On $(\mathcal{P}, \mathcal{Q})$-structured spaces

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$(\mathcal{P}, \mathcal{Q})$-structured spaces

Definition

Let $\mathcal{P}$ and $\mathcal{Q}$ be some classes of topological spaces. A topological space X will be called $(\mathcal{P}, \mathcal{Q})$-*structured* if there exists a subspace Y of X (called a $\mathcal{P}$-kernel of X) such that $Y \in \mathcal{P}$ and for every open neighborhood U of Y in X, the subspace $X \setminus U$ belongs to $\mathcal{Q}$.

$(\mathcal{P}, \mathcal{Q})$-structured spaces

We denote by $\mathcal{M}$ the class of metrizable separable spaces, $\mathcal{K}$ the class of compact spaces, $L\Sigma$ the class of Lindelöf-Σ spaces. $\mathcal{P}$ and $\mathcal{Q}$ will be subclasses of $L\Sigma$.

The $(L\Sigma, L\Sigma)$-structured spaces are also called *Charming spaces*.

Some basic results

Proposition

Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of topological spaces closed under countinuous images, closed subspaces, and such that $\mathcal{M} \subseteq \mathcal{P} \subseteq \mathcal{Q} \subseteq L\Sigma$. Then

1. Every continuous image of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space.

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2. Every closed subspace of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space.
3. If $\mathcal{P}$ and $\mathcal{Q}$ are closed under perfect preimages. Then every perfect preimage of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space.

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2. Every closed subspace of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space.
3. If $\mathcal{P}$ and $\mathcal{Q}$ are closed under perfect preimages. Then every perfect preimage of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space.
4. Every $(\mathcal{P}, \mathcal{Q})$-structured space is Lindelöf.

Some basic results

Corollary

1. Every continuous image of a $(L\Sigma, L\Sigma)$-structured space is a $(L\Sigma, L\Sigma)$-structured space.
2. Every closed subspace of a $(L\Sigma, L\Sigma)$-structured space is a $(L\Sigma, L\Sigma)$-structured space.
3. Every perfect preimage of a $(L\Sigma, L\Sigma)$-structured space is a $(L\Sigma, L\Sigma)$-structured space.
4. Every $(L\Sigma, L\Sigma)$-structured space is Lindelöf.

Some basic results

Remark

Since a multivalued mapping $p : X \to Y$ is compact-valued upper semicontinuous iff it is a composition of the inverse of a perfect mapping and a continuous function, we have that the image of a $(\mathcal{P}, \mathcal{Q})$-structured space under a compact-valued upper semicontinuous mapping is a $(\mathcal{P}, \mathcal{Q})$-structured space.

Some basic results

Remark

1. The product $X \times K$ of a $(\mathcal{P}, \mathcal{Q})$-structured space X and a compact space K is a $(\mathcal{P}, \mathcal{Q})$-structured space.

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2. If X is a $(\mathcal{P}, \mathcal{Q})$-structured space, then the Alexandroff duplicate $AD(X)$ of X is a $(\mathcal{P}, \mathcal{Q})$-structured space.

Countable...

If we assume that $\mathcal{P}$ and $\mathcal{Q}$ are closed under countable unions we have

Proposition

1. Let $\{X_n : n \in \mathbb{N}\}$ be a family of $(\mathcal{P}, \mathcal{Q})$-structured spaces. Then $X = \bigoplus\{X_n : n \in \mathbb{N}\}$ is $(\mathcal{P}, \mathcal{Q})$-structured.

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2. Let X be a space and $\{X_n : n \in \mathbb{N}\}$ a family of $(\mathcal{P}, \mathcal{Q})$-structured subspaces of X. Then $\bigcup\{X_n : n \in \mathbb{N}\}$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.

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3. Let X be a $(\mathcal{P}, \mathcal{Q})$-structured space. Then every F_σ subspace of X is a $(\mathcal{P}, \mathcal{Q})$-structured space.

Countable...

Corollary

1. Let $\{X_n : n \in \mathbb{N}\}$ be a family of $(L\Sigma, L\Sigma)$-structured spaces. Then $X = \bigoplus\{X_n : n \in \mathbb{N}\}$ is $(L\Sigma, L\Sigma)$-structured.
2. Let X be a space and $\{X_n : n \in \mathbb{N}\}$ a family of $(L\Sigma, L\Sigma)$-structured subspaces of X. Then $\bigcup\{X_n : n \in \mathbb{N}\}$ is a $(L\Sigma, L\Sigma)$-structured space.
3. Let X be a $(L\Sigma, L\Sigma)$-structured space. Then every F_σ subspace of X is a $(L\Sigma, L\Sigma)$-structured space.

Two examples

A $(\mathcal{M},\mathcal{M})$-structured no $L\Sigma$ space

Let $Y = L_\kappa$ (with $\kappa \geq \omega_1$) the Lindelöfication of the discrete space of cardinality κ, then Y is a $(\mathcal{M},\mathcal{M})$-structured space that is not $L\Sigma$.

Two examples

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Let $Y = L_\kappa$ (with $\kappa \geq \omega_1$) the Lindelöfication of the discrete space of cardinality κ, then Y is a $(\mathcal{M}, \mathcal{M})$-structured space that is not $L\Sigma$.

