |H|$ and so we are allowed to use H to index the family of all sequences of closed discrete subsets of D; let $\{G_x : x \in H\}$ be such family (with $G_x = \langle G_{x,n} : n < \omega \rangle$, say).

For every fixed $n < \omega$ and $x \in H$, let $U_{x,n}$ be the open neighbourhood of x given by $U_{x,n} = X \setminus \Big((H \setminus \{x\}) \cup G_{x,n} \Big)$ and consider the open cover $\mathcal{U}_n = \{X \setminus H\} \cup \{U_{x,n} : x \in H\}$.

It is easy to check that D and the sequence $\langle \mathcal{U}_n : n < \omega \rangle$ witness that X is not selectively (a).

Ψ-spaces are separable, and $\mathcal{A}$ is closed discrete in $\Psi(\mathcal{A})$; so, if $\Psi(\mathcal{A})$ is selectively (a) then $|\mathcal{A}| < \mathfrak{c}$.

In general, **separable selectively (a) spaces cannot include closed discrete subsets of size $\mathfrak{c}$.**

This lead us to the following result:

da Silva, 2013

Under **CH**, separable, Moore, selectively (a)-spaces are metrizable.

The proof goes easily, considering the boldfaced phrase of this slide and the following result (due to van Douwen, Reed, Roscoe and Tree): "If X is a Moore space such that $w(X)$ does not have countable cofinality, then there is a closed discrete subset D of X such that $|D| = w(X)$".

More consistency results

For now on, we focus on consistency results related to equivalence and non-equivalence of the properties under investigation, restricted to the class of Ψ-spaces.

First, we remark the following:

Consistency of the equivalence

Assume **CH** and let $\Psi(\mathcal{A})$ be a Ψ-space. Then both properties under investigation – property (a) and its selective version – are equivalent to the countability of the almost disjoint family $\mathcal{A}$.

Indeed: under **CH**, Matveev's result – and its selective version – avoid the existence of uncountable a.d. families whose corresponding space satisfy (a) or selectively (a). On the other hand, countable a. d. families always correspond to **metrizable** Ψ-spaces !

CH is independent of the equivalence between being (*a*) and being selectively (*a*)

Here we use Martin's Axiom/small cardinals for obtaining models of $\neg$**CH** were the properties are equivalent for Ψ-spaces.

It is well-known that $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$ (a classical result from Bell).

Szeptycki and Vaughan (1998) have considered a σ-centered p.o. to prove within **ZFC** that if $|\mathcal{A}| < \mathfrak{p}$ then $\Psi(\mathcal{A})$ has property (*a*).

So, we have the following:

The equivalence is consistent with ZFC + $\neg$CH (da Silva, 2013)

If $\mathfrak{p} = \mathfrak{c}$, then a Ψ-space satisfies property (*a*) if, and only if, satisfies its selective version.

In fact, this also shows that even "$2^{\aleph_0} < 2^{\aleph_1}$" is independent of the referred equivalence.

A ZFC result

To give a framework for further consistency results, we now prove that, in a certain way, the role played by $\mathfrak{p}$ in the context of (a)-spaces is played by $\mathfrak{d}$ in the context of selectively (a)-spaces.

Proposition (da Silva, 2013)

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an infinite a. d. family.

(i) If $|\mathcal{A}| < \mathfrak{d}$, then $\Psi(\mathcal{A})$ is selectively (a).

(ii) Suppose $\mathcal{A}$ is maximal. Then $\Psi(\mathcal{A})$ is selectively (a) if, and only if, $|\mathcal{A}| < \mathfrak{d}$.

Sketch of the proof

For the first part:

- $\mathcal{A}$ a.d. family of size $|\mathcal{A}| < \mathfrak{d}$, $\langle \mathcal{U}_n : n < \omega \rangle$ arbitrary sequence of open covers of X.

- For $A \in \mathcal{A}$ and $n < \omega$, let $U_{A,n}$ be an open neighbourhood of A which belongs to $\mathcal{U}_n$.

- $\mathcal{F} = \{f_A : A \in \mathcal{A}\} \subseteq {}^{\omega}\omega$ defined by putting $f_A(n) = \min(U_{A,n} \cap \omega)$ for every $A \in \mathcal{A}$ and $n < \omega$.

- As $|\mathcal{F}| \leqslant |\mathcal{A}| < \mathfrak{d}$, there is $f : \omega \to \omega$ such that for every $A \in \mathcal{A}$ there is $m < \omega$ such that $f_A(m) < f(m)$.

- Define $A_n = \{k < \omega : 0 \leqslant k \leqslant f(n)\} \cup \{n\}$ and we are done.