A Lindelöf no $(L\Sigma, L\Sigma)$-structured space

Let $Y = L_\kappa \times L_\kappa$ (with $\kappa \geq \omega_1$) the square of the Lindelöfication of the discrete space of cardinality κ, then Y is not a $(L\Sigma, L\Sigma)$-structured space.

Products

Proposition

Let X be a $(\mathcal{P}, \mathcal{Q})$-structured space and Z a σ-compact space. Then $X \times Z$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.

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Proposition

Let X belong to $\mathcal{P}$ and κ an infinite cardinal. Then $L_\kappa \times X$ is a $(\mathcal{P}, L\Sigma)$-structured space.

Products

Remark

The previous proposition let us construct $(L\Sigma, L\Sigma)$-structured spaces that are not $L\Sigma$-spaces. In fact, given a Lindelöf-Σ space X, we have that $L_{\omega_1} \times X$ is a $(L\Sigma, L\Sigma)$-structured space that is not $L\Sigma$ and there is a $L\Sigma$-kernel of $L_{\omega_1} \times X$ homeomorphic to X.

Stability

An important property of the $L\Sigma$-spaces is that they are stable.

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Recall that a topological space X is κ-stable if for every continuous image Y of X, if Y condenses onto a space Z with $w(Z) \leq \kappa$ we have that $nw(Y) \leq \kappa$. A space is *stable* if it is κ-stable for every infinite cardinal κ.

Stability

We can give a partial result of stability for $(L\Sigma, L\Sigma)$-structured spaces.

Proposition

Let X be a $(\mathcal{K}, L\Sigma)$-structured space. Then X is stable.

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Corollary

Let X be a $(\mathcal{K}, L\Sigma)$-structured space. Then $C_p(X)$ is monolithic.

$C_p(X)$ and $(L\Sigma, L\Sigma)$-structured spaces

Proposition

Let X be a Tychonoff space. Then $C_p(X)$ is a $(\mathcal{K}, L\Sigma)$-structured space if and only if $C_p(X)$ is a Lindelöf-Σ space.

$C_p(X)$ and $(L\Sigma, L\Sigma)$-structured spaces

Theorem

Let X be a Tychonoff space. Then $C_p(X)$ is a $(\mathcal{K}, L\Sigma)$-structured space if and only if $C_p(X)$ is a Lindelöf-Σ space.

Proposition

Let $\mathcal{P}$ and $\mathcal{Q}$ subclasses of $L\Sigma$. If $\mathcal{Q}$ is closed under countable unions and $C_p(X)$ is a $(\mathcal{P}, \mathcal{Q})$-structured space with a non dense $\mathcal{P}$-kernel, then $C_p(X)$ is a $L\Sigma$-space.

$C_p(X)$ and $(L\Sigma, L\Sigma)$-structured spaces

Theorem

Let $\mathcal{P}$ be a subclass of $L\Sigma$ that includes $\mathcal{M}$. Let X be a compact space. Then $C_p(X)$ is $(\mathcal{P}, L\Sigma)$-structured space if and only if $C_p(X)$ is a Lindelöf Σ-space.

$C_p(X)$ and $(L\Sigma, L\Sigma)$-structured spaces

Theorem (Guerrero-Tkachuk)

Given a space X and a closed-hereditary property $\mathcal{P}$, if $C_p(X, \mathcal{I})$ has a closed clousure-preserving cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$, then $C_p(X, \mathcal{I})$ also has the property $\mathcal{P}$.

$C_p(X)$ and $(L\Sigma, L\Sigma)$-structured spaces

Given that the property of be a $(\mathcal{P}, \mathcal{Q})$-structured space is preserved by closed subspaces, we have the next result.

$C_p(X)$ and $(L\Sigma, L\Sigma)$-structured spaces

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Corollary

Let X be a Tychonoff space and suppose that $C_p(X, \mathcal{I})$ has a clousure preserving closed cover $\mathcal{C}$ such that every element of it is a $(\mathcal{P}, \mathcal{Q})$-structured space, then $C_p(X, \mathcal{I})$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.

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Corollary

Let X be a Tychonoff space and suppose that $C_p(X, \mathcal{I})$ has a clousure preserving closed cover $\mathcal{C}$ such that every element of it is a $(\mathcal{P}, \mathcal{Q})$-structured space, then $C_p(X, \mathcal{I})$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.

Corollary

Let X be a pseudocompact Tychonoff space. Suppose that $C_p(X, \mathcal{I})$ has a clousure preserving closed cover $\mathcal{C}$ such that every element of it is a $(\mathcal{P}, \mathcal{Q})$-structured space, then $C_p(X)$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.

Some questions...

1. Are the $(\mathcal{P}, \mathcal{Q})$-structured spaces stable?
2. If $C_p(X)$ is a $(\mathcal{P}, \mathcal{Q})$-structured space must be a $L\Sigma$ space?
3. Is the product of a $(\mathcal{P}, \mathcal{Q})$-structured space and a $L\Sigma$ space a $(\mathcal{P}, \mathcal{Q})$-structured space?

Obrigado!