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For the second part:

- Let $\mathcal{A}$ be a maximal a.d. family $\mathcal{A}$, $|\mathcal{A}| \geqslant \mathfrak{d}$. We show that $\Psi(\mathcal{A})$ is not selectively (a).
- Take: $\mathcal{A}' \subseteq \mathcal{A}$ with $|\mathcal{A}'| = \mathfrak{d}$, say $\mathcal{A}' = \{A_\alpha : \alpha < \mathfrak{d}\}$, and $\{f_\alpha : \alpha < \mathfrak{d}\}$ a dominating family in the pointwisely defined order.
- For each $n < \omega$, take the open cover $\mathcal{U}_n = \{\{A_\alpha\} \cup (A_\alpha \setminus f_\alpha(n)) : \alpha < \mathfrak{d}\} \cup \{X \setminus \mathcal{A}'\}$.
- If $\langle P_n : n < \omega \rangle$ is an arbitrary sequence of closed discrete subsets of the dense set ω , the **maximality** of $\mathcal{A}$ ensures that each one of the P_n's is a **finite** set.
- If $g : \omega \to \omega$ is defined by putting $g(n) = \sup(P_n) + 1$, take $\xi < \mathfrak{d}$ such that $g(n) \leqslant f_\xi(n)$ for every $n < \omega$. Then we have that $A_\xi \notin \bigcup\{St(P_n, \mathcal{U}_n) : n < \omega\}$, so we are done.

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And more consistency results

It is well known that MAD families are not (*a*)-spaces. It follows that:

Corollary (da Silva, 2013)

It is consistent that there are selectively (*a*) spaces, constructed from almost disjoint families, which are not (*a*)-spaces.

Indeed, one has just to consider an infinite MAD family of minimal size in a model of $\mathfrak{a} < \mathfrak{d}$.

And more consistency results

By just mimicking the proof of the first part of the preceding proposition, we also get the following:

Corollary (da Silva, 2013)

Let X be a T_1 separable space with $|X| < \mathfrak{d}$ and suppose X has the following property:

(*) Any dense subset of X has a countable, dense subset.

Under these assumptions, X is a selectively (a) space.

In particular, it is consistent that T_1 spaces satisfying (*) and with size less than $\mathfrak{c}$ are all selectively (a).

Examples of spaces satisfying (*): spaces with a countable base, or even first countable separable spaces; hereditarily separable spaces; separable spaces with a dense set of isolated points; and so on.

Notes and Questions

Adding $\aleph_{\omega_1}$ Cohen reals to a model of **GCH**, one has $\aleph_1 = \mathfrak{a} < \mathfrak{d} = \mathfrak{c}$ and $2^{\aleph_0} < 2^{\aleph_1}$ in the extension. So we have:

Proposition (da Silva, 2013)

The following statement is consistent with **ZFC** $+ 2^{\aleph_0} < 2^{\aleph_1}$:

"There is a Ψ-space which is selectively (a) but does not satisfy property (a)"

Proposition (da Silva, 2013)

The following statement is consistent with **ZFC** $+ 2^{\aleph_0} < 2^{\aleph_1}$:

"There is a separable, selectively (a)-space with an uncountable closed discrete subset."

Notes and Questions

The interest of the 1st proposition of the previous slide is: the equivalence between our properties, restricted to Ψ-spaces, is independent of $2^{\aleph_0} < 2^{\aleph_1}$ (before, we have shown the independency in the other way around...).

The interest of the 2nd proposition of the previous slide is: it is still an open question (due to the speaker) whether $2^{\aleph_0} < 2^{\aleph_1}$ alone suffices to avoid the existence of separable (a)-spaces with uncountable closed discrete subsets. So, we could see that, for selectively (a)-spaces, it doesn't !!!

This is an example of an open problem on (a)-spaces which, after relaxing for selectively (a)-spaces, is settled.

Selective versions of previous questions on (a)-spaces

However, there are open questions in the literature, previously posed for (a), which still can be formulated for selectively (a).

Three surviving questions

- (posed for (a)-spaces by the speaker)

  Is it consistent that there is an a.d. family $\mathcal{A}$ of size $\mathfrak{d}$ such that $\Psi(\mathcal{A})$ is selectively (a) ?

- (posed for (a)-spaces by Szeptycki)

  If $\Psi(\mathcal{A})$ is normal, is it a selectively (a)-space ?

- (posed for (a)-spaces by the speaker)

  If $\Psi(\mathcal{A})$ is countably paracompact, is it a selectively (a)-space ?

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Are we talking of small cardinals ???

To finish, notice that all consistency results of this paper were given in terms of small cardinals.

Is there a way to describe precisely the possibilities of equivalence between our two properties, when restricted to Ψ-spaces, by using such cardinals ?

The final problem

Find a statement φ, if any, enunciated in terms of small cardinals, such that (a) and selectively (a) are equivalent properties for Ψ-spaces if, and only if, φ holds.

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Hope see you soon in Salvador